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A HOMOMORPHISM OF HARISH-CHANDRA AND DIRECT IMAGES OF $\mathcal{D}$-MODULES

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Abstract. Harish-Chandra defined a homomorphism $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$ of algebras of invariant polynomial differential operators. The construction and existence of δ are somewhat mysterious. We show how δ naturally arises when one considers matters in the context of $\mathcal{D}$-modules.

0. Introduction

0.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with adjoint group $G = \mathrm{Aut}(\mathfrak{g})^\circ$, Cartan subalgebra $\mathfrak{h}$, and Weyl group W. Harish-Chandra [4], [5] defined a homomorphism $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$ of algebras of invariant polynomial differential operators. We show that δ arises naturally in the following way: Consider the diagram

$$(*)\qquad \begin{array}{ccc} & & \mathfrak{g} \\ & & \big\downarrow \chi \\ \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{h}/W \end{array}$$

where π and χ are the quotient morphisms by W and the adjoint action of G, respectively. The sheaf of regular functions $\mathcal{O}_\mathfrak{h}$ on $\mathfrak{h}$ is a $\mathcal{D}_\mathfrak{h}$-module, where $\mathcal{D}_\mathfrak{h}$ denotes the sheaf of differential operators on $\mathfrak{h}$. Then we have the direct image $\pi_+\mathcal{O}_\mathfrak{h}$ and its pull back $\mathcal{M} := \chi^*\pi_+\mathcal{O}_\mathfrak{h}$, which is a $\mathcal{D}_\mathfrak{g}$-module. The G-invariant global sections $\Gamma(\mathfrak{g}, \mathcal{M})^G$ contain a copy of $\mathcal{O}(\mathfrak{h})$ and are a $(\mathcal{D}(\mathfrak{g})^G = \Gamma(\mathfrak{g}, \mathcal{D}_\mathfrak{g})^G)$-module.

Theorem. *$\mathcal{D}(\mathfrak{g})^G$ acts on the image of $\mathcal{O}(\mathfrak{h})$ in $\Gamma(\mathfrak{g}, \mathcal{M})^G$ via δ.*

Our methods are elementary and consist of a change of variables theorem for étale maps and a result about lifting differential operators on a quotient.

0.2. The $\mathcal{D}(\mathfrak{g})^G$-module $\Gamma(\mathfrak{g}, \mathcal{M})^G$ is larger than $\mathcal{O}(\mathfrak{h})$. Using less elementary means, we show how one can construct a $\mathcal{D}(\mathfrak{g})^G$-module which identifies canonically with $\mathcal{O}(\mathfrak{h})$: Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$, and let $B \supset H$ be the corresponding subgroups of G. The adjoint action $G \times \mathfrak{g} \to \mathfrak{g}$ induces a morphism φ from $\tilde{\mathfrak{g}} := G \times^B \mathfrak{b}$ to $\mathfrak{g}$ (the Grothendieck-Springer resolution). The projection

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$\mathfrak{b} \to \mathfrak{b}/\mathfrak{n} \simeq \mathfrak{h}$ induces a smooth map $\theta : \tilde{\mathfrak{g}} \to \mathfrak{h}$, where $\mathfrak{n}$ is the maximal nilpotent subalgebra of $\mathfrak{b}$. There is a commutative diagram

$$(**) \qquad \begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{\varphi} & \mathfrak{g} \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\chi} \\ \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{h}/W \end{array}$$

Now we obtain a $\mathcal{D}(\mathfrak{g})^G$-module from $\mathcal{O}_{\mathfrak{h}}$ by going the other way around. Let $\mathcal{N}$ denote $\varphi_+\theta^*\mathcal{O}_{\mathfrak{h}} \simeq \varphi_+\mathcal{O}_{\tilde{\mathfrak{g}}}$.

Theorem. (1) *The space of invariants $\Gamma(\mathfrak{g},\mathcal{N})^G$ identifies canonically with the polynomial functions $\mathcal{O}(\mathfrak{h})$ on $\mathfrak{h}$:*

$$\Gamma(\mathfrak{g},\mathcal{N})^G \;=\; \mathcal{O}(\mathfrak{h})\,,$$

where $\mathcal{D}(\mathfrak{g})^G$ acts on $\mathcal{O}(\mathfrak{h})$ via δ.

