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COMMUTATOR CONDITIONS IMPLYING THE CONVERGENCE OF THE LIE–TROTTER PRODUCTS

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ABSTRACT. In this paper we investigate commutator conditions for two strongly continuous semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ on a Banach space implying the convergence of the Lie–Trotter products $[T(\frac{t}{n})S(\frac{t}{n})]^n$. The results are then applied to various examples and, in particular, to the Ornstein–Uhlenbeck operator.

1. INTRODUCTION

In 1959 H.F. Trotter [13] obtained an explicit product formula for semigroups whose generator is the sum of two generators. Using this idea, more general product formulas were considered by P.R. Chernoff [4, Ch. 1, 2]. Following this approach one can prove the theorem below (cf. [6, Ch. III, Cor. 5.8]).

Theorem 1. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be strongly continuous semigroups on a Banach space E with generators $(A, D(A))$ and $(B, D(B))$, respectively, satisfying the stability condition*

$$\| \left[T(\tfrac{t}{n})S(\tfrac{t}{n})\right]^n \| \leq Me^{\omega t} \tag{1}$$

for all $t \geq 0, n \in \mathbb{N}$, and some constants $M \geq 1, \omega \in \mathbb{R}$. Consider the sum $A + B$ on a subspace $D \subseteq D(A) \cap D(B)$ and assume that D and $(\lambda_0 - A - B)D$ are dense in E for some $\lambda_0 > \omega$.

Then the closure of $A + B$ exists and generates a strongly continuous semigroup $(U(t))_{t\geq 0}$ given by the Lie–Trotter product formula

$$U(t)f = \lim_{n\to\infty} \left[T(\tfrac{t}{n})S(\tfrac{t}{n})\right]^n f \tag{2}$$

where the limit exists for all $f \in E$ uniformly for t in compact intervals in $\mathbb{R}_+$.

In [8] we gave an example of operators A and B such that the closure of the sum of A and B is a generator, but (1), and hence (2), is violated. Thus, to obtain the Lie–Trotter formula (2) a stability condition as in (1) is necessary.

On the other hand, in Theorem 1 one requires the range condition $\overline{(\lambda_0 - A - B)D} = E$, which is sometimes hard to verify. The aim of this work is to replace the

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range condition by assumptions on the commutator of $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, of A and B, and their resolvents, respectively. At first glance, the commutator condition and introducing a new norm $|||\cdot|||$ seems quite sophisticated. However, in the applications of our results in Section 4 and 5 we will see that we can use very simple norms. In particular, for the Ornstein–Uhlenbeck operator a natural Sobolev norm will work.

With our results we obtain a new approach for the Ornstein-Uhlenbeck semigroup in finite dimension as the limit of the Lie–Trotter products of a degenerate diffusion semigroup and a semigroup induced by a flow.

2. Commutator conditions for semigroups and generators

To use commutator conditions to establish the convergence of the Lie–Trotter products, we modify the stability estimate (1).

Definition 2. Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be semigroups on a normed vector space $(F, |||\cdot|||)$. The semigroup $(T(t))_{t\geq 0}$ is called **exponentially bounded** if

$$|||T(t)||| \leq Me^{\omega t}$$

for all $t \geq 0$ and some constants $M \geq 1,\ \omega \in \mathbb{R}$.

The semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are called **locally Trotter–stable** if there exist constants $t_0 > 0$ and $M_{t_0} \geq 1$ such that

$$|||\left[T(\tfrac{t}{n})S(\tfrac{t}{n})\right]^n||| \leq M_{t_0} \tag{3}$$

for all $t \in [0, t_0]$ and $n \in \mathbb{N}$.

With this concept, we are able to state our main result.

Theorem 3. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be strongly continuous semigroups on a Banach space E and suppose that there exists a normed vector space $(F, |||\cdot|||)$ which is densely embedded in E and invariant under both semigroups such that the following conditions hold*:

(a) *The two semigroups are exponentially bounded on F and locally Trotter–stable on E and F.*

(b) *The commutator condition*

$$\|T(t)S(t)f - S(t)T(t)f\| \leq t^{\alpha} M_1 |||f|||$$

is satisfied for all $f \in F$, and $t \in [0, \delta]$ with some $\delta > 0$, and for some constants $\alpha > 1$ and $M_1 \geq 0$.

Then the Lie–Trotter product formula holds, i.e.,

$$U(t)f := \lim_{n\to\infty} \left[T(\tfrac{t}{n})S(\tfrac{t}{n})\right]^n f$$

exists for all $f \in E$ uniformly for t in compact intervals in $\mathbb{R}_+$. Moreover, $(U(t))_{t\geq 0}$ is a strongly continuous semigroup with generator $(G, D(G))$ such that

$$G \supseteq \tfrac{d}{dt}\left[T(t)S(t)\right]_{|t=0} \supseteq A + B\ ,$$

where each operator is defined on its natural domain.

Proof. We denote the Lie–Trotter products by

$$P_n(t) := \left[T(\tfrac{t}{n})S(\tfrac{t}{n})\right]^n$$

for $n \in \mathbb{N}$ and $t \geq 0$. Let $t_0 > 0$ such that the stability condition (3) is fulfilled on $(E, \|\cdot\|)$ and $(F, |||\cdot|||)$. Therefore, we obtain

$$\|P_n(t)\| = \|P_k(\tfrac{kt}{n})P_l(\tfrac{lt}{n})\| \leq M_{t_0}^2$$

for $t \in [0, \frac{3t_0}{2}]$ and $n = k + l,\ k, l \in \mathbb{N}$, such that $\frac{kt}{n},\ \frac{lt}{n} \leq t_0,\ n \geq 2$. The analogous estimate also holds for $|||\cdot|||$. By induction we conclude that

$$\|P_n(t)\| \leq M_T \quad \text{and} \quad |||P_n(t)||| \leq M_T \tag{4}$$

for $(n, t) \in \mathbb{N} \times [0, T]$, $T > 0$, and a constant $M_T \geq 0$.

