A HOMOMORPHISM OF HARISH-CHANDRA AND DIRECT IMAGES OF \( D \)-MODULES

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Abstract. Harish-Chandra defined a homomorphism \( \delta : D(g)^G \to D(h)^W \) of algebras of invariant polynomial differential operators. The construction and existence of \( \delta \) are somewhat mysterious. We show how \( \delta \) naturally arises when one considers matters in the context of \( D \)-modules.

0. Introduction

0.1. Let \( g \) be a complex semisimple Lie algebra with adjoint group \( G = \text{Aut}(g)^\circ \), Cartan subalgebra \( h \), and Weyl group \( W \). Harish-Chandra \([4],[5]\) defined a homomorphism \( \delta : D(g)^G \to D(h)^W \) of algebras of invariant polynomial differential operators. We show that \( \delta \) arises naturally in the following way: Consider the diagram

\[
\begin{array}{ccc}
g & \xrightarrow{\chi} & h \\
\downarrow{\pi} & & \downarrow{\pi/W} \\
h & \to & h/W 
\end{array}
\]

where \( \pi \) and \( \chi \) are the quotient morphisms by \( W \) and the adjoint action of \( G \), respectively. The sheaf of regular functions \( O_h \) on \( h \) is a \( D_h \)-module, where \( D_h \) denotes the sheaf of differential operators on \( h \). Then we have the direct image \( \pi_* O_h \) and its pull back \( M := \chi^* \pi_* O_h \), which is a \( D_g \)-module. The \( G \)-invariant global sections \( \Gamma(g,M)^G \) contain a copy of \( O(h) \) and are a \( (D(g)^G = \Gamma(g,D_g)^G) \)-module.

Theorem. \( D(g)^G \) acts on the image of \( O(h) \) in \( \Gamma(g,M)^G \) via \( \delta \).

Our methods are elementary and consist of a change of variables theorem for \( \acute{e} \)tale maps and a result about lifting differential operators on a quotient.

0.2. The \( D(g)^G \)-module \( \Gamma(g,M)^G \) is larger than \( O(h) \). Using less elementary means, we show how one can construct a \( D(g)^G \)-module which identifies canonically with \( O(h) \): Fix a Borel subalgebra \( b \) of \( g \) containing \( h \), and let \( B \supset H \) be the corresponding subgroups of \( G \). The adjoint action \( G \times g \to g \) induces a morphism \( \varphi \) from \( \tilde{g} := G \times B \) to \( g \) (the Grothendieck-Springer resolution). The projection

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positive roots. Let 

\[ \text{quotient map } h \]

\[ \text{De} \]

Define 

\[ \text{The map } S \]

isomorphism of vector spaces. Suppose that 

\[ \text{V} \]

\[ \text{The multiplication map } S \]

the symmetric algebra 

\[ \text{Let } 1.1. \]

\[ \text{D} \]

denote 

\[ \text{Now we obtain a } D(g)^G \text{-module from } \mathcal{O}_h \text{ by going the other way around. Let } N \text{ denote } \varphi + \theta' \mathcal{O}_h \simeq \varphi + \mathcal{O}_g. \]

**Theorem.** (1) The space of invariants \( \Gamma(g,N)^G \) identifies canonically with the polynomial functions \( \mathcal{O}(h) \) on \( h \):

\[ \Gamma(g,N)^G = \mathcal{O}(h), \]

where \( D(g)^G \) acts on \( \mathcal{O}(h) \) via \( \delta \).

(2) As a \( D(g) \)-module, \( N \) is generated by \( \Gamma(g,N)^G \). Thus the inclusion \( \mathcal{O}(h) \rightarrow \Gamma(g,M)^G \) induces an inclusion of \( (D(g),G) \)-modules

\[ 0 \rightarrow N \rightarrow M. \]

The proof uses results of Evens [3], Hotta and Kashiwara [7] and Hunziker and Wallach [8].

