COMMUTATOR CONDITIONS IMPLYING THE CONVERGENCE OF THE LIE–TROTTER PRODUCTS

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Abstract. In this paper we investigate commutator conditions for two strongly continuous semigroups \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) on a Banach space implying the convergence of the Lie–Trotter products \([T(\frac{1}{n}) S(\frac{1}{n})]^n\). The results are then applied to various examples and, in particular, to the Ornstein–Uhlenbeck operator.

1. Introduction

In 1959 H.F. Trotter [13] obtained an explicit product formula for semigroups whose generator is the sum of two generators. Using this idea, more general product formulas were considered by P.R. Chernoff [4, Ch. 1, 2]. Following this approach one can prove the theorem below (cf. [6, Ch. III, Cor. 5.8]).

Theorem 1. Let \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) be strongly continuous semigroups on a Banach space \(E\) with generators \((A, D(A))\) and \((B, D(B))\), respectively, satisfying the stability condition

\[
\| [T(\frac{1}{n}) S(\frac{1}{n})]^n \| \leq M e^{\omega t}
\]

for all \(t \geq 0, n \in \mathbb{N}, \) and some constants \(M \geq 1, \omega \in \mathbb{R}\). Consider the sum \(A + B\) on a subspace \(D \subseteq D(A) \cap D(B)\) and assume that \(D\) and \((\lambda_0 - A - B)D\) are dense in \(E\) for some \(\lambda_0 > \omega\).

Then the closure of \(A + B\) exists and generates a strongly continuous semigroup \((U(t))_{t \geq 0}\) given by the Lie–Trotter product formula

\[
U(t)f = \lim_{n \to \infty} [T(\frac{1}{n}) S(\frac{1}{n})]^n f
\]

where the limit exists for all \(f \in E\) uniformly for \(t\) in compact intervals in \(\mathbb{R}_+\).

In [8] we gave an example of operators \(A\) and \(B\) such that the closure of the sum of \(A\) and \(B\) is a generator, but [11], and hence [2], is violated. Thus, to obtain the Lie–Trotter formula [2] a stability condition as in [11] is necessary.

On the other hand, in Theorem [11] one requires the range condition \((\lambda_0 - A - B)D = E\), which is sometimes hard to verify. The aim of this work is to replace the
range condition by assumptions on the commutator of \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\), of \(A\) and \(B\), and their resolvents, respectively. At first glance, the commutator condition and introducing a new norm \(||| \cdot |||\) seems quite sophisticated. However, in the applications of our results in Section 4 and 5 we will see that we can use very simple norms. In particular, for the Ornstein–Uhlenbeck operator a natural Sobolev norm will work.

With our results we obtain a new approach for the Ornstein-Uhlenbeck semigroup in finite dimension as the limit of the Lie–Trotter products of a degenerate diffusion semigroup and a semigroup induced by a flow.

2. Commutator conditions for semigroups and generators

To use commutator conditions to establish the convergence of the Lie–Trotter products, we modify the stability estimate (1).

**Definition 2.** Let \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) be semigroups on a normed vector space \((F, ||| \cdot |||)\). The semigroup \((T(t))_{t \geq 0}\) is called exponentially bounded if

\[ |||T(t)||| \leq M e^{\omega t} \]

for all \(t \geq 0\) and some constants \(M \geq 1\), \(\omega \in \mathbb{R}\).

The semigroups \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) are called locally Trotter–stable if there exist constants \(t_0 > 0\) and \(M_{t_0} \geq 1\) such that

\[ |||T(t)S(t) - S(t)T(t)||| \leq M_{t_0} \]

for all \(t \in [0, t_0]\) and \(n \in \mathbb{N}\).

With this concept, we are able to state our main result.

**Theorem 3.** Let \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) be strongly continuous semigroups on a Banach space \(E\) and suppose that there exists a normed vector space \((F, ||| \cdot |||)\) which is densely embedded in \(E\) and invariant under both semigroups such that the following conditions hold:

(a) The two semigroups are exponentially bounded on \(F\) and locally Trotter–stable on \(E\) and \(F\).

(b) The commutator condition

\[ |||T(t)S(t)f - S(t)T(t)f||| \leq t^\alpha M_1 |||f||| \]

is satisfied for all \(f \in F\), and \(t \in [0, \delta]\) with some \(\delta > 0\), and for some constants \(\alpha > 1\) and \(M_1 \geq 0\).

Then the Lie–Trotter product formula holds, i.e.,

\[ U(t)f := \lim_{n \to \infty} \left[ T(t)S(t) \right]^n f \]

exists for all \(f \in E\) uniformly for \(t\) in compact intervals in \(\mathbb{R}_+\). Moreover, \((U(t))_{t \geq 0}\) is a strongly continuous semigroup with generator \((G, D(G))\) such that

\[ G \supseteq \frac{\alpha}{\alpha \delta} [T(t)S(t)]_{t=0} \supseteq A + B \]

where each operator is defined on its natural domain.