Now fix $T > 0$ and let $t \in [0, T]$ and $f \in F$. We choose $k, m, n \in \mathbb{N}$ with $0 < k \leq m$ such that $\frac{T}{m} \leq \delta$ and $m = k\,n$. Then, we obtain from the commutator condition (b) that

$$\begin{aligned}
&\|T(\tfrac{t}{m})S(\tfrac{jt}{m})f - S(\tfrac{jt}{m})T(\tfrac{t}{m})f\| \\
&\quad \leq \sum_{l=0}^{j-1} \|S(\tfrac{lt}{m})\left(T(\tfrac{t}{m})S(\tfrac{t}{m}) - S(\tfrac{t}{m})T(\tfrac{t}{m})\right) S(\tfrac{(j-1-l)t}{m})f\| \\
&\quad \leq j\left(\tfrac{t}{m}\right)^{\alpha} M_2 |||f|||
\end{aligned}$$

for $j \in \mathbb{N}$ and some constant $M_2 \geq 0$. Hence, by forming a telescope sum, we have

$$\begin{aligned}
&\|\left[T(\tfrac{kt}{m})S(\tfrac{kt}{m})\right] f - \left[T(\tfrac{t}{m})S(\tfrac{t}{m})\right]^k f\| \\
&\quad = \sum_{j=1}^{k-1} \|T(\tfrac{tj}{m})\left(T(\tfrac{t}{m})S(\tfrac{tj}{m}) - S(\tfrac{tj}{m})T(\tfrac{t}{m})\right) S(\tfrac{t}{m})\left[T(\tfrac{t}{m})S(\tfrac{t}{m})\right]^{k-1-j} f\| \\
&\quad \leq M_3 \sum_{j=1}^{k-1} j \cdot \left(\tfrac{t}{m}\right)^{\alpha} |||f||| \\
&\quad = M_3\ \frac{t^{\alpha}(k-1)k}{2\,m^{\alpha}} |||f|||
\end{aligned}$$

for a suitable constant $M_3 \geq 0$. Note that

$$|||\left[T(\tfrac{t}{m})S(\tfrac{t}{m})\right]^j||| = |||\left[T\left(\tfrac{tj/m}{j}\right) S\left(\tfrac{tj/m}{j}\right)\right]^j||| \leq M_T$$

for $j \in \mathbb{N}$ and $1 \leq j \leq m$. We now conclude that

$$\begin{aligned}
&\|P_n(t)f - P_m(t)f\| = \|\left[T(\tfrac{kt}{m})S(\tfrac{kt}{m})\right]^n f - \left[T(\tfrac{t}{m})S(\tfrac{t}{m})\right]^{k\cdot n} f\| \\
&\quad \leq \sum_{l=0}^{n-1} \|\left[T(\tfrac{kt}{m})S(\tfrac{kt}{m})\right]^l \Big(T(\tfrac{kt}{m})S(\tfrac{kt}{m}) - \left[T(\tfrac{t}{m})S(\tfrac{t}{m})\right]^k\Big) \left[T(\tfrac{t}{m})S(\tfrac{t}{m})\right]^{k\cdot(n-1-l)} f\| \\
&\quad \leq M_4\ \frac{t^{\alpha} n(k-1)k}{2\,m^{\alpha}} |||f||| \\
&\quad \leq M_4\ \frac{t^{\alpha} k}{2\,m^{\alpha-1}} |||f||| \\
&\quad \leq M_5 \frac{k}{2m^{\alpha-1}} |||f|||
\end{aligned} \tag{5}$$

for suitable constants $M_4, M_5 \geq 0$.

Thus, for arbitrary $i, j \in \mathbb{N}$ and $0 \le t \le T$, we have

$$\begin{aligned}\|P_{2^i}(t)f - P_{2^j}(t)f\| &\le \sum_{l=i}^{j-1} \|P_{2^l}(t)f - P_{2^{l+1}}(t)f\| \\ &\le M_5 \sum_{l=i}^{j-1} \left(\frac{1}{2^{\alpha-1}}\right)^{l+1} |||f||| \ .\end{aligned}$$

The last expression converges to 0 as $i, j \to \infty$, since $\alpha > 1$. Hence,

$$U(t)f := \lim_{i\to\infty} P_{2^i}(t)f \tag{6}$$

exists for $f \in F$ uniformly for $t \in [0, T]$. Due to (4) we can extend $U(t)$ to a bounded linear operator on E.

For arbitrary $n \in \mathbb{N}$, we obtain from (5) that

(7)

$$\begin{aligned}&\|P_n(t)f - U(t)f\| \\ &\quad\le \|P_n(t)f - P_{n2^l}(t)f\| + \|P_{n2^l}(t)f - P_{2^l}(t)f\| + \|P_{2^l}(t)f - U(t)f\| \\ &\quad\le \sum_{k=0}^{l-1} \|P_{n2^k}(t)f - P_{n2^{k+1}}(t)f\| + \|P_{n2^l}(t)f - P_{2^l}(t)f\| + \|P_{2^l}(t)f - U(t)f\| \\ &\quad\le \frac{M_5}{n^{\alpha-1}} \sum_{k=0}^{l-1} \left(\frac{1}{2^{\alpha-1}}\right)^{k+1} |||f||| + M_5 \ \frac{n}{(n2^l)^{\alpha-1}} |||f||| + \|P_{2^l}(t)f - U(t)f\| \\ &\quad\le \left(\frac{2^{\alpha-1}M_5}{(2^{\alpha-1}-1)n^{\alpha-1}} + \frac{M_5 n^{2-\alpha}}{2^{l(\alpha-1)}}\right) |||f||| + \|P_{2^l}(t)f - U(t)f\|\end{aligned}$$

for $l \in \mathbb{N}$.