(2) *As a $\mathcal{D}_{\mathfrak{g}}$-module, $\mathcal{N}$ is generated by $\Gamma(\mathfrak{g},\mathcal{N})^G$. Thus the inclusion $\mathcal{O}(\mathfrak{h}) \to \Gamma(\mathfrak{g},\mathcal{M})^G$ induces an inclusion of $(\mathcal{D}_{\mathfrak{g}}, G)$-modules*

$$0 \to \mathcal{N} \to \mathcal{M}\ .$$

The proof uses results of Evens [3], Hotta and Kashiwara [7] and Hunziker and Wallach [8].

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1. The homomorphism δ

1.1. Let V be a finite dimensional complex vector space with coordinate ring $\mathcal{O}(V)$. Let $\mathcal{D}(V)$ denote the algebra of polynomial differential operators on V. We identify the symmetric algebra $S(V)$ with the constant coefficient differential operators. The multiplication map $\mu\colon \mathcal{O}(V) \otimes S(V) \to \mathcal{D}(V)$ defined by $f \otimes P \mapsto f \cdot P$ is an isomorphism of vector spaces. Suppose that V is a rational G-module. Then $\mathcal{O}(V)$, $S(V)$, and $\mathcal{D}(V)$ are locally finite rational G-modules and φ is a G-isomorphism.

1.2. The map $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^W$ has the following properties:

($\delta 1$) δ is an algebra homomorphism.

($\delta 2$) On $\mathcal{O}(\mathfrak{g})^G$, δ is the isomorphism given by restriction $\mathcal{O}(\mathfrak{g})^G \xrightarrow{\sim} \mathcal{O}(\mathfrak{h})^W$.

($\delta 3$) On $S(\mathfrak{g})^G$, δ is the isomorphism $S(\mathfrak{g})^G \xrightarrow{\sim} S(\mathfrak{h})^W$ induced by the canonical projection $\mathfrak{g} \to \mathfrak{h}$.

($\delta 4$) The kernel of δ is the ideal $I = \{P \in \mathcal{D}(\mathfrak{g})^G \mid P(f) = 0 \text{ for all } f \in \mathcal{O}(\mathfrak{g})^G\}$.

Harish-Chandra's construction of δ is as follows: Fix a system $R_+ \subset \mathfrak{h}^*$ of positive roots. Let $\sigma = \prod_{\alpha\in R_+} \alpha$. The set of regular elements $\mathfrak{h}' \subset \mathfrak{h}$ equals the subset $\mathfrak{h}_\sigma$ where σ does not vanish. The Weyl group W acts freely on $\mathfrak{h}'$ and the quotient map $\pi : \mathfrak{h}' \to \mathfrak{h}'/W$ is a covering (proper étale map). This implies that there is an isomorphism $\mathcal{D}(\mathfrak{h}')^W \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}'/W)$. We then obtain a homomorphism $\delta'\colon \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h}')^W$ as follows: Let $\mathfrak{g}'$ denote the regular semisimple elements of $\mathfrak{g}$, and let ρ denote the isomorphisms of $\mathcal{O}(\mathfrak{g})^G$ and $\mathcal{O}(\mathfrak{g}')^G$ with $\mathcal{O}(\mathfrak{h})^W$ and $\mathcal{O}(\mathfrak{h}')^W$ induced by restriction. Then

$$\delta'(P)(f) = \rho(P(\rho^{-1}(f))),\ \text{for } P \in \mathcal{D}(\mathfrak{g})^G,\ f \in \mathcal{O}(\mathfrak{h}')^W.$$

Define $\delta = m_\sigma \circ \delta' \circ m_\sigma^{-1}$ where m_σ denotes multiplication by σ. Assuming that $\delta(\mathcal{D}(\mathfrak{g})^G)$ lies in $\mathcal{D}(\mathfrak{h})^W \subset \mathcal{D}(\mathfrak{h}')^W$ (the hard part), one easily sees that ($\delta 1$), ($\delta 2$)

and ($\delta 4$) hold, and they imply ($\delta 3$). See [12] for a short proof of the existence of δ along these lines. A completely different construction of δ is given in Hunziker and Wallach [8].