**Proof.** We denote the Lie–Trotter products by

\[ P_n(t) := \left[ T(t)S(t) \right]^n \]
for $n \in \mathbb{N}$ and $t \geq 0$. Let $t_0 > 0$ such that the stability condition \((3)\) is fulfilled on 
\((E, \| \cdot \|)\) and \((F, \| \cdot \|)\). Therefore, we obtain
\[
\| P_n(t) \| = \| P_k(\frac{t}{m}) P_l(\frac{t}{n}) \| \leq M_{t_0}^2
\]
for $t \in [0, \frac{3t_0}{2}]$ and $n = k + l$, $k, l \in \mathbb{N}$, such that $\frac{k}{n}, \frac{l}{n} \leq t_0$, $n \geq 2$. The analogous estimate also holds for \(B_t \cdot \| \). By induction we conclude that
\[
(4) \quad \| P_n(t) \| \leq M_T \quad \text{and} \quad \| P_n(t) \| \leq M_T
\]
for $(n, t) \in \mathbb{N} \times [0, T]$, $T > 0$, and a constant $M_T \geq 0$.

Now fix $T > 0$ and let $t \in [0, T]$ and $f \in F$. We choose $k, m, n \in \mathbb{N}$ with $0 < k \leq m$ such that $\frac{t}{m} \leq \delta$ and $m = kn$. Then, we obtain from the commutator condition \((b)\) that
\[
\| T(\frac{1}{m})S(\frac{1}{m}) f - S(\frac{1}{m})T(\frac{1}{m}) f \|
\leq \sum_{j=0}^{k-1} \| S(\frac{1}{m}) (T(\frac{1}{m})S(\frac{1}{m}) - S(\frac{1}{m})T(\frac{1}{m})) S(\frac{1}{m}) (T(\frac{1}{m})S(\frac{1}{m}))^{k-1-j} f \|
\leq j \left( \frac{1}{m} \right)^{\alpha} M_2 \| f \|
\]
for $j \in \mathbb{N}$ and some constant $M_2 \geq 0$. Hence, by forming a telescope sum, we have
\[
\| [T(\frac{1}{m})S(\frac{1}{m})]^{j-1} f - [T(\frac{1}{m})S(\frac{1}{m})]^{k-j} f \|
= \sum_{j=1}^{k-1} \| T(\frac{1}{m}) (T(\frac{1}{m})S(\frac{1}{m}) - S(\frac{1}{m})T(\frac{1}{m})) S(\frac{1}{m}) [T(\frac{1}{m})S(\frac{1}{m})]^{k-1-j} f \|
\leq M_3 \sum_{j=1}^{k-1} j \cdot \left( \frac{1}{m} \right)^{\alpha} \| f \|
= M_3 \frac{t^{\alpha}(k-1)k}{2m^{\alpha}} \| f \|
\]
for a suitable constant $M_3 \geq 0$. Note that
\[
\| [T(\frac{1}{m})S(\frac{1}{m})]^{j} f \| = \| T\left(\frac{t^{m}}{m}j\right)S\left(\frac{t^{m}}{m}j\right) \| \leq M_T
\]
for $j \in \mathbb{N}$ and $1 \leq j \leq m$. We now conclude that
\[
(5) \quad \| P_n(t) f - P_m(t) f \| = \| [T(\frac{t}{m})S(\frac{t}{m})]^{n} f - [T(\frac{t}{m})S(\frac{t}{m})]^{k-n} f \|
\leq \sum_{l=0}^{n-1} \| [T(\frac{1}{m})S(\frac{1}{m})]^{j} \left( T(\frac{t}{m})S(\frac{t}{m}) - [T(\frac{1}{m})S(\frac{1}{m})]^{k-l} \right) [T(\frac{1}{m})S(\frac{1}{m})]^{k-(n-l)} f \|
\leq M_4 \frac{t^{\alpha} k(k-1)k}{2m^{\alpha}} \| f \|
\leq M_4 \frac{t^{\alpha} k}{2m^{\alpha-1}} \| f \|
\leq M_5 \frac{k}{2m^{\alpha-1}} \| f \|
\]
for suitable constants $M_4, M_5 \geq 0$. 

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Thus, for arbitrary \(i, j \in \mathbb{N}\) and \(0 \leq t \leq T\), we have

\[
\|P_{2^i}(t)f - P_{2^j}(t)f\| \leq \sum_{l=i}^{j-1} \|P_{2^l}(t)f - P_{2^{l+1}}(t)f\| \leq M_5 \sum_{l=i}^{j-1} \left( \frac{1}{2^{l-1}} \right)^{l+1} \|f\|.
\]

The last expression converges to 0 as \(i, j \to \infty\), since \(\alpha > 1\). Hence,

\[
(6) \quad U(t)f := \lim_{i \to \infty} P_{2^i}(t)f
\]

exists for \(f \in F\) uniformly for \(t \in [0, T]\). Due to (4) we can extend \(U(t)\) to a bounded linear operator on \(E\).

For arbitrary \(n \in \mathbb{N}\), we obtain from (5) that

\[
(7) \quad \|P_n(t)f - U(t)f\| \\
\leq \|P_n(t)f - P_{n2^i}(t)f\| + \|P_{n2^i}(t)f - P_{2^i}(t)f\| + \|P_{2^i}(t)f - U(t)f\| \\
\leq \sum_{k=0}^{l-1} \|P_{n2^k}(t)f - P_{n2^{k+1}}(t)f\| + \|P_{n2^k}(t)f - P_{2^k}(t)f\| + \|P_{2^k}(t)f - U(t)f\| \\
\leq \frac{M_5}{n^{\alpha-1}} \sum_{k=0}^{l-1} \left( \frac{1}{2^{\alpha-1}} \right)^{k+1} \|f\| + M_5 \frac{n}{(n2^l)^{\alpha-1}} \|f\| + \|P_{2^l}(t)f - U(t)f\| \\
\leq \left( \frac{2^{\alpha-1}M_5}{(2^{\alpha-1} - 1)n^{\alpha-1}} + \frac{M_5n^{2-\alpha}}{2^{l(\alpha-1)}} \right) \|f\| + \|P_{2^l}(t)f - U(t)f\|
\]

for \(l \in \mathbb{N}\).