Fix $\epsilon > 0$. By the definition of the operators $U(t)$, there exists $l_0 \in \mathbb{N}$ such that

$$\|P_{2^l}(t)f - U(t)f\| \le \epsilon$$

for $f \in F$ and $l \ge l_0$. Further, for $n \in \mathbb{N}$ with $\left(\frac{1}{n}\right)^{\alpha-1} \le \frac{\epsilon(2^{\alpha-1}-1)}{2^{\alpha-1}M_5}$ and $l \ge \max\{l_0, l_1\}$ such that $2^{l_1(\alpha-1)} \ge \left(\frac{M_5 n^{2-\alpha}}{\epsilon}\right)$, we have

$$\begin{aligned}\|P_n(t)f - U(t)f\| &\le \left(\frac{2^{\alpha-1}M_5}{(2^{\alpha-1}-1)n^{\alpha-1}} + \frac{M_5 n^{2-\alpha}}{2^{l(\alpha-1)}}\right) |||f||| + \|P_{2^l}(t)f - U(t)f\| \\ &\le 3\epsilon(\|f\| + |||f|||)\end{aligned}$$

for $f \in F$ uniformly for $t \in [0, T]$. So the stability condition (4) implies that $P_n(t)$ converges strongly in E uniformly for $t \in [0, T]$.

To show the semigroup law for $(U(t))_{t\ge 0}$ and for rational numbers t, we use an idea from the paper of Chernoff (cf. [4, Thm. 2.5.1]). To that purpose, let $f \in E$

and $\epsilon > 0$. Then

$$\begin{aligned}
&\|U(t)f - U(\tfrac{t}{2})U(\tfrac{t}{2})f\| \\
&\qquad \le \|U(t)f - \left[T(\tfrac{t}{2n})\ S(\tfrac{t}{2n})\right]^{2n} f\| + \|\left[T(\tfrac{t}{2n})\ S(\tfrac{t}{2n})\right]^{2n} f - U(\tfrac{t}{2})U(\tfrac{t}{2})f\| \\
&\qquad \le \epsilon + \|\left[T(\tfrac{t}{2n})S(\tfrac{t}{2n})\right]^{n} \left(\left[T(\tfrac{t}{2n})S(\tfrac{t}{2n})\right]^{n} f - U(\tfrac{t}{2})\right) f\| \\
&\qquad\qquad + \|\left(\left[T(\tfrac{t}{2n})S(\tfrac{t}{2n})\right]^{n} - U(\tfrac{t}{2})\right) U(\tfrac{t}{2})f\| \\
&\qquad \le 3\epsilon
\end{aligned}$$

for n sufficiently large. This proves that $U(t) = U(\frac{t}{2})U(\frac{t}{2})$. In an analogous way, one can show $U(nt) = U(t)^n$ for $n \in \mathbb{N}$, from which one obtains the semigroup law for rational numbers.

By the uniform convergence of the Lie–Trotter products on compact intervals, we obtain that $(U(t))_{t\ge 0}$ is a strongly continuous semigroup. The assertion concerning the generator G follows from [4, Prop. 4.1]. □

In the following, we show how commutator conditions on the generators A and B imply condition (b) of Theorem 3. These conditions can be verified in some interesting applications (see Section 4 and 5).

Definition 4. Let $(A, D(A))$ and $(B, D(B))$ be linear operators on a Banach space E. The operator

$$C := AB - BA \qquad \text{with domain}$$
$$D(C) := D(AB) \cap D(BA) = \{f \in D(A) \cap D(B) : Bf \in D(A) \text{ and } Af \in D(B)\}$$

is called the **commutator** of A and B.

In the following we denote by $\|\cdot\|_C$ the **C–norm** defined by

$$\|f\|_C := \|f\| + \|Cf\|$$

for all $f \in D(C)$.

The next lemma allows to relate the commutator of $(T(t))_{t\ge 0}$ and $(S(t))_{t\ge 0}$ to the commutator C.

Lemma 5. *Let $(A, D(A))$ and $(B, D(B))$ be generators of strongly continuous semigroups $(T(t))_{t\ge 0}$ and $(S(t))_{t\ge 0}$ on a Banach space E, and let $(C, D(C))$ be the commutator of A and B. Suppose that there exists a subspace $F \subseteq D(C)$ which is dense in E and invariant under $(T(t))_{t\ge 0}$ and $(S(t))_{t\ge 0}$ such that for some $\delta > 0$ and $t \in [0, \delta]$ the functions*

$$\begin{cases} s \mapsto AS(t)T(s)f\ ,\ s \mapsto CS(s)f \text{ are continuous and} \\ s \mapsto AS(s)f \text{ is differentiable} \end{cases}$$

in E for $s \in [0, \delta]$ and for all $f \in F$. Then the identity

$$T(t)S(t)f - S(t)T(t)f = \int_0^t T(s)\left(\int_0^t S(t-r)CS(r)T(t-s)f\,dr\right)ds \tag{8}$$

holds for all $f \in F$ and $t \in [0, \delta]$.

Proof. Let $t \in [0, \delta]$ with $\delta > 0$. We apply the fundamental theorem of calculus to the continuously differentiable functions

$$s \mapsto T(s)S(t)T(t-s)f \quad \text{and} \quad r \mapsto S(t-r)AS(r)f$$

for $0 \leq r, s \leq \delta$ and $f \in F$. Since the operator A is closed, we obtain by the differentiability of the second function that

$$\frac{d}{dr} S(t-r)AS(r)f = S(t-r)CS(r)f$$

for $0 \leq r \leq \delta$ and $f \in F$. Thus, we have

$$\begin{aligned} T(t)S(t)f - S(t)T(t)f &= \int_0^t \frac{d}{ds} \left\{ T(s)S(t)T(t-s)f \right\} ds \\ &= \int_0^t T(s) \left(AS(t) - S(t)A \right) T(t-s) f ds \\ &= \int_0^t T(s) \left(\int_0^t \frac{d}{dr} \left\{ S(t-r)AS(r)T(t-s)f \right\} dr \right) ds \\ &= \int_0^t T(s) \left(\int_0^t S(t-r)CS(r)T(t-s)f dr \right) ds \end{aligned}$$

for $f \in F$ and $t \in [0, \delta]$. □

As an easy consequence of Lemma 5 we obtain the following result.