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1. The Homomorphism \( \delta \)

1.1. Let \( V \) be a finite dimensional complex vector space with coordinate ring \( \mathcal{O}(V) \). Let \( D(V) \) denote the algebra of polynomial differential operators on \( V \). We identify the symmetric algebra \( S(V) \) with the constant coefficient differential operators. The multiplication map \( \mu: \mathcal{O}(V) \otimes S(V) \rightarrow D(V) \) defined by \( f \otimes P \mapsto f \cdot P \) is an isomorphism of vector spaces. Suppose that \( V \) is a rational \( G \)-module. Then \( \mathcal{O}(V) \), \( S(V) \), and \( D(V) \) are locally finite rational \( G \)-modules and \( \varphi \) is a \( G \)-isomorphism.

1.2. The map \( \delta: D(g)^G \rightarrow D(h)^W \) has the following properties:

(\( \delta_1 \)) \( \delta \) is an algebra homomorphism.

(\( \delta_2 \)) On \( \mathcal{O}(g)^G \), \( \delta \) is the isomorphism given by restriction \( \mathcal{O}(g)^G \simeq \mathcal{O}(h)^W \).

(\( \delta_3 \)) On \( S(g)^G \), \( \delta \) is the isomorphism \( S(g)^G \simeq S(h)^W \) induced by the canonical projection \( g \rightarrow h \).

(\( \delta_4 \)) The kernel of \( \delta \) is the ideal \( I = \{ P \in D(g)^G \mid P(f) = 0 \text{ for all } f \in \mathcal{O}(g)^G \} \).

Harish-Chandra’s construction of \( \delta \) is as follows: Fix a system \( R_+ \subset h^* \) of positive roots. Let \( \sigma = \prod_{\alpha \in R_+} \alpha \). The set of regular elements \( h^* \subset h \) equals the subset \( h^*_\sigma \) where \( \sigma \) does not vanish. The Weyl group \( W \) acts freely on \( h^* \) and the quotient map \( \pi: h^* \rightarrow h^*/W \) is a covering (proper étale map). This implies that there is an isomorphism \( D(h)^W \simeq D(h^*/W) \). We then obtain a homomorphism \( \delta': D(g)^G \rightarrow D(h)^W \) as follows: Let \( g^* \) denote the regular semisimple elements of \( g \), and let \( \rho \) denote the isomorphisms of \( \mathcal{O}(g)^G \) and \( \mathcal{O}(g)^G \) with \( \mathcal{O}(h)^W \) and \( \mathcal{O}(h)^W \) induced by restriction. Then

\[ \delta'(P)(f) = \rho(P(\rho^{-1}(f))), \text{ for } P \in D(g)^G, f \in \mathcal{O}(h)^W. \]

Define \( \delta = m_{\sigma} \circ \delta' \circ m_{\sigma}^{-1} \) where \( m_{\sigma} \) denotes multiplication by \( \sigma \). Assuming that \( \delta(D(g)^G) \) lies in \( D(h)^W \subset D(h^*)^W \) (the hard part), one easily sees that (\( \delta_1 \), (\( \delta_2 \)
and (δ4) hold, and they imply (δ3). See [12] for a short proof of the existence of δ along these lines. A completely different construction of δ is given in Hunziker and Wallach [8].

In the next two sections, we show how the twist by σ arises naturally.

2. An algebraic change of variables formula

2.1. Let X be a (smooth) complex algebraic variety. We denote by \( \mathcal{O}_X \) the sheaf of regular functions and by \( \mathcal{D}_X \) the sheaf of algebraic differential operators. As usual, \( \mathcal{O}(X) = \Gamma(X, \mathcal{O}_X) \) and \( \mathcal{D}(X) = \Gamma(X, \mathcal{D}_X) \). If \( \pi : X \to Y \) is a morphism between varieties we denote by \( \pi_+ \) the usual functors between \( \mathcal{O} \)-modules. The inverse image of a \( \mathcal{D}_Y \)-module (looked upon as an \( \mathcal{O}_Y \)-module) always carries a natural structure of a \( \mathcal{D}_X \)-module (see [11, Chapter VI, §4]). The notion of direct image is, however, much more complicated for \( \mathcal{D} \)-modules. In general, to define a direct image functor \( \pi_+ \) one has to work in the derived category, i.e., one has to work with complexes of \( \mathcal{D} \)-modules (see [11, Chapter VI, §5]). The morphisms π that we are considering in this section are affine, in which case the functor \( \pi_+ \) exists on the module level. If \( \pi \) is étale, then \( \pi_+ \) is isomorphic to the functor \( \pi_* \), and we give this isomorphism explicitly below.