Fix \(\epsilon > 0\). By the definition of the operators \(U(t)\), there exists \(l_0 \in \mathbb{N}\) such that

\[
\|P_{2^l}(t)f - U(t)f\| \leq \epsilon
\]

for \(f \in F\) and \(l \geq l_0\). Further, for \(n \in \mathbb{N}\) with \(\left( \frac{1}{n} \right)^{\alpha-1} \leq \frac{(2^{\alpha-1} - 1)}{2^{\alpha-1}M_5}\) and \(l \geq \max\{l_0, l_1\}\) such that \(2^{l_1(\alpha-1)} \geq \left( \frac{M_5n^{2-\alpha}}{\epsilon^2} \right)\), we have

\[
\|P_n(t)f - U(t)f\| \leq \left( \frac{2^{\alpha-1}M_5}{(2^{\alpha-1} - 1)n^{\alpha-1}} + \frac{M_5n^{2-\alpha}}{2^{l(\alpha-1)}} \right) \|f\| + \|P_{2^l}(t)f - U(t)f\| \\
\leq 3\epsilon (\|f\| + \|f\|)
\]

for \(f \in F\) uniformly for \(t \in [0, T]\). So the stability condition (4) implies that \(P_n(t)\) converges strongly in \(E\) uniformly for \(t \in [0, T]\).

To show the semigroup law for \((U(t))_{t \geq 0}\) and for rational numbers \(t\), we use an idea from the paper of Chernoff (cf. [3, Thm. 2.5.1]). To that purpose, let \(f \in E\).
and $\epsilon > 0$. Then

\[
\|U(t)f - U(\frac{t}{k})U(\frac{1}{k})f\| \\
\leq \|U(t)f - [T(\frac{1}{2k}) S(\frac{1}{2n})]^{2n} f\| + \| [T(\frac{1}{2k}) S(\frac{1}{2n})]^{2n} f - U(\frac{1}{k})U(\frac{1}{k})f\| \\
\leq \epsilon + \| [T(\frac{1}{2k}) S(\frac{1}{2n})]^{2n} (\frac{1}{2n}) f\| \\
+ \| (\frac{1}{2n}) U(\frac{1}{k}) U(\frac{1}{k}) f\| \\
\leq 3\epsilon
\]

for $n$ sufficiently large. This proves that $U(t) = U(\frac{1}{k})U(\frac{1}{k})$. In an analogous way, one can show $U(nt) = U(t)^n$ for $n \in \mathbb{N}$, from which one obtains the semigroup law for rational numbers.

By the uniform convergence of the Lie–Trotter products on compact intervals, we obtain that $(U(t))_{t \geq 0}$ is a strongly continuous semigroup. The assertion concerning the generator $G$ follows from [4, Prop. 4.1].

In the following, we show how commutator conditions on the generators $A$ and $B$ imply condition (b) of Theorem 3. These conditions can be verified in some interesting applications (see Section 4 and 5).

**Definition 4.** Let $(A, D(A))$ and $(B, D(B))$ be linear operators on a Banach space $E$. The operator

\[ C := AB - BA \quad \text{with domain} \]
\[ D(C) := D(AB) \cap D(BA) = \{ f \in D(A) \cap D(B) : Bf \in D(A) \text{ and } Af \in D(B) \} \]

is called the **commutator** of $A$ and $B$.

In the following we denote by $\| \cdot \|_C$ the **C–norm** defined by

\[ \|f\|_C := \|f\| + \|Cf\| \]

for all $f \in D(C)$.

The next lemma allows to relate the commutator of $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ to the commutator $C$.

**Lemma 5.** Let $(A, D(A))$ and $(B, D(B))$ be generators of strongly continuous semigroups $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ on a Banach space $E$, and let $(C, D(C))$ be the commutator of $A$ and $B$. Suppose that there exists a subspace $F \subseteq D(C)$ which is dense in $E$ and invariant under $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ such that for some $\delta > 0$ and $t \in [0, \delta]$ the functions

\[
\left\{ \begin{array}{lcl}
    s \mapsto AS(t)S(t)s & \text{and} & s \mapsto CS(s) f \\
    s \mapsto AS(s) f \text{ is differentiable}
\end{array} \right. \\
\text{in } E \text{ for } s \in [0, \delta] \text{ and for all } f \in F. \text{ Then the identity}
\]
\[ (8) \quad T(t)S(t)f - S(t)T(t)f = \int_0^t T(s) \left( \int_0^s S(t - r)CS(r)T(t - s)f dr \right) ds \]

holds for all $f \in F$ and $t \in [0, \delta]$.

**Proof.** Let $t \in [0, \delta]$ with $\delta > 0$. We apply the fundamental theorem of calculus to the continuously differentiable functions

\[ s \mapsto T(s)S(t)T(t - s)f \quad \text{and} \quad r \mapsto S(t - r)AS(r)f \]
for $0 \leq r, s \leq \delta$ and $f \in F$. Since the operator $A$ is closed, we obtain by the differentiability of the second function that

$$\frac{d}{dr} S(t-r)AS(r)f = S(t-r)CS(r)f$$

for $0 \leq r \leq \delta$ and $f \in F$. Thus, we have

$$T(t)S(t)f - S(t)T(t)f = \int_0^t \frac{d}{ds} \{T(s)S(t)T(t-s)f\} \, ds = \int_0^t T(s) (AS(t) - S(t)A) T(t-s) f \, ds$$

$$= \int_0^t T(s) \left( \int_0^t \frac{d}{dr} \{S(t-r)AS(r)T(t-s)f\} \, dr \right) ds$$

$$= \int_0^t T(s) \left( \int_0^t S(t-r)CS(r)T(t-s)f \, dr \right) ds$$

for $f \in F$ and $t \in [0, \delta]$.