Corollary 6. *Let $(A, D(A))$ and $(B, D(B))$ be generators of strongly continuous semigroups $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ on a Banach space E and let $(C, D(C))$ be the commutator of A and B. Suppose that:*

(a) *there exists a normed vector space $(F, |||\cdot|||)$, $F \subseteq D(C)$, which is dense in E and invariant under both semigroups, and the norm $|||\cdot|||$ is finer than the C-norm $\|\cdot\|_C$,*
(b) *there exists $\delta > 0$ such that the functions*

$$\begin{cases} s \mapsto AS(t)T(s)f \ , \ s \mapsto CS(s)f \text{ are continuous and} \\ s \mapsto AS(s)f \text{ is differentiable} \end{cases}$$

in E for $s, t \in [0, \delta]$ and $f \in F$, and
(c) *the two semigroups are exponentially bounded on F and locally Trotter–stable on E and F.*

Then the conclusion of Theorem 3 holds.

Proof. By the exponential boundedness of the semigroups and Lemma 5 we obtain the estimate

$$\begin{aligned} \|T(t)S(t)f - S(t)T(t)f\| &= \| \int_0^t T(s) \left(\int_0^t S(t-r)CS(r)T(t-s)f dr \right) ds \| \\ &\leq Mt^2 \sup_{0 \leq r, s \leq t} \|S(r)T(t-s)f\|_C \\ &\leq \tilde{M} t^2 |||f||| \end{aligned}$$

for $f \in F$, $t \in [0, \delta]$ and some constants $M, \tilde{M} \geq 0$. Thus, assumption (b) of Theorem 3 is fulfilled. Since condition (a) of Theorem 3 was assumed explicitly, the assertion follows. □

3. Commutator conditions for resolvent operators

In the following we express the commutator condition on the semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ in Theorem 3 by a corresponding condition on the powers of the resolvent operators $R(\lambda, A)$ and $R(\mu, B)$, respectively. In special cases, only a commutator condition on $R(\lambda, A)$ and $R(\mu, B)$ will be necessary.

First, we state the following result on the asymptotic behaviour of the gamma function.

Lemma 7. *The gamma function defined by*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \qquad \textit{for } \mathrm{Re} z > 0$$

satisfies

$$\lim_{n\to\infty} n^{b-a}\frac{\Gamma(n+a)}{\Gamma(n+b)} = 1 \qquad \textit{for } a, b \in \mathbb{R}\ .$$

Proof. By Stirling's Formula we can conclude that

$$\Gamma(x+1) = \sqrt{2\pi}x^{x+1/2}e^{-x}(1+\phi(x))$$

for $x \in \mathbb{R}$, where ϕ is a function converging to zero as $\frac{1}{x}$ (see [1, p. 652]). Therefore, we have

$$n^{b-a}\frac{\Gamma(n+a)}{\Gamma(n+b)} = \left(\frac{n+a-1}{n+b-1}\right)^{n+a-1/2}\left(\frac{n}{n+b-1}\right)^{b-a} e^{b-a}\frac{1+\phi(n+a-1)}{1+\phi(n+b-1)}$$

for $a, b \in \mathbb{R}$ which converges to 1 as $n \to \infty$. □

In the sequel, we assume without loss of generality that $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are bounded semigroups. Note that we now use a commutator condition on the semigroups which is slightly different from the one in Theorem 3.

Theorem 8. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be bounded strongly continuous semigroups on a Banach space E with generators $(A, D(A))$ and $(B, D(B))$, respectively, and suppose that there exists a normed vector space $(F, |||\cdot|||)$ which is embedded in E. Furthermore, let $\alpha, \beta \geq 0$ and $M \geq 0$. Then the following statements are equivalent*:

(i) *The semigroups satisfy the commutator condition*

$$\|T(t)S(s)f - S(s)T(t)f\| \leq t^\alpha s^\beta M|||f||| \tag{9}$$

for all $f \in F$ and $s, t \geq 0$.

(ii) *The resolvent operators satisfy the commutator condition*

$$\|R(\lambda, A)^n R(\mu, B)^n f - R(\mu, B)^n R(\lambda, A)^n f\| \leq \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)!^2\lambda^{\alpha+n}\mu^{\beta+n}}M|||f|||$$

for all $f \in F$, $\lambda, \mu > 0$ and $n \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii). Let $f \in F$ and $\lambda, \mu > 0$. We use the Laplace representation of the resolvent (see [6, Ch. II, Thm. 1.10]) and obtain

$$\begin{aligned}
&\|R(\lambda, A)^n R(\mu, B)^n f - R(\mu, B)^n R(\lambda, A)^n f\| \\
&\qquad \leq \frac{1}{(n-1)!^2} \int_0^\infty \int_0^\infty r^{n-1} s^{n-1} e^{-\lambda r} e^{-\mu s} \|T(r)S(s)f - S(s)T(r)f\| \, ds dr \\
&\qquad \leq \frac{M}{(n-1)!^2} \int_0^\infty \int_0^\infty r^{n-1+\alpha} e^{-\lambda r} s^{n-1+\beta} e^{-\mu s} dr ds \, |||f||| \\
&\qquad = \frac{M}{(n-1)!^2 \lambda^{\alpha+n} \mu^{\beta+n}} \int_0^\infty r^{n-1+\alpha} e^{-r} dr \int_0^\infty s^{n-1+\beta} e^{-s} ds \, |||f||| \\
&\qquad = \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)!^2 \lambda^{\alpha+n} \mu^{\beta+n}} M |||f|||
\end{aligned}$$

for all $n \in \mathbb{N}$.