2.2. Let \( X = Y = \mathbb{C}^n \). We choose linear coordinates \( x_1, \ldots, x_n \) on \( X \) and \( y_1, \ldots, y_n \) on \( Y \). Then \( \mathcal{O}(X) = \mathbb{C}[x_1, \ldots, x_n] \) and \( \mathcal{O}(Y) = \mathbb{C}[y_1, \ldots, y_n] \). Let \( \pi : X \to Y \) be a polynomial map with coordinate functions \( u_1 = \pi^* y_1, \ldots, u_n = \pi^* y_n \). Set \( a_{ij} = \frac{\partial u_i}{\partial x_j} \), \( i, j = 1, \ldots, n \), and let \( \sigma = \det(a_{ij}) \) denote the Jacobian.

Then \( \pi \) is étale on the open subset \( X_\sigma = \{ x \in X \mid \sigma(x) \neq 0 \} \). Let \( (b_{ij}) \) denote the matrix of cofactors of \( (a_{ij}) \). Then \( (b_{ij}) \) is the matrix with entries in \( \mathcal{O}(X) \) such that

\[
\sum_{k=1}^n a_{ik} b_{kj} = \sigma \delta_{ij}.
\]

We define vector fields \( \partial_{u_j} \) on \( X_\sigma \) by \( \partial_{u_j} = \sigma^{-1} \sum_{i=1}^n b_{ij} \partial_{x_i} \), where \( \partial_{x_i} = \partial/\partial x_i \), \( i = 1, \ldots, n \). Then we have

\[
\partial_{u_j}(u_k) = \frac{1}{\sigma} \sum_{i=1}^n b_{ij} \frac{\partial u_k}{\partial x_i} = \frac{1}{\sigma} \sum_{i=1}^n b_{ij} a_{ki} = \delta_{kj}.
\]

2.3. Let \( \mathcal{D}(X \to Y) := \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{D}(Y) \). It is a \( (\mathcal{D}(X), \mathcal{D}(Y)) \)-bimodule, where the right \( \mathcal{D}(Y) \)-module structure is the obvious one given by right multiplication. The left action of \( \mathcal{O}(X) \) is given by left multiplication, and the partial derivatives \( \partial_{x_i} \) act by

\[
\partial_{x_i}(f \otimes P) = \frac{\partial f}{\partial x_i} \otimes P + \sum_{k=1}^n f \frac{\partial u_k}{\partial x_i} \otimes \partial_{y_k} P, \quad i = 1, \ldots, n.
\]

Suppose that we localize at \( X_\sigma \), i.e., we consider the subset of \( X \) where \( \pi \) is étale. We have an injective homomorphism \( \eta : \mathcal{D}(Y) \to \mathcal{D}(X_\sigma) \) where \( \eta(y_i) = \pi^*(y_i) = u_i \) and \( \eta(\partial_{y_i}) = \partial_{u_i} \), \( i = 1, \ldots, n \). We consider \( \mathcal{D}(X_\sigma) \) as a \( (\mathcal{D}(X_\sigma), \mathcal{D}(Y)) \)-bimodule (via \( \eta \) for the \( \mathcal{D}(Y) \)-module structure). Then