As an easy consequence of Lemma 5 we obtain the following result.

**Corollary 6.** Let $(A, D(A))$ and $(B, D(B))$ be generators of strongly continuous semigroups $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ on a Banach space $E$ and let $(C, D(C))$ be the commutator of $A$ and $B$. Suppose that:

(a) there exists a normed vector space $(F, ||||)$, $F \subseteq D(C)$, which is dense in $E$ and invariant under both semigroups, and the norm $||||$ is finer than the $C$-norm $\cdot ||$,

(b) there exists $\delta > 0$ such that the functions

\[
\begin{cases}
  s \mapsto AS(t)T(s)f, & s \mapsto CS(s)f \\
  s \mapsto AS(s)f
\end{cases}
\]

are continuous and $s \mapsto AS(s)f$ is differentiable in $E$ for $s, t \in [0, \delta]$ and $f \in F$, and

(c) the two semigroups are exponentially bounded on $F$ and locally Trotter-stable on $E$ and $F$.

Then the conclusion of Theorem 3 holds.

**Proof.** By the exponential boundedness of the semigroups and Lemma 5 we obtain the estimate

$$\|T(t)S(t)f - S(t)T(t)f\| = \| \int_0^t T(s) \left( \int_0^t S(t-r)CS(r)T(t-s)f \, dr \right) ds \|$$

$$\leq Mt^2 \sup_{0 \leq r, s \leq t} \|S(r)T(t-s)f\|_C$$

$$\leq \tilde{M}t^2 ||f||$$

for $f \in F$, $t \in [0, \delta]$ and some constants $M, \tilde{M} \geq 0$. Thus, assumption (b) of Theorem 3 is fulfilled. Since condition (a) of Theorem 3 was assumed explicitly, the assertion follows. \[\square\]
3. Commutator conditions for resolvent operators

In the following we express the commutator condition on the semigroups \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) in Theorem 3 by a corresponding condition on the powers of the resolvent operators \(R(\lambda, A)\) and \(R(\mu, B)\), respectively. In special cases, only a commutator condition on \(R(\lambda, A)\) and \(R(\mu, B)\) will be necessary.

First, we state the following result on the asymptotic behaviour of the gamma function.

**Lemma 7.** The gamma function defined by

\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad \text{for Re} z > 0
\]

satisfies

\[
\lim_{n \to \infty} n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)} = 1 \quad \text{for } a, b \in \mathbb{R}.
\]

**Proof.** By Stirling’s Formula we can conclude that

\[
\Gamma(x + 1) = \sqrt{2\pi} x^{x+1/2}e^{-x}(1 + \phi(x))
\]

for \(x \in \mathbb{R}\), where \(\phi\) is a function converging to zero as \(\frac{x}{\log x}\) (see [1] p. 652). Therefore, we have

\[
n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)} = \left(\frac{n+a-1}{n+b-1}\right)^{n+a-1/2} \left(\frac{n}{n+b-1}\right)^{b-a} \frac{1 + \phi(n+a-1)}{1 + \phi(n+b-1)}
\]

for \(a, b \in \mathbb{R}\) which converges to 1 as \(n \to \infty\). 

In the sequel, we assume without loss of generality that \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) are bounded semigroups. Note that we now use a commutator condition on the semigroups which is slightly different from the one in Theorem 3.

**Theorem 8.** Let \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) be bounded strongly continuous semigroups on a Banach space \(E\) with generators \((A, D(A))\) and \((B, D(B))\), respectively, and suppose that there exists a normed vector space \((F, \|\cdot\|)\) which is embedded in \(E\). Furthermore, let \(\alpha, \beta \geq 0\) and \(M \geq 0\). Then the following statements are equivalent:

(i) The semigroups satisfy the commutator condition

\[
\|T(t)S(s)f - S(s)T(t)f\| \leq t^\alpha s^\beta M||f||
\]

for all \(f \in F\) and \(s, t \geq 0\).

(ii) The resolvent operators satisfy the commutator condition

\[
\|R(\lambda, A)^nR(\mu, B)^n f - R(\mu, B)^n R(\lambda, A)^n f\| \leq \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)!^{2n}} \lambda^{\alpha+n} \mu^{\beta+n} M||f||
\]

for all \(f \in F\), \(\lambda, \mu > 0\) and \(n \in \mathbb{N}\).
Proof. (i) ⇒ (ii). Let \( f \in F \) and \( \lambda, \mu > 0 \). We use the Laplace representation of the resolvent (see [6, Ch. II, Thm. 1.10]) and obtain

\[
\| R(\lambda, A)^n R(\mu, B) f - R(\lambda, B)^n R(\lambda, A)^n f \| \\
\leq \frac{1}{(n-1)^2} \int_0^\infty \int_0^{\infty} r^{n-1}s^{n-1}e^{-\lambda r}e^{-\mu s}\|T(r)S(s)f - S(s)T(r)f\| \, dsdr \\
\leq \frac{M}{(n-1)^2} \int_0^\infty \int_0^{\infty} r^{n-1+\alpha}e^{-\lambda r}s^{n-1+\beta}e^{-\mu s}drds \|f\| \\
= \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)^2\lambda^{\alpha+n}\mu^{\beta+n}} \int_0^\infty r^{n-1+\alpha}e^{-r}dr \int_0^\infty s^{n-1+\beta}e^{-s}ds \|f\| \\
= \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)^2\lambda^{\alpha+n}\mu^{\beta+n}} |M| \|f\| \\
\text{for all } n \in \mathbb{N}. 
\]