(ii) $\Rightarrow$ (i). Recall that by the Post–Widder Inversion Formula

$$T(t)f = \lim_{n\to\infty} \left[\tfrac{n}{t} R(\tfrac{n}{t}, A)\right]^n f$$

for $f \in E$ uniformly for t in compact intervals in $\mathbb{R}_+$ ([6, Ch. III, Cor. 5.5]). Let $f \in F$ and $s, t \geq 0$. Applying Lemma 7 we conclude that

$$\begin{aligned}
&\frac{n^{2n}}{t^n s^n} \|R(\tfrac{n}{t}, A)^n R(\tfrac{n}{s}, B)^n f - R(\tfrac{n}{s}, B)^n R(\tfrac{n}{t}, A)^n f\| \\
&\qquad \leq \frac{n^{2n}}{t^n s^n} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)!^2 \left(\frac{n}{t}\right)^{\alpha+n} \left(\frac{n}{s}\right)^{\beta+n}} M |||f||| \\
&\qquad = n^{-(\alpha+\beta)} t^\alpha s^\beta \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\Gamma(n)^2} M |||f||| \\
&\qquad \longrightarrow t^\alpha s^\beta M |||f|||
\end{aligned}$$

as $n \to \infty$ which implies (i). □

We now discuss some cases where it is enough to impose a commutator condition on the resolvent operators and not on all their powers.

Proposition 9. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be bounded strongly continuous semigroups on a Banach space E with generators $(A, D(A))$ and $(B, D(B))$, respectively. Let $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 2$. Then the following statements are equivalent*:

(i) *The semigroups satisfy the commutator condition*

$$\|T(t)S(s)f - S(s)T(t)f\| \leq t^\alpha s^\beta M_1 \|f\|$$

for all $f \in E$, $s, t \geq 0$, and a constant $M_1 \geq 0$.

(ii) *The resolvent operators satisfy the commutator condition*

$$\|R(\lambda, A)R(\mu, B)f - R(\mu, B)R(\lambda, A)f\| \leq \frac{M_2}{\lambda^{\alpha+1}\mu^{\beta+1}} \|f\|$$

for all $f \in E$, $\lambda, \mu > 0$, $n \in \mathbb{N}$, and a constant $M_2 \geq 0$.

Moreover, if $\alpha + \beta > 2$, then the semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ commute.

Proof. The implication (i) $\Rightarrow$ (ii) is proved analogously as in Theorem 8.

(ii) $\Rightarrow$ (i). For simplicity, we define

$$\mathcal{R}_A(\tfrac{n}{t}) := \tfrac{n}{t} R(\tfrac{n}{t}, A) \ , \mathcal{R}_B(\tfrac{n}{t}) := \tfrac{n}{t} R(\tfrac{n}{t}, B)$$

and the commutator

$$\left[\mathcal{R}_A(\tfrac{n}{t}), \mathcal{R}_B(\tfrac{n}{t})\right] := \tfrac{n}{t}R(\tfrac{n}{t}, A)\tfrac{n}{t}R(\tfrac{n}{t}, B) - \tfrac{n}{t}R(\tfrac{n}{t}, B)\tfrac{n}{t}R(\tfrac{n}{t}, A)$$

for $n \in \mathbb{N}$ and $t \geq 0$. Since the semigroups are bounded there exists a constant $M \geq 0$ such that

$$\| \left(\mathcal{R}_A(\tfrac{n}{t})\right)^j \|, \| \left(\mathcal{R}_B(\tfrac{n}{s})\right)^j \| \leq M$$

for all $j, n \in \mathbb{N}$ such that $1 \leq j \leq n$, and for all $t, s \geq 0$. By assumption, we have

$$\| \left[\mathcal{R}_A(\tfrac{n}{t}), \mathcal{R}_B(\tfrac{n}{t})\right] \| \leq M_2 \left(\tfrac{t}{n}\right)^{\alpha+\beta}$$

for all $n \in \mathbb{N},\ t \geq 0$ and some constant $M_2 \geq 0$. Let $f \in E$ and $t, s \geq 0$. Then

$$\begin{aligned} &\|T(t)S(s)f - S(s)T(t)f\| \\ &\quad = \lim_{n\to\infty} \| \left(\mathcal{R}_A(\tfrac{n}{t})\right)^n \left(\mathcal{R}_B(\tfrac{n}{s})\right)^n f - \left(\mathcal{R}_B(\tfrac{n}{s})\right)^n \left(\mathcal{R}_A(\tfrac{n}{t})\right)^n f\| \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\left(\mathcal{R}_A(\tfrac{n}{t})\right)^n \left(\mathcal{R}_B(\tfrac{n}{s})\right)^n f - \left(\mathcal{R}_B(\tfrac{n}{s})\right)^n \left(\mathcal{R}_A(\tfrac{n}{t})\right)^n f \\ &\quad = \mathcal{R}_A(\tfrac{n}{t}) \left[\left(\mathcal{R}_A(\tfrac{n}{t})\right)^{n-1}, \left(\mathcal{R}_B(\tfrac{n}{s})\right)^{n-1}\right] \mathcal{R}_B(\tfrac{n}{s})f \\ &\qquad + \left[\mathcal{R}_A(\tfrac{n}{t}), \left(\mathcal{R}_B(\tfrac{n}{s})\right)^{n-1}\right] \left(\mathcal{R}_A(\tfrac{n}{t})\right)^{n-1} \mathcal{R}_B(\tfrac{n}{s})f \\ &\qquad + \left(\mathcal{R}_B(\tfrac{n}{s})\right)^{n-1} \left[\left(\mathcal{R}_A(\tfrac{n}{t})\right)^n, \mathcal{R}_B(\tfrac{n}{s})\right] f \end{aligned}$$