**Proposition.** The map \( \eta \) induces an isomorphism of \( (\mathcal{D}(X_\sigma), \mathcal{D}(Y)) \)-bimodules:

\[
\mathcal{O}(X_\sigma) \otimes_{\mathcal{O}(Y)} \mathcal{D}(Y) \simeq \mathcal{D}(X_\sigma), \quad f \otimes P \mapsto f \eta(P), \quad f \in \mathcal{O}(X_\sigma), \quad P \in \mathcal{D}(Y).
\]
2.4. Recall that formal transpose $P \mapsto P^t$ is an anti-involution of $\mathcal{D}(X)$ and $\mathcal{D}(Y)$. It is the identity on $\mathcal{O}(X)$ and $\mathcal{O}(Y)$, and $\partial^t_{x_i} = -\partial_{x_i}$ and $\partial^t_{y_i} = -\partial_{y_i}$, $i = 1, \ldots, n$. We now need to consider $\mathcal{D}(Y \leftarrow X)$ which is the transpose of $\mathcal{D}(X \rightarrow Y)$ for both the $\mathcal{D}(X)$- and $\mathcal{D}(Y)$-actions. As an $\mathcal{O}(X)$- and $\mathcal{O}(Y)$-module it is the same as $\mathcal{D}(X \rightarrow Y)$, but the actions of the $\partial_{x_i}$ and $\partial_{y_i}$ are transposed. If $M$ is a left $\mathcal{D}(X)$-module, we define the pushforward $\pi_+ M := \mathcal{D}(Y \leftarrow X) \otimes_{\mathcal{D}(X)} M$. Then $\pi_+ M$ is a left $\mathcal{D}(Y)$-module. If we restrict to $X_\sigma$, we have our identification of $\mathcal{D}(X_\sigma \rightarrow Y)$ with $\mathcal{D}(X_\sigma)$ where $\mathcal{D}(Y)$ acts via $\eta$. Let $M_\sigma$ denote $\mathcal{D}(X_\sigma) \otimes_{\mathcal{D}(X)} M$. Then

$$\pi_+ M_\sigma = \mathcal{D}(Y \leftarrow X_\sigma) \otimes_{\mathcal{D}(X_\sigma)} M_\sigma \simeq \mathcal{D}(X_\sigma) \otimes_{\mathcal{D}(X_\sigma)} M_\sigma \simeq M'_\sigma,$$

where $M'_\sigma$ is a copy of $M_\sigma$ on which elements $P \in \mathcal{D}(Y)$ act via $\eta(P^t)^t$.

**Lemma.** Let $1 \leq i \leq n$. Then $y_i \in \mathcal{D}(Y)$ acts on $M'_\sigma$ as multiplication by $u_i$ and $\partial_{y_i}$ acts as the differential operator $m_\sigma \cdot \partial_{u_i} m_\sigma$.

**Proof.** The action of $y_i$ is via $\eta(y_i) = u_i$. The action of $\partial_{y_i}$ is by the transpose of $\eta(-\partial_{y_i}) \in \mathcal{D}(X_\sigma)$. The transposed action of $\mathcal{D}(X)$ on $\mathcal{O}(X)$ can be seen as the action on smooth differential $n$-forms via (the negative of) the Lie derivative. In other words,

$$(-\partial^t_{x_i}(f)) \omega_0 = \text{Lie}(\partial_{x_i})(f \omega_0),$$

where $\omega_0 := dx_1 \wedge \cdots \wedge dx_n$. Set $\omega := \sigma \omega_0$. Then $\omega = du_1 \wedge \cdots \wedge du_n$, so that

$$\text{Lie}(\partial_{u_i})(f \sigma \omega_0) = \text{Lie}(\partial_{u_i})(f \omega) = \frac{\partial f}{\partial u_i} \omega = \sigma \frac{\partial f}{\partial u_i} \omega_0,$$

proving the Lemma.

**Corollary.** Consider $M_\sigma$ as a $\mathcal{D}(Y)$-module via the action above. Then there is a commutative diagram of $\mathcal{O}(Y)$-modules

$$
\begin{array}{ccc}
M & \xrightarrow{\alpha} & \pi_+ M \\
\downarrow & & \downarrow \\
M_\sigma & \xrightarrow{\beta} & \pi_+ M_\sigma
\end{array}
$$

where $\alpha$ is a homomorphism of $\mathcal{O}(Y)$-modules and $\beta$ is an isomorphism of $\mathcal{D}(Y)$-modules. Here $\alpha$ sends an element $m \in M$ to $1 \otimes 1 \otimes m \in \mathcal{D}(Y \leftarrow X) \otimes_{\mathcal{D}(X)} M$, and similarly for $\beta$.