In the next two sections, we show how the twist by σ arises naturally.

2. An algebraic change of variables formula

2.1. Let X be a (smooth) complex algebraic variety. We denote by $\mathcal{O}_X$ the sheaf of regular functions and by $\mathcal{D}_X$ the sheaf of algebraic differential operators. As usual, $\mathcal{O}(X) = \Gamma(X, \mathcal{O}_X)$ and $\mathcal{D}(X) = \Gamma(X, \mathcal{D}_X)$. If $\pi : X \to Y$ is a morphism between varieties we denote by π_*, π^* the usual functors between $\mathcal{O}$-modules. The inverse image of a $\mathcal{D}_Y$-module (looked upon as an $\mathcal{O}_Y$-module) always carries a natural structure of a $\mathcal{D}_X$-module (see [1, Chapter VI, §4]). The notion of direct image is, however, much more complicated for $\mathcal{D}$-modules. In general, to define a direct image functor π_+ one has to work in the derived category, i.e., one has to work with complexes of $\mathcal{D}$-modules (see [1, Chapter VI, §5]). The morphisms π that we are considering in this section are affine, in which case the functor π_+ exists on the module level. If π is étale, then π_+ is isomorphic to the functor π_*, and we give this isomorphism explicitly below.

2.2. Let $X = Y = \mathbb{C}^n$. We choose linear coordinates $x_1, \ldots, x_n$ on X and $y_1, \ldots, y_n$ on Y. Then $\mathcal{O}(X) = \mathbb{C}[x_1, \ldots, x_n]$ and $\mathcal{O}(Y) = \mathbb{C}[y_1, \ldots, y_n]$. Let $\pi : X \to Y$ be a polynomial map with coordinate functions $u_1 = \pi^* y_1, \ldots, u_n = \pi^* y_n$. Set $a_{ij} = \frac{\partial u_i}{\partial x_j}$, $i, j = 1, \ldots, n$, and let $\sigma = \det(a_{ij})$ denote the Jacobian. Then π is étale on the open subset $X_\sigma = \{x \in X \mid \sigma(x) \neq 0\}$. Let (b_{ij}) denote the matrix of cofactors of (a_{ij}). Then (b_{ij}) is the matrix with entries in $\mathcal{O}(X)$ such that

$$\sum_{k=1}^{n} a_{ik} b_{kj} = \sigma \delta_{ij}.$$

We define vector fields ∂_{u_j} on X_σ by $\partial_{u_j} = \sigma^{-1} \sum_{i=1}^{n} b_{ij} \partial_{x_i}$, where $\partial_{x_i} = \partial / \partial x_i$, $i = 1, \ldots, n$. Then we have

$$\partial_{u_j}(u_k) = \frac{1}{\sigma} \sum_{i=1}^{n} b_{ij} \frac{\partial u_k}{\partial x_i} = \frac{1}{\sigma} \sum_{i=1}^{n} b_{ij} a_{ki} = \delta_{kj}.$$

2.3. Let $\mathcal{D}(X \to Y) := \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{D}(Y)$. It is a $(\mathcal{D}(X), \mathcal{D}(Y))$-bimodule, where the right $\mathcal{D}(Y)$-module structure is the obvious one given by right multiplication. The left action of $\mathcal{O}(X)$ is given by left multiplication, and the partial derivatives ∂_{x_i} act by

$$\partial_{x_i}.(f \otimes P) \ = \ \frac{\partial f}{\partial x_i} \otimes P \ + \ \sum_{k=1}^{n} f \, \frac{\partial u_k}{\partial x_i} \otimes \partial_{y_k} P, \quad i = 1, \ldots, n.$$