for all $n \in \mathbb{N}$. Moreover, we can estimate

$$\begin{aligned} \| \left[\left(\mathcal{R}_A(\tfrac{n}{t})\right)^n, \left(\mathcal{R}_B(\tfrac{n}{s})\right)\right] f\| &\leq \sum_{j=0}^{n-1} \| \left(\mathcal{R}_A(\tfrac{n}{t})\right)^{n-1-j} \left[\mathcal{R}_A(\tfrac{n}{t}), \mathcal{R}_B(\tfrac{n}{s})\right] \left(\mathcal{R}_A(\tfrac{n}{t})\right)^j f\| \\ &\leq nM^2 M_2 \left(\tfrac{t}{n}\right)^\alpha \left(\tfrac{s}{n}\right)^\beta \|f\| \ , \\ \| \left[\mathcal{R}_A(\tfrac{n}{t}), \left(\mathcal{R}_B(\tfrac{n}{s})\right)^{n-1}\right] f\| &\leq (n-1)M^2 M_2 \left(\tfrac{t}{n}\right)^\alpha \left(\tfrac{s}{n}\right)^\beta \|f\| \ , \end{aligned}$$

and by induction

$$\| \left[\left(\mathcal{R}_A(\tfrac{n}{t})\right)^j, \left(\mathcal{R}_B(\tfrac{n}{s})\right)^j\right] f\| \leq j^2 M^2 M_2 \left(\tfrac{t}{n}\right)^\alpha \left(\tfrac{s}{n}\right)^\beta \|f\|$$

for each $j, n \in \mathbb{N}$ such that $1 \leq j \leq n$. Therefore, equation (10) yields

$$\begin{aligned} &\|T(t)S(s)f - S(s)T(t)f\| \\ &\quad = \lim_{n\to\infty} \| \left(\mathcal{R}_A(\tfrac{n}{t})\right)^n \left(\mathcal{R}_B(\tfrac{n}{s})\right)^n f - \left(\mathcal{R}_B(\tfrac{n}{s})\right)^n \left(\mathcal{R}_A(\tfrac{n}{t})\right)^n f\| \\ &\quad \leq \lim_{n\to\infty} \frac{n^2 t^\alpha s^\beta}{n^{\alpha+\beta}} M^2 M_2 \|f\| \\ &\quad \leq t^\alpha s^\beta M_1 \|f\| \end{aligned} \tag{11}$$

for some constant $M_1 \geq 0$.

Using the estimate (11), it follows that the semigroups commute if $\alpha+\beta > 2$. □

A similar commutator condition for the resolvent operators in the context of nonautonomous evolution equations was studied in [9].

4. Applications to semigroups from quantum mechanics

First, we apply our results to groups arising in quantum mechanics. Let $(T(t))_{t\in\mathbb{R}}$ and $(S(t))_{t\in\mathbb{R}}$ be a pair of unitary groups on a complex Hilbert space H satisfying the so–called Weyl relation (see [10, p. 274])

$$T(t)S(s) = e^{ist}S(s)T(t) \text{ for all } s,t \in \mathbb{R}\ . \tag{12}$$

These semigroups are evidently exponentially bounded and locally Trotter–stable on H. Moreover, condition (b) of Theorem 3 follows from the estimate

$$\|T(t)S(t)f - S(t)T(t)f\| = \|\left(e^{it^2}-1\right)S(t)T(t)f\| \le t^2\sum_{k=1}^{\infty}\frac{(t^2)^{k-1}}{k!}\|f\| \le t^2e^{t^2}\,\|f\|$$

for $f \in H$ and $t \ge 0$. Therefore, the Lie–Trotter product formula holds.

Based on this relation, P.E.T. Jørgensen, R.T. Moore [7, Ch. 11] and H. Suzuki [12] treated generalized Weyl relations leading to the following result.

Corollary 10. *Let $(T(t))_{t\ge0}$, $(S(t))_{t\ge0}$ and $(V(t))_{t\ge0}$ be strongly continuous semigroups on a Banach space E with generators $(A, D(A)), (B, D(B))$ and $(C, D(C))$, respectively. Suppose that there exists a subspace $F \subseteq D(C)$ which is dense in E and invariant under the semigroups $(T(t))_{t\ge0}$ and $(S(t))_{t\ge0}$ such that the following conditions hold:*

(a) *The semigroups $(T(t))_{t\ge0}$ and $(S(t))_{t\ge0}$ are exponentially bounded on $(F, \|\cdot\|_C)$ and locally Trotter–stable on both E and $(F, \|\cdot\|_C)$.*
(b) *The generalized Weyl relation holds, i.e.*

$$T(t)S(t)f = V(t^2)S(t)T(t)f$$

for all $f \in E$ and $t \ge 0$.

Then the conclusion of Theorem 3 holds.

Proof. Fix $\delta > 0$. We can estimate the commutator of $(T(t))_{t\ge0}$ and $(S(t))_{t\ge0}$ as

$$\begin{aligned}\|T(t)S(t)f - S(t)T(t)f\| &= \|\left(V(t^2) - Id\right)[S(t)T(t)]\,f\| \\ &= \|\int_0^{t^2} V(s)C\,[S(t)T(t)]\,f ds\| \\ &\le t^2M\|f\|_C\end{aligned}$$

for $f \in F$, $t \in [0,\delta]$ and some constant $M \ge 0$. So Theorem 3 implies the assertion. □