3. The inverse image of a $\mathcal{D}$-module on a quotient

3.1. Let $G$ be a complex reductive group and $Z$ an affine $G$-variety. There is a quotient variety $Z/G$ and a surjection $\varpi : Z \rightarrow Z/G$ such that $\varpi^*$ identifies with the inclusion $\mathcal{O}(Z)^G \rightarrow \mathcal{O}(Z)$. There is a canonical homomorphism $\varpi_* : \mathcal{D}(Z)^G \rightarrow \mathcal{D}(Z/G)$ given by restriction of operators $P \mapsto P|_{\mathcal{O}(Z)^G}$.

3.2. **Lemma.** Assume that $Z$ and $Z/G$ are smooth. Let $\mathcal{L}$ be a $\mathcal{D}_{Z/G}$-module. Then $\varpi^* \mathcal{L}$ is a $(\mathcal{D}_Z, G)$-module and

$$\Gamma(Z, \varpi^* \mathcal{L})^G = \Gamma(Z/G, \mathcal{L})$$

as $\mathcal{D}(Z)^G$-modules,

where $\mathcal{D}(Z)^G$ acts on $\Gamma(Z/G, \mathcal{L})$ via $\varpi_*$. 

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Proof. Considered as an \( \mathcal{O}(Z) \)-module,
\[
\Gamma(Z, \varpi^* \mathcal{L}) = \mathcal{O}(Z) \otimes_{\mathcal{O}(Z)^G} \Gamma(Z/G, \mathcal{L}).
\]
The \( G \)-action is given by the action on the left factor. Clearly, the canonical map
\[
\Gamma(Z/G, \mathcal{L}) \to \Gamma(Z, \varpi^* \mathcal{L})^G, \, s \mapsto 1 \otimes s
\]
is an isomorphism of \( \mathcal{O}(Z)^G \)-modules. We will prove that it is also an isomorphism of \( \mathcal{D}(Z)^G \)-modules.

Using Luna’s slice theorem (see [11] §4), we can reduce to the case that \( Z/G = \mathbb{C}^n \) is a vector space (the only case we actually need, anyway). Then \( \mathcal{O}(Z)^G = \mathbb{C}[u_1, \ldots, u_n] \), where the \( u_i \) are algebraically independent functions. Let \( y_1, \ldots, y_n \) denote coordinate functions on \( \mathbb{C}^n \) such that \( u_i = \varpi^* y_i \). By definition, a vector field \( \theta \) on \( Z \) acts by Leibniz’s rule on \( \Gamma(Z, \varpi^* \mathcal{L}) \):
\[
\theta(f \otimes s) = \theta(f) \otimes s + \sum_{i=1}^n f(u_i) \otimes \partial_{y_i}(s).
\]

Let \( P \in \mathcal{D}(Z)^G \) and \( 1 \otimes s \in \Gamma(Z/G, \mathcal{L}) \). By the formula above
\[
P(1 \otimes s) = \sum_{\beta=(\beta_1, \ldots, \beta_n)} f_\beta \otimes \partial^\beta_y(s),
\]
where the functions \( f_\beta \in \mathcal{O}(Z) \) only depend upon \( P \). Here we use the usual multi-index notation \( \partial^\beta_y = \partial^{\beta_1}_{y_1} \cdots \partial^{\beta_n}_{y_n} \). Since \( P(1 \otimes s) \) and the \( \varpi^* \partial^\beta_y(s) \) are \( G \)-invariant, we may replace the \( f_\beta \) by their projections to the invariants (Reynolds operator).

From the case \( \mathcal{L} = \mathcal{O}_{Z/G} \) it is clear that \( P \) is the image of \( P \) under \( \varpi^* \).

\[\square\]

Proof of Theorem 0.1. Recall diagram (\( \ast \)) of the introduction. Set \( M = \mathcal{O}(h') \) and let \( \pi : h' \to h/W \simeq g/G \) be the natural étale morphism with Jacobian \( \sigma \). Let \( Q \in \mathcal{D}(g/G) \) and \( f \in \mathcal{O}(h') \). The Corollary in 2.4 shows that \( Q \) sends \( f' := 1 \otimes 1 \otimes f \in \pi_* \mathcal{O}(h') \) to \( 1 \otimes 1 \otimes (m_{\sigma^{-1}} \circ Q \circ m_{\sigma})f \). The construction of \( \delta \) (see 1.2) and Lemma 3.2 show that \( P \in \mathcal{D}(g)^G \) sends \( \gamma^* f' \) to \( \chi^* (1 \otimes 1 \otimes (m_{\sigma^{-1}} \circ \delta(P) \circ m_{\sigma})f) = \chi^* (1 \otimes 1 \otimes (\delta^\gamma(P))f) \). Thus \( \mathcal{D}(g)^G \) acts on \( \mathcal{O}(h') \simeq \Gamma(g, \mathcal{O}(h'))^G \) via \( \delta \). Since \( \mathcal{O}(h) \to \mathcal{O}(h') \) is injective, we obtain the Theorem in 0.1.