(ii) ⇒ (i). Recall that by the Post–Widder Inversion Formula

\[
T(t)f = \lim_{n \to \infty} \left[ \frac{t}{\pi} R(\frac{\lambda}{\pi}, A) \right]^n f
\]

for \( f \in E \) uniformly for \( t \) in compact intervals in \( \mathbb{R}_+ \) (\([6\text{, Ch. III, Cor. 5.5}]\)). Let \( f \in F \) and \( s, t \geq 0 \). Applying Lemma 7, we conclude that

\[
\frac{n^2}{t^{n}s^n} \| R(\frac{\lambda}{\pi}, A)^n R(\frac{\mu}{\pi}, B)^n f - R(\lambda, B)^n R(\lambda, A)^n f \| \\
\leq \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)^2 \lambda^{\alpha+n}\mu^{\beta+n}} M \|f\| \\
= \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)^2 \lambda^{\alpha+n}\mu^{\beta+n}} M \|f\| \\
\to t^{\alpha}s^{\beta}M \|f\| \\
as n \to \infty \text{ which implies (i).} \]

We now discuss some cases where it is enough to impose a commutator condition on the resolvent operators and not on all their powers.

**Proposition 9.** Let \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) be bounded strongly continuous semigroups on a Banach space \( E \) with generators \((A, D(A))\) and \((B, D(B))\), respectively. Let \( \alpha, \beta \geq 0 \) such that \( \alpha + \beta \geq 2 \). Then the following statements are equivalent:

(i) The semigroups satisfy the commutator condition
\[
\| T(t)S(s)f - S(s)T(t)f \| \leq t^{\alpha}s^{\beta}M_1 \|f\|
\]
for all \( f \in E \), \( s, t \geq 0 \), and a constant \( M_1 \geq 0 \).

(ii) The resolvent operators satisfy the commutator condition
\[
\| R(\lambda, A)R(\mu, B)f - R(\mu, B)R(\lambda, A)f \| \leq \frac{M_2}{\lambda^{\alpha+1}\mu^{\beta+1}} \|f\|
\]
for all \( f \in E \), \( \lambda, \mu > 0 \), \( n \in \mathbb{N} \), and a constant \( M_2 \geq 0 \). Moreover, if \( \alpha + \beta > 2 \), then the semigroups \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) commute.