Suppose that we localize at X_σ, i.e., we consider the subset of X where π is étale. We have an injective homomorphism $\eta : \mathcal{D}(Y) \to \mathcal{D}(X_\sigma)$ where $\eta(y_i) = \pi^*(y_i) = u_i$ and $\eta(\partial_{y_i}) = \partial_{u_i}$, $i = 1, \ldots, n$. We consider $\mathcal{D}(X_\sigma)$ as a $(\mathcal{D}(X_\sigma), \mathcal{D}(Y))$-bimodule (via η for the $\mathcal{D}(Y)$-module structure). Then

Proposition. *The map η induces an isomorphism of $(\mathcal{D}(X_\sigma), \mathcal{D}(Y))$-bimodules:*

$$\mathcal{O}(X_\sigma) \otimes_{\mathcal{O}(Y)} \mathcal{D}(Y) \simeq \mathcal{D}(X_\sigma), \quad f \otimes P \mapsto f\eta(P), \ f \in \mathcal{O}(X_\sigma), \ P \in \mathcal{D}(Y).$$

2.4. Recall that formal transpose $P \mapsto P^t$ is an anti-involution of $\mathcal{D}(X)$ and $\mathcal{D}(Y)$. It is the identity on $\mathcal{O}(X)$ and $\mathcal{O}(Y)$, and $\partial_{x_i}^t = -\partial_{x_i}$ and $\partial_{y_i}^t = -\partial_{y_i}$, $i = 1, \dots, n$. We now need to consider $\mathcal{D}(Y \leftarrow X)$ which is the transpose of $\mathcal{D}(X \to Y)$ for both the $\mathcal{D}(X)$- and $\mathcal{D}(Y)$-actions. As an $\mathcal{O}(X)$- and $\mathcal{O}(Y)$-module it is the same as $\mathcal{D}(X \to Y)$, but the actions of the ∂_{x_i} and ∂_{y_i} are transposed. If M is a left $\mathcal{D}(X)$-module, we define the pushforward $\pi_+ M := \mathcal{D}(Y \leftarrow X) \otimes_{\mathcal{D}(X)} M$. Then $\pi_+ M$ is a left $\mathcal{D}(Y)$-module. If we restrict to X_σ, we have our identification of $\mathcal{D}(X_\sigma \to Y)$ with $\mathcal{D}(X_\sigma)$ where $\mathcal{D}(Y)$ acts via η. Let M_σ denote $\mathcal{D}(X_\sigma) \otimes_{\mathcal{D}(X)} M$. Then

$$\pi_+ M_\sigma = \mathcal{D}(Y \leftarrow X_\sigma) \otimes_{\mathcal{D}(X_\sigma)} M_\sigma \simeq \mathcal{D}(X_\sigma) \otimes_{\mathcal{D}(X_\sigma)} M_\sigma \simeq M'_\sigma,$$

where M'_σ is a copy of M_σ on which elements $P \in \mathcal{D}(Y)$ act via $\eta(P^t)^t$.

Lemma. *Let $1 \le i \le n$. Then $y_i \in \mathcal{D}(Y)$ acts on M'_σ as multiplication by u_i and ∂_{y_i} acts as the differential operator $m_{\sigma^{-1}} \partial_{u_i} m_\sigma$.*

Proof. The action of y_i is via $\eta(y_i) = u_i$. The action of ∂_{y_i} is by the transpose of $\eta(-\partial_{y_i}) \in \mathcal{D}(X_\sigma)$. The transposed action of $\mathcal{D}(X)$ on $\mathcal{O}(X)$ can be seen as the action on smooth differential n-forms via (the negative of) the Lie derivative. In other words,

$$(-\partial_{x_i}^t(f))\omega_0 = \mathrm{Lie}(\partial_{x_i})(f\omega_0),$$

where $\omega_0 := dx_1 \wedge \cdots \wedge dx_n$. Set $\omega := \sigma\omega_0$. Then $\omega = du_1 \wedge \cdots \wedge du_n$, so that

$$\mathrm{Lie}(\partial_{u_i})(f\sigma\omega_0) = \mathrm{Lie}(\partial_{u_i})(f\omega) = \frac{\partial f}{\partial u_i}\omega = \sigma\frac{\partial f}{\partial u_i}\omega_0,$$

proving the Lemma. □

Corollary. *Consider M_σ as a $\mathcal{D}(Y)$-module via the action above. Then there is a commutative diagram of $\mathcal{O}(Y)$-modules*