\[\square\]

Note that we have not shown that \( \mathcal{D}(g)^G \) leaves \( \mathcal{O}(h) \subset \mathcal{O}(h') \) stable. This still requires a separate proof (see [12] §3, for example).

3.3. For the rest of this paper we assume that \( \mathcal{D}(g)^G \) leaves \( \mathcal{O}(h) \) stable. We show how to replace \( M = \chi^* \pi_* \mathcal{O}(h) \) by another module whose \( G \)-invariants are \( \mathcal{O}(h) \) on the nose.

4. The homomorphism \( \delta \) and the Grothendieck-Springer resolution

4.1. In [8], Wallach and Hunziker gave a different construction of \( \delta \). Starting with a triangular decomposition \( g = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n} \) they define a homomorphism \( \gamma : \mathcal{D}(g)_H \to \mathcal{D}(h) \) such that \( \mathcal{D}(g)^G \) lands inside \( \mathcal{D}(h)^W \). From the construction of \( \gamma \) it is immediate that properties (\( \delta_1 \)), (\( \delta_2 \)) and (\( \delta_3 \)) hold. The hard part is to show that (\( \delta_4 \)) holds, i.e., that \( \gamma \) vanishes on the ideal \( (\mathcal{D}(g) \tau(g))^G \). In the proof of this fact in [8], a certain \( (\mathcal{D}_g, G) \)-module was introduced, which we will denote by \( N_0 \). The construction of \( N_0 \) was such that the space of \( G \)-invariant global sections, looked
upon as a $\mathcal{D}(\mathfrak{g})^{G}$-module, identifies canonically with the polynomial functions on $\mathfrak{h}$:

$$\Gamma(\mathfrak{g}, N_0)^G = \mathcal{O}(\mathfrak{h}).$$

Here the $\mathcal{D}(\mathfrak{g})^G$-action on $\mathcal{O}(\mathfrak{h})$ is given via $\gamma$.

4.2. In [3], Evens related the module $N_0$ to constructions of Hotta and Kashiwara [7]. Recall diagram (**) of 0.2 (also reproduced in 4.3 below).

**Theorem** (Evens [3, x1]). The $(\mathcal{D}_\mathfrak{g}, G)$-module $N_0$ is naturally isomorphic to the direct image $N$ of the regular functions on $\mathfrak{g}$ by $\varphi$:

$$N_0 \simeq N = \varphi_+ \mathcal{O}_{\mathfrak{g}}.$$

**Remark.** A priori, $\varphi_+ \mathcal{O}_{\mathfrak{g}}$ is an object in the derived category. However, by a result due to Hotta and Kashiwara [7, Corollary 4.23], $h^i(\varphi_+ \mathcal{O}_{\mathfrak{g}}) = 0$ for $i \neq 0$ and hence we may regard $\varphi_+ \mathcal{O}_{\mathfrak{g}}$ as a $\mathcal{D}_{\mathfrak{g}}$-module.

4.3. We now want to relate the module $N = \varphi_+ \mathcal{O}_{\mathfrak{g}}$ to the compatible $(\mathcal{D}_\mathfrak{g}, G)$-module $M = \chi^* \pi_+ \mathcal{O}_{\mathfrak{h}}$ that we studied in §3. Recall the Grothendieck-Springer resolution diagram:

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\varphi} & \mathfrak{g} \\
\theta \downarrow & & \downarrow \chi \\
\mathfrak{h} & \xrightarrow{\pi} & \mathfrak{h}/W
\end{array}
\]

The morphism $\varphi \times \theta : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{h}$ is a birational map onto the fiber product $\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h} = \{ (x, t) \mid u(x) = u(t) \text{ for all } u \in \mathcal{O}(\mathfrak{g})^G \}$. Let $\mathfrak{g}'$ be the set of regular semisimple elements in $\mathfrak{g}$ and let $\mathfrak{g}' = \theta^{-1}(\mathfrak{h}')$. Then $(\varphi \times \theta)|_{\mathfrak{g}'} : \mathfrak{g}' \sim \mathfrak{g}' \times_{\mathfrak{h}/W} \mathfrak{h}'$ is an isomorphism. This isomorphism is the key to the following lemma that is proved by base change.