**Proof.** The implication (i) ⇒ (ii) is proved analogously as in Theorem 8.

(ii) ⇒ (i). For simplicity, we define
\[
\mathcal{R}_A(\frac{\lambda}{\pi}) := \frac{\lambda}{\pi} R(\frac{\lambda}{\pi}, A) \ , \ \mathcal{R}_B(\frac{\mu}{\pi}) := \frac{\mu}{\pi} R(\frac{\mu}{\pi}, B)
\]
and the commutator
\[ \left[ \mathcal{R}_A \left( \frac{\partial}{\partial t} \right), \mathcal{R}_B \left( \frac{\partial}{\partial t} \right) \right] := \frac{\partial}{\partial t} \mathcal{R}_A (\frac{\partial}{\partial t}, A) \mathcal{R}_B (\frac{\partial}{\partial t}, B) - \frac{\partial}{\partial t} \mathcal{R}_B (\frac{\partial}{\partial t}, B) \mathcal{R}_A (\frac{\partial}{\partial t}, A) \]
for \( n \in \mathbb{N} \) and \( t \geq 0 \). Since the semigroups are bounded there exists a constant \( M \geq 0 \) such that
\[
\| (\mathcal{R}_A (\frac{\partial}{\partial t}))^j \|, \| (\mathcal{R}_B (\frac{\partial}{\partial t}))^j \| \leq M
\]
for all \( j, n \in \mathbb{N} \) such that \( 1 \leq j \leq n \), and for all \( t, s \geq 0 \). By assumption, we have
\[
\| \left[ \mathcal{R}_A \left( \frac{\partial}{\partial t} \right), \mathcal{R}_B \left( \frac{\partial}{\partial t} \right) \right] \| \leq M_2 \left( \frac{t}{n} \right)^{\alpha + \beta}
\]
for all \( n \in \mathbb{N}, t \geq 0 \) and some constant \( M_2 \geq 0 \). Let \( f \in E \) and \( t, s \geq 0 \). Then
\[
\| T(t)S(s)f - S(s)T(t)f \|
= \lim_{n \to \infty} \| (\mathcal{R}_A (\frac{\partial}{\partial t}))^n (\mathcal{R}_B (\frac{\partial}{\partial t}))^n f - (\mathcal{R}_B (\frac{\partial}{\partial t}))^n (\mathcal{R}_A (\frac{\partial}{\partial t}))^n f \|
\]
and
\[
(\mathcal{R}_A (\frac{\partial}{\partial t}))^n (\mathcal{R}_B (\frac{\partial}{\partial t}))^n f - (\mathcal{R}_B (\frac{\partial}{\partial t}))^n (\mathcal{R}_A (\frac{\partial}{\partial t}))^n f
= (\mathcal{R}_A (\frac{\partial}{\partial t})) \left[ (\mathcal{R}_A (\frac{\partial}{\partial t}))^{-1}, (\mathcal{R}_B (\frac{\partial}{\partial t}))^{-1} \right] (\mathcal{R}_B (\frac{\partial}{\partial t})) f
+ (\mathcal{R}_A (\frac{\partial}{\partial t})) (\mathcal{R}_B (\frac{\partial}{\partial t}))^{-1} (\mathcal{R}_A (\frac{\partial}{\partial t}))^{-1} (\mathcal{R}_B (\frac{\partial}{\partial t})) f
+ (\mathcal{R}_B (\frac{\partial}{\partial t}))^{-1} [ (\mathcal{R}_A (\frac{\partial}{\partial t})) (\mathcal{R}_B (\frac{\partial}{\partial t})) ] f
\]
for all \( n \in \mathbb{N} \). Moreover, we can estimate
\[
\| \left[ (\mathcal{R}_A (\frac{\partial}{\partial t}))^n, (\mathcal{R}_B (\frac{\partial}{\partial t}))^n \right] \| \leq \sum_{j=0}^{n-1} \| (\mathcal{R}_A (\frac{\partial}{\partial t}))^{n-1-j} (\mathcal{R}_A (\frac{\partial}{\partial t}))^j \| f \|
\leq nM^2M_2 \left( \frac{t}{n} \right)^{\alpha} \left( \frac{s}{n} \right)^{\beta} \| f \|,
\|
[ (\mathcal{R}_A (\frac{\partial}{\partial t}))^n, (\mathcal{R}_B (\frac{\partial}{\partial t}))^n ] \| \leq (n-1)M^2M_2 \left( \frac{t}{n} \right)^{\alpha} \left( \frac{s}{n} \right)^{\beta} \| f \|,
\]
and by induction
\[
\| \left[ (\mathcal{R}_A (\frac{\partial}{\partial t}))^j, (\mathcal{R}_B (\frac{\partial}{\partial t}))^j \right] \| \leq j^2M^2M_2 \left( \frac{t}{n} \right)^{\alpha} \left( \frac{s}{n} \right)^{\beta} \| f \|
\]
for each \( j, n \in \mathbb{N} \) such that \( 1 \leq j \leq n \). Therefore, equation (10) yields
\[
\| T(t)S(s)f - S(s)T(t)f \|
= \lim_{n \to \infty} \| (\mathcal{R}_A (\frac{\partial}{\partial t}))^n (\mathcal{R}_B (\frac{\partial}{\partial t}))^n f - (\mathcal{R}_B (\frac{\partial}{\partial t}))^n (\mathcal{R}_A (\frac{\partial}{\partial t}))^n f \|
\leq \lim_{n \to \infty} \frac{n^2t^\alpha s^\beta}{n^{\alpha+\beta}} M^2M_2 \| f \|
\leq t^\alpha s^\beta M_2 \| f \|
\]
for some constant \( M_2 \geq 0 \).

Using the estimate (11), it follows that the semigroups commute if \( \alpha + \beta > 2 \).

A similar commutator condition for the resolvent operators in the context of nonautonomous evolution equations was studied in [9].
4. Applications to semigroups from quantum mechanics

First, we apply our results to groups arising in quantum mechanics. Let \((T(t))_{t \in \mathbb{R}}\) and \((S(t))_{t \in \mathbb{R}}\) be a pair of unitary groups on a complex Hilbert space \(H\) satisfying the so-called Weyl relation (see [10, p. 274])

\[
T(t)S(s) = e^{ist}S(s)T(t) \quad \text{for all } s, t \in \mathbb{R}.
\]

These semigroups are evidently exponentially bounded and locally Trotter–stable on \(H\). Moreover, condition (b) of Theorem 3 follows from the estimate

\[
\|T(t)S(t)f - S(t)T(t)f\| = \|e^{it^2} - 1\| S(t)T(t)f \leq \int_0^t \frac{(t')^{k-1}}{k!} \|f\| \leq t^2 e^{t^2} \|f\|
\]

for \(f \in H\) and \(t \geq 0\). Therefore, the Lie–Trotter product formula holds.

Based on this relation, P.E.T. Jørgensen, R.T. Moore [7, Ch. 11] and H. Suzuki [12] treated generalized Weyl relations leading to the following result.

**Corollary 10.** Let \((T(t))_{t \geq 0}, (S(t))_{t \geq 0}\) and \((V(t))_{t \geq 0}\) be strongly continuous semigroups on a Banach space \(E\) with generators \((A, D(A)), (B, D(B))\) and \((C, D(C))\), respectively. Suppose that there exists a subspace \(F \subseteq D(C)\) which is dense in \(E\) and invariant under the semigroups \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) such that the following conditions hold:

(a) The semigroups \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) are exponentially bounded on \((F, \| \cdot \|_C)\) and locally Trotter–stable on both \(E\) and \((F, \| \cdot \|_C)\).

(b) The generalized Weyl relation holds, i.e.

\[
T(t)S(t)f = V(t^2)S(t)T(t)f
\]

for all \(f \in E\) and \(t \geq 0\).

Then the conclusion of Theorem 3 holds.