5. Applications to Ornstein-Uhlenbeck operators

In this section we consider the (finite–dimensional) Ornstein–Uhlenbeck operator which has been studied e.g. in [5]. Let $E := C_0(\mathbb{R}^d)$ or $L^p(\mathbb{R}^d)$, $1 \le p < \infty$. For any symmetric, positive semi–definite matrix $A := (a_{ij})$ and a matrix $B := (b_{ij}) \in \mathcal{L}(\mathbb{R}^d)$, *the Ornstein–Uhlenbeck operator* is defined by

$$[\mathcal{O}f](x) := \sum_{i,j=1}^{d} a_{ij}D_{ij}f(x) + \sum_{i,j=1}^{d} b_{ij}x_jD_if(x) = \langle\nabla, A\nabla f(x)\rangle + \langle Bx, \nabla f(x)\rangle \tag{13}$$

for each $f \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d$, the Schwartz space of rapidly decreasing functions, $x \in \mathbb{R}^d$, and $\nabla := (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$. Without loss of generality, we write A as

$\mathrm{diag}(a_1, \dots, a_d)$ where $a_k > 0$ for $1 \le k \le j$, $a_{j+1} = \cdots = a_d = 0$ and define the operators $\mathcal{A}$ and $\mathcal{B}$ as the closure of

$$\mathcal{A}f := \langle \nabla, A\nabla f\rangle \quad \text{and} \quad [\mathcal{B}f](x) := \langle Bx, \nabla f(x)\rangle$$

for $f \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, respectively (cf. [14, Ch. II.6]). The operator $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\ge 0}$ given by

(14)

$$\mathcal{T}(t)f(x) = \frac{1}{(4\pi t \prod_{i=1}^{j} a_i)^{j/2}} \int_{\mathbb{R}^j} e^{-\sum_{i=1}^{j} \frac{|x_i - s_i|^2}{4a_i t}} f(s_1, \cdots, s_j, x_{j+1}, \cdots, x_d) ds_1 \cdots ds_j$$

for $t > 0, x \in \mathbb{R}^d$, and $f \in E$ (see [3]). Furthermore, the operator $(\mathcal{B}, D(\mathcal{B}))$ generates the strongly continuous semigroup $(\mathcal{S}(t))_{t\ge 0}$ given by

$$\mathcal{S}(t)f(x) = f(e^{tB}x) \tag{15}$$

for all $f \in E$ and $x \in \mathbb{R}^d$ (see [6, Ch. II, Sec. 3.28]).

These semigroups and their generators have the following useful properties.

Lemma 11. *Let $(\mathcal{T}(t))_{t\ge 0}$, $(\mathcal{S}(t))_{t\ge 0}$ and their generators $(\mathcal{A}, D(\mathcal{A}))$ and $(\mathcal{B}, D(\mathcal{B}))$ be as above. Then the following properties hold*:

(a) *The semigroups $(\mathcal{T}(t))_{t\ge 0}$ and $(\mathcal{S}(t))_{t\ge 0}$ are locally Trotter–stable on E.*

(b) *The commutator $\mathcal{C}$ of $(\mathcal{A}, D(\mathcal{A}))$ and $(\mathcal{B}, D(\mathcal{B}))$ is given by*

$$\mathcal{C}f := \mathcal{A}\mathcal{B}f - \mathcal{B}\mathcal{A}f = 2\langle BA\nabla, \nabla f\rangle$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

(c) *For $t \ge 0$ the functions*

$$\begin{cases} s \mapsto \mathcal{A}\mathcal{S}(t)\mathcal{T}(s)f\ , s \mapsto \mathcal{C}\mathcal{S}(s)f \text{ are continuous and} \\ s \mapsto \mathcal{A}\mathcal{S}(s)f \text{ is differentiable} \end{cases}$$

in E for $s \ge 0$ and for all $f \in \mathcal{S}(\mathbb{R}^d)$.

(d) *The operator $\langle BA\nabla, \nabla\rangle$ commutes with $(\mathcal{T}(t))_{t\ge 0}$ on $\mathcal{S}(\mathbb{R}^d)$ and*

$$\langle BA\nabla, \nabla\mathcal{S}(t)f\rangle = \mathcal{S}(t)\langle e^{tB}BAe^{tB^\star}\nabla, \nabla f\rangle$$

for $f \in \mathcal{S}(\mathbb{R}^d)$, where $B^\star$ denotes the transpose of the matrix B.

Proof. Clearly, the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is invariant under both semigroups and both generators $\mathcal{A}$ and $\mathcal{B}$. Moreover, $(\mathcal{T}(t))_{t\ge 0}$ is a semigroup of contractions on E and

$$\|\mathcal{S}(t)f\|_{C_0} \le \|f\|_{C_0}\ ,\ \|\mathcal{S}(t)f\|_{L^p} \le e^{t\omega}\|f\|_{L^p}$$

for $f \in E$ and $\omega := \frac{tr(B)}{p}$. It follows that the semigroups are locally Trotter–stable on E.

To determine the commutator of $(\mathcal{A}, D(\mathcal{A}))$ and $(\mathcal{B}, D(\mathcal{B}))$ on $\mathcal{S}(\mathbb{R}^d)$, we note that

$$\frac{\partial}{\partial x_k}\langle Bx, \nabla f(x)\rangle = \langle Be_k, \nabla f(x)\rangle + \langle Bx, \frac{\partial}{\partial x_k}\nabla f(x)\rangle$$

for $1 \le k \le d$, $f \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d$, and the k-th unit vector e_k. Hence, we have

$$\frac{\partial^2}{\partial^2 x_k}\langle Bx, \nabla f(x)\rangle = 2\langle Be_k, \frac{\partial}{\partial x_k}\nabla f(x)\rangle + \langle Bx, \nabla\frac{\partial^2}{\partial x_k^2}f(x)\rangle \tag{16}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Therefore, we obtain from identity (16) that

$$\begin{aligned}[\mathcal{AB}f - \mathcal{BA}f](x) &= 2\sum_{k=1}^{d}\langle Be_k a_k \frac{\partial}{\partial x_k}, \nabla f(x)\rangle + \sum_{k=1}^{d}\langle Bx, a_k \frac{\partial^2}{\partial x_k^2}\nabla f(x)\rangle \\ &\quad - \langle Bx, \nabla \sum_{k=1}^{d} a_k \frac{\partial^2}{\partial x_k^2} f(x)\rangle \\ &= 2\langle BA\nabla, \nabla f(x)\rangle\end{aligned}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

Clearly, the operators $\mathcal{A}$ and $\mathcal{B}$ are continuous for the usual topology on the space $\mathcal{S}(\mathbb{R}^d)$ induced by the family of seminorms

$$p_{\alpha,n} = \sup_{x\in\mathbb{R}^d} |x^n D^\alpha f(x)| \ .$$

By a straightforward computation, using [11, Thm. 7.2 and Thm. 7.4], we obtain that the statements in (c) are fulfilled for the usual topology on $\mathcal{S}(\mathbb{R}^d)$. Since this topology is finer than the topology on E, assertion (c) is proved.