**Lemma.** We have a natural isomorphism of compatible $(\mathcal{D}_{\mathfrak{g}'}, G)$-modules:

$$N|_{\mathfrak{g}'} \simeq M|_{\mathfrak{g}'}.$$

**Proof.** Consider the following commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{g}' & \xrightarrow{\varphi} & \mathfrak{g}' \\
\theta \downarrow & & \downarrow \chi \\
\mathfrak{h}' & \xrightarrow{\pi} & \mathfrak{h}/W
\end{array}
\]

The vertical arrows $j$ are the inclusion maps. Every face except the bottom face of this cube is a Cartesian diagram. Note also that every map in this diagram is $G$-equivariant if we define trivial $G$-actions on $\mathfrak{h}$ and $\mathfrak{h}/W$, respectively. We then have the following natural isomorphisms of functors (in the equivariant derived
category):
\[ j^! \circ \chi^! \circ \pi_+ = \chi^! \circ j^! \circ \pi_+ \quad \text{(composition law, right face)} \]
\[ = \chi^! \circ \pi_+ \circ j^! \quad \text{(base change, front face)} \]
\[ = \varphi^+ \circ \theta^! \circ j^! \quad \text{(base change, top face)} \]
\[ = \varphi^+ \circ j^! \circ \theta^! \quad \text{(composition law, left face)} \]
\[ = j^! \circ \varphi^+ \circ \theta^! \quad \text{(base change, back face)}. \]

(We refer the reader to the appendix below for notation and basic background on operations on \( \mathcal{D} \)-modules.) Thus we obtain a natural isomorphism
\[ (\chi^! \pi_+ \mathcal{O}_b)|_{g'} \simeq (\varphi^+ \theta^! \mathcal{O}_b)|_{g'}. \]
Since \( \theta^! \mathcal{O}_b = \theta^* \mathcal{O}_b[d_{g/b}] = \mathcal{O}_{\widetilde{g}}[d_{g/b}], \) the lemma follows. ⧫

**An interpretation of Lemma 4.3.** Recall that we have a natural injective homomorphism \( \mathcal{O}(\mathfrak{h}) \to \Gamma(\mathfrak{g}, \mathcal{M})^G \) of \( \mathcal{D}(\mathfrak{g})^G \)-modules, where \( \mathcal{D}(\mathfrak{g})^G \) acts on \( \mathcal{O}(\mathfrak{h}) \) via \( \delta \). On the other hand, we have a natural isomorphism \( \mathcal{O}(\mathfrak{h}) \simeq \Gamma(\mathfrak{g}, \mathcal{N})^G, \) where \( \mathcal{D}(\mathfrak{g})^G \) acts on \( \mathcal{O}(\mathfrak{h}) \) via \( \gamma \). The isomorphism \( \mathcal{N}|_{g'} \simeq \mathcal{M}|_{g'} \) identifies \( \Gamma(\mathfrak{g}', \mathcal{M})^G \) with \( \Gamma(\mathfrak{g}', \mathcal{N})^G \). We therefore have a geometric explanation that \( \gamma = \delta \). More precisely, in light of the proof of the Lemma in 4.3 we may say that \( \gamma = \delta \) via base change.