**Proof.** Fix \(\delta > 0\). We can estimate the commutator of \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) as

\[
\|T(t)S(t)f - S(t)T(t)f\| = \| (V(t^2) - Id) [S(t)T(t)] f \| = \| \int_0^t e^{s^2} V(s)C [S(t)T(t)] f ds \| \leq t^2 M \|f\|_C
\]

for \(f \in F\), \(t \in [0, \delta]\) and some constant \(M \geq 0\). So Theorem 3 implies the assertion.

\(\Box\)

5. Applications to Ornstein–Uhlenbeck operators

In this section we consider the (finite–dimensional) Ornstein–Uhlenbeck operator which has been studied e.g. in [5]. Let \(E := C_0(\mathbb{R}^d)\) or \(L^p(\mathbb{R}^d), 1 \leq p < \infty\). For any symmetric, positive semi–definite matrix \(A := (a_{ij})\) and a matrix \(B := (b_{ij}) \in \mathcal{L}(\mathbb{R}^d)\), the Ornstein–Uhlenbeck operator is defined by

\[
[Of](x) := \sum_{i,j=1}^d a_{ij} D_{ij}f(x) + \sum_{i,j=1}^d b_{ij} x_j D_i f(x) = \langle \nabla, Af(x) \rangle + \langle Bx, \nabla f(x) \rangle
\]

for each \(f \in S(\mathbb{R}^d), x \in \mathbb{R}^d\), the Schwartz space of rapidly decreasing functions, \(x \in \mathbb{R}^d\), and \(\nabla := (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})\). Without loss of generality, we write \(A\) as
diag(a_1, \ldots, a_d) where a_k > 0 for 1 \leq k \leq j, a_{j+1} = \cdots = a_d = 0 and define the operators \( A \) and \( B \) as the closure of
\[
A f := \langle \nabla, A \nabla f \rangle \quad \text{and} \quad [B f](x) := \langle B x, \nabla f(x) \rangle
\]
for \( f \in S(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \), respectively (cf. [13, Ch. II, Ch.6]). The operator \((A, D(A))\) generates a strongly continuous semigroup \((T(t))_{t \geq 0}\) given by
\[
T(t)f(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{\sum_{i=1}^{d} |x_i-s_i|^2}{4at}} f(s_1, \ldots, s_j, x_{j+1}, \ldots, x_d)ds_1 \cdots ds_j
\]
for \( t > 0, x \in \mathbb{R}^d \), and \( f \in E \) (see [4]). Furthermore, the operator \((B, D(B))\) generates the strongly continuous semigroup \((S(t))_{t \geq 0}\) given by
\[
S(t)f(x) = f(e^{tB}x)
\]
for all \( f \in E \) and \( x \in \mathbb{R}^d \) (see [4, Ch. II, Sec. 3.28]).

These semigroups and their generators have the following useful properties.

**Lemma 11.** Let \((T(t))_{t \geq 0}, (S(t))_{t \geq 0}\) and their generators \((A, D(A))\) and \((B, D(B))\) be as above. Then the following properties hold:

(a) The semigroups \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) are locally Trotter–stable on \( E \).

(b) The commutator \( C \) of \((A, D(A))\) and \((B, D(B))\) is given by
\[
C f := ABf - BAf = 2\langle B A \nabla, \nabla f \rangle
\]
for all \( f \in S(\mathbb{R}^d) \).

(c) For \( t \geq 0 \) the functions
\[
\{ \begin{aligned} 
 s & \mapsto AS(t)T(s)f, \\ s & \mapsto CS(s)f \end{aligned} \}
\]
are continuous and
\[
\{ \begin{aligned} 
 s & \mapsto AS(s)f \end{aligned} \}
\]
is differentiable in \( E \) for \( s \geq 0 \) and for all \( f \in S(\mathbb{R}^d) \).

(d) The operator \( \langle B A \nabla, \nabla \rangle \) commutes with \((T(t))_{t \geq 0}\) on \( S(\mathbb{R}^d) \) and
\[
\langle B A \nabla, \nabla S(t)f \rangle = S(t)\langle e^{tB} B A e^{tB^*} \nabla, \nabla f \rangle
\]
for \( f \in S(\mathbb{R}^d) \), where \( B^* \) denotes the transpose of the matrix \( B \).

**Proof.** Clearly, the Schwartz space \( S(\mathbb{R}^d) \) is invariant under both semigroups and both generators \( A \) and \( B \). Moreover, \((T(t))_{t \geq 0}\) is a semigroup of contractions on \( E \) and
\[
\|S(t)f\|_{C_0} \leq \|f\|_{C_0} \quad \text{and} \quad \|S(t)f\|_{L^p} \leq e^{\omega t} \|f\|_{L^p}
\]
for \( f \in E \) and \( \omega := \frac{\max(B)}{p} \). It follows that the semigroups are locally Trotter–stable on \( E \).