Observe that $(\mathcal{T}(t))_{t\geq 0}$ is a convolution semigroup and that convolution commutes with the differential operators $\frac{\partial}{\partial x_k}$ $(1 \leq k \leq d)$. Therefore, the commutator of $(\mathcal{A}, D(\mathcal{A}))$ and $(\mathcal{B}, D(\mathcal{B}))$ commutes with $(\mathcal{T}(t))_{t\geq 0}$. For the semigroup $(\mathcal{S}(t))_{t\geq 0}$ we have the following commutator relation

$$\nabla\mathcal{S}(t)f = \sum_{k=1}^{d}\mathcal{S}(t)\langle e^{tB}e_k, \nabla f\rangle e_k \ ,$$

and therefore

$$\langle BA\nabla, \nabla\mathcal{S}(t)f\rangle = \mathcal{S}(t)\langle e^{tB}BAe^{tB^\star}\nabla, \nabla f\rangle$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. □

We can now apply Corollary 6. The fact that the closure of $\mathcal{A}+\mathcal{B}$ is a generator seems to be known (cf. [5]).

Proposition 12. *Let $(\mathcal{T}(t))_{t\geq 0}$, $(\mathcal{S}(t))_{t\geq 0}$ be the strongly continuous semigroups on E given by* (14) *and* (15) *generated by $(\mathcal{A}, D(\mathcal{A}))$and $(\mathcal{B}, D(\mathcal{B}))$, respectively. Then the conclusion of Theorem 3 holds.*

Proof. On $\mathcal{S}(\mathbb{R}^d)$ we define the norm

$$|||f||| := \|f\| + \sum_{1\leq i,j\leq d}\left\|\frac{\partial^2}{\partial_i\partial_j}f\right\|$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. This norm is finer than the $\mathcal{C}$–norm $\|\cdot\|_{\mathcal{C}}$.

In the next step, we show the exponential boundedness of the semigroups $(\mathcal{T}(t))_{t\geq 0}$ and $(\mathcal{S}(t))_{t\geq 0}$ on $(\mathcal{S}(\mathbb{R}^d), |||\cdot|||)$. This is obvious for $(\mathcal{T}(t))_{t\geq 0}$ which is $|||\cdot|||$-contractive on $\mathcal{S}(\mathbb{R}^d)$. On the other hand, we have

$$\|\frac{\partial^2}{\partial_i\partial_j}\mathcal{S}(t)f\| = \|\mathcal{S}(t)\langle e^{tB}e_{ij}e^{tB^\star}\nabla, \nabla f\rangle\| \leq e^{2t\tilde{\omega}}\|\mathcal{S}(t)\|\cdot|||f|||$$

where

$$e_{ij} := \begin{array}{cc} \begin{array}{c} \downarrow i \end{array} & \\ \begin{pmatrix} 0 \cdots 0 \cdots 0 \\ \vdots \quad \vdots \quad \vdots \\ 0 \cdots 1 \cdots 0 \\ \vdots \quad \vdots \quad \vdots \\ 0 \cdots 0 \cdots 0 \end{pmatrix} & \begin{array}{c} \\ \\ \leftarrow \\ \\ \\ j \end{array} \end{array}$$

and $\tilde{\omega} := \max_{1 \le i,j \le d} |b_{ij}|$. Therefore, we can estimate

$$|||\mathcal{S}(t)f||| \le d^2 e^{2t\tilde{\omega}} \|\mathcal{S}(t)\| \cdot |||f|||$$

for $f \in \mathcal{S}(\mathbb{R}^d)$.

Finally, this implies the local Trotter–stability on $(\mathcal{S}(\mathbb{R}^d), |||\cdot|||)$

$$||| \left[\mathcal{T}(\tfrac{t}{n})\mathcal{S}(\tfrac{t}{n})\right]^n ||| \le e^{2t\tilde{\omega}} |||f|||$$

if $E = C_0(\mathbb{R}^d)$, and

$$||| \left[\mathcal{T}(\tfrac{t}{n})\mathcal{S}(\tfrac{t}{n})\right]^n ||| \le e^{t(\omega+2\tilde{\omega})} |||f|||$$

if $E = L^p(\mathbb{R}^d)$. We can now apply Corollary 6 which concludes the proof. □

As a concrete example we mention a semigroup appearing in mathematical finance ([2]).

Examples 13. On the space $E := C_0(\mathbb{R}^2)$ or $L^p(\mathbb{R}^2)$ $(1 \le p < \infty)$ the Cauchy problem

$$\left\{ \begin{array}{rcl} \frac{d}{dt}u(t,x,y) & = & \frac{\partial^2}{\partial x^2}u(t,x,y) + x\frac{\partial}{\partial y}u(t,x,y) \ , \quad t \ge 0 \ , \\ u(0,x,y) & = & f(x,y) \ , \end{array} \right. \tag{17}$$

is investigated in [2]. Taking

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

the Cauchy problem (17) can be reformulated as an Ornstein–Uhlenbeck operator $\mathcal{O}$ defined in (13). Therefore, all assumptions of Proposition 12 are satisfied and $\mathcal{O}$ generates a semigroup given as the limit of the Lie–Trotter products.

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