**Proof of Theorem 0.2.** We have part (1) of the theorem, i.e., that \( \Gamma(\mathfrak{g}, \mathcal{N})^G \simeq \mathcal{O}(\mathfrak{h}) \). It was shown by Hotta and Kashiwara that \( N \simeq \varphi^+ \mathcal{O}_b \) is a regular holonomic \( \mathcal{D}_g \)-module equal to the minimal extension of \( (\varphi^+ \mathcal{O}_b)|_{g'} \). Thus the restriction map \( \Gamma(\mathfrak{g}, \mathcal{N}) \to \Gamma(\mathfrak{g}', \mathcal{N}) \) is injective. It also follows from their work that \( \Gamma(\mathfrak{g}, \mathcal{N}) \) is the quotient of \( \mathcal{D}(\mathfrak{g}) \) by a \( G \)-stable left ideal, in particular, \( \mathcal{N} \) is generated by \( \Gamma(\mathfrak{g}, \mathcal{N})^G \). Thus the canonical inclusion \( \mathcal{O}(\mathfrak{h}) \to \Gamma(\mathfrak{g}, \mathcal{M})^G \) induces an inclusion \( \mathcal{N} \to \mathcal{M} \) of \( (\mathcal{D}_g, G) \)-modules, and we have part (2).

**APPENDIX: OPERATIONS ON \( \mathcal{D} \)-MODULES**

**A.1.** If \( X \) is a smooth variety we denote by \( D^b(D_X) \) the derived category of bounded complexes of \( D_X \)-modules. If \( \mathcal{M} \in D^b(D_X) \) we write \( h^i(\mathcal{M}) \) for the \( i \)-th cohomology module of the complex. We will identify the category of \( D_X \)-modules with the full category of complexes \( \mathcal{M} \in D^b(D_X) \) such that \( h^i(\mathcal{M}) = 0 \) for \( i \neq 0 \).

If \( \pi : X \to Y \) is a morphism between smooth varieties, then there are functors
\[ \pi_- : D^b(D_X) \to D^b(D_Y), \quad \pi^! : D^b(D_Y) \to D^b(D_X). \]
If \( \psi : Y \to Z \) is another morphism between smooth varieties we have the following composition laws:
\[ (\psi \circ \pi)_+ = \psi_+ \circ \pi_+, \quad (\psi \circ \pi)^! = \pi^! \circ \psi^!. \]

**A.2.** Now suppose that \( X \) is a smooth \( G \)-variety, where \( G \) is some linear algebraic group. Bernstein defined an equivariant derived category \( D^b_G(D_X) \) together with a forgetful functor \( D^b_G(D_X) \to D^b(D_X) \). If \( X \) is an affine \( G \)-variety and if \( \mathcal{M} \) is an object in \( D^b_G(D_X) \), then the \( D_X \)-modules \( h^i(\mathcal{M}) \) inherit a structure of a compatible
\((D_X, G)\)-module. If \(\pi : X \to Y\) is a \(G\)-equivariant morphism between smooth \(G\)-varieties, then there are functors \(\pi_+ : D_c^b(D_X) \to D_c^b(D_Y)\) and \(\pi^! : D_c^b(D_Y) \to D_c^b(D_X)\).

**A.3.** We recall the principle of base change for operations on \(\mathcal{D}\)-modules (see [6, Chapter 1]). Consider a Cartesian square of smooth varieties:

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{\pi} & Z \\
\downarrow \varphi & & \downarrow \varphi \\
X & \xrightarrow{\pi} & Y
\end{array}
\]

Cartesian means that

\[
\tilde{Z} = X \times_Y Z = \{(x, z) \in X \times Z \mid \pi(x) = \varphi(z)\}.
\]

Then we have a natural isomorphism of functors from \(D^b(D_X)\) to \(D^b(D_Z)\):

\[
\varphi^! \circ \pi_+ = \tilde{\pi}_+ \circ \varphi^!.
\]

For equivariant maps we have natural isomorphisms of functors between equivariant derived categories.

**A.4.** Let \(\varphi : Z \to Y\) be a smooth morphism and let \(d_{Z/Y} = \dim(Z) - \dim(Y)\). Then \(h^i \circ \chi^! = 0\) for \(i \neq d_{Z/Y}\) and for every \(\mathcal{D}_Y\)-module \(\mathcal{L}\), \(\chi^! \mathcal{L} = \chi^* \mathcal{L} [d_{Z/Y}]\). Here, as usual, if \(C\) is a complex, then \(C[d]\) denotes the shift of the complex by \(d\).

**References**


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