To determine the commutator of \((A, D(A))\) and \((B, D(B))\) on \( S(\mathbb{R}^d) \), we note that
\[
\frac{\partial}{\partial x_k} \langle B x, \nabla f(x) \rangle = \langle B e_k, \nabla f(x) \rangle + \langle B x, \frac{\partial}{\partial x_k} \nabla f(x) \rangle
\]
for \( 1 \leq k \leq d, f \in S(\mathbb{R}^d), x \in \mathbb{R}^d, \) and the \( k \)-th unit vector \( e_k \). Hence, we have
\[
\frac{\partial^2}{\partial^2 x_k} \langle B x, \nabla f(x) \rangle = 2\langle B e_k, \frac{\partial}{\partial x_k} \nabla f(x) \rangle + \langle B x, \frac{\partial^2}{\partial x_k^2} f(x) \rangle
\]
for $f \in S(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Therefore, we obtain from identity (10) that

$$[ABf - BAf](x) = 2 \sum_{k=1}^{d} \langle Bc_k a_k \frac{\partial}{\partial x_k} \nabla f(x) \rangle + \sum_{k=1}^{d} \langle Bx, a_k \frac{\partial^2}{\partial x_k^2} \nabla f(x) \rangle$$

$$- \langle Bx, \nabla \sum_{k=1}^{d} a_k \frac{\partial^2}{\partial x_k^2} f(x) \rangle$$

$$= 2\langle BA \nabla, \nabla f(x) \rangle$$

for $f \in S(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

Clearly, the operators $A$ and $B$ are continuous for the usual topology on the space $S(\mathbb{R}^d)$ induced by the family of seminorms

$$p_{\alpha,n} = \sup_{x \in \mathbb{R}^d} |x^n D^n f(x)| .$$

By a straightforward computation, using \cite{11} Thm. 7.2 and Thm. 7.4, we obtain that the statements in (c) are fulfilled for the usual topology on $S(\mathbb{R}^d)$. Since this topology is finer than the topology on $E$, assertion (c) is proved.

Observe that $(T(t))_{t \geq 0}$ is a convolution semigroup and that convolution commutes with the differential operators $\frac{\partial}{\partial x_k} (1 \leq k \leq d)$. Therefore, the commutator of $(A, D(A))$ and $(B, D(B))$ commutes with $(T(t))_{t \geq 0}$. For the semigroup $(S(t))_{t \geq 0}$ we have the following commutator relation

$$\nabla S(t) f = \sum_{k=1}^{d} S(t) (e^{tB} e_k, \nabla f) e_k ,$$

and therefore

$$\langle BA \nabla, \nabla S(t) f \rangle = S(t) \langle e^{tB} BA e^{tB^*} \nabla, \nabla f \rangle$$

for $f \in S(\mathbb{R}^d)$.

We can now apply Corollary \[3\] The fact that the closure of $A + B$ is a generator seems to be known (cf. \cite{13}).

**Proposition 12.** Let $(T(t))_{t \geq 0}$, $(S(t))_{t \geq 0}$ be the strongly continuous semigroups on $E$ given by \cite{14} and \cite{15} generated by $(A, D(A))$ and $(B, D(B))$, respectively. Then the conclusion of Theorem \[3\] holds.

**Proof.** On $S(\mathbb{R}^d)$ we define the norm

$$|||f||| := \|f\| + \sum_{1 \leq i, j \leq d} \left\| \frac{\partial^2}{\partial x_i \partial x_j} f \right\|$$

for $f \in S(\mathbb{R}^d)$. This norm is finer than the $C$–norm $\| \cdot \|_C$.

In the next step, we show the exponential boundedness of the semigroups $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ on $S(\mathbb{R}^d; ||| \cdot |||)$. This is obvious for $(T(t))_{t \geq 0}$ which is $||| \cdot |||\text{-contractive}$ on $S(\mathbb{R}^d)$. On the other hand, we have

$$\left\| \frac{\partial^2}{\partial x_i \partial x_j} S(t)f \right\| = \| S(t)(e^{tB} e_{ij} e^{tB^*} \nabla, \nabla f) \| \leq e^{2\omega t} \| S(t) \| \cdot |||f|||$$
where
\[
e_{ij} := \begin{pmatrix}
\downarrow_i \\
0 \cdots 0 \cdots 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 \cdots 1 \cdots 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 \cdots 0 \cdots 0
\end{pmatrix}
\]
and \( \hat{\omega} := \max_{1 \leq i, j \leq d} |b_{ij}| \). Therefore, we can estimate
\[
|||S(t)f||| \leq d^2 e^{2t\hat{\omega}} |||S(t)||| \cdot |||f|||
\]
for \( f \in S(\mathbb{R}^d) \).

Finally, this implies the local Trotter–stability on \((S(\mathbb{R}^d), ||| \cdot |||)\)
\[
||| \left[ T\left( \frac{t}{n} \right) S\left( \frac{1}{n} \right) \right]^n ||| \leq e^{2t\hat{\omega}} |||f|||
\]
if \( E = C_0(\mathbb{R}^d) \), and
\[
||| \left[ T\left( \frac{t}{n} \right) S\left( \frac{1}{n} \right) \right]^n ||| \leq e^{t(\omega + 2\hat{\omega})} |||f|||
\]
if \( E = L^p(\mathbb{R}^d) \). We can now apply Corollary \([3]\) which concludes the proof.

As a concrete example we mention a semigroup appearing in mathematical finance (\([2]\)).

**Examples 13.** On the space \( E := C_0(\mathbb{R}^2) \) or \( L^p(\mathbb{R}^2) \) (\( 1 \leq p < \infty \)) the Cauchy problem
\[
\begin{align*}
\frac{d}{dt} u(t, x, y) &= \frac{\partial}{\partial x} u(t, x, y) + x \frac{\partial}{\partial y} u(t, x, y), \quad t \geq 0, \\
u(0, x, y) &= f(x, y),
\end{align*}
\]
is investigated in \([2]\). Taking
\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
the Cauchy problem \((17)\) can be reformulated as an Ornstein–Uhlenbeck operator \( O \) defined in \([13]\). Therefore, all assumptions of Proposition \([12]\) are satisfied and \( O \) generates a semigroup given as the limit of the Lie–Trotter products.

**References**


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