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & \pi_+ M \\ \big\downarrow & & \big\downarrow \\ M_\sigma & \xrightarrow[\sim]{\beta} & \pi_+ M_\sigma \end{array}$$

where α is a homomorphism of $\mathcal{O}(Y)$-modules and β is an isomorphism of $\mathcal{D}(Y)$-modules. Here α sends an element $m \in M$ to $1 \otimes 1 \otimes m \in \mathcal{D}(Y \leftarrow X) \otimes_{\mathcal{D}(X)} M$, and similarly for β.

3. The inverse image of a $\mathcal{D}$-module on a quotient

3.1. Let G be a complex reductive group and Z an affine G-variety. There is a quotient variety $Z/\!\!/G$ and a surjection $\varpi : Z \to Z/\!\!/G$ such that ϖ^* identifies with the inclusion $\mathcal{O}(Z)^G \to \mathcal{O}(Z)$. There is a canonical homomorphism $\varpi_* : \mathcal{D}(Z)^G \to \mathcal{D}(Z/\!\!/G)$ given by restriction of operators $P \mapsto P|_{\mathcal{O}(Z)^G}$.

3.2. Lemma. *Assume that Z and $Z/\!\!/G$ are smooth. Let $\mathcal{L}$ be a $\mathcal{D}_{Z/\!\!/G}$-module. Then $\varpi^*\mathcal{L}$ is a $(\mathcal{D}_Z, G)$-module and*

$$\Gamma(Z, \varpi^*\mathcal{L})^G \;=\; \Gamma(Z/\!\!/G, \mathcal{L}) \text{ as } \mathcal{D}(Z)^G\text{-modules,}$$

where $\mathcal{D}(Z)^G$ acts on $\Gamma(Z/\!\!/G, \mathcal{L})$ via ϖ_*.

Proof. Considered as an $\mathcal{O}(Z)$-module,

$$\Gamma(Z,\varpi^*\mathcal{L}) \;=\; \mathcal{O}(Z)\otimes_{\mathcal{O}(Z)^G}\Gamma(Z/\!\!/G,\mathcal{L})\ .$$

The G-action is given by the action on the left factor. Clearly, the canonical map $\Gamma(Z/\!\!/G,\mathcal{L})\to\Gamma(Z,\varpi^*\mathcal{L})^G$, $s\mapsto 1\otimes s$ is an isomorphism of $\mathcal{O}(Z)^G$-modules. We will prove that it is also an isomorphism of $\mathcal{D}(Z)^G$-modules.

Using Luna's slice theorem (see [11, §4]), we can reduce to the case that $Z/\!\!/G=\mathbb{C}^n$ is a vector space (the only case we actually need, anyway). Then $\mathcal{O}(Z)^G=\mathbb{C}[u_1,\dots,u_n]$, where the u_i are algebraically independent functions. Let $y_1,\dots,y_n$ denote coordinate functions on $\mathbb{C}^n$ such that $u_i=\varpi^*y_i$. By definition, a vector field θ on Z acts by Leibniz's rule on $\Gamma(Z,\varpi^*\mathcal{L})$:

$$\theta(f\otimes s)\;=\;\theta(f)\otimes s\;+\;\sum_{i=1}^{n} f\,\theta(u_i)\otimes\partial_{y_i}(s)\ .$$

Let $P\in\mathcal{D}(Z)^G$ and $1\otimes s\in\Gamma(Z/\!\!/G,\mathcal{L})$. By the formula above

$$P(1\otimes s)\;=\;\sum_{\beta=(\beta_1,\dots,\beta_n)} f_\beta\otimes\partial_y^\beta(s)\ ,$$

where the functions $f_\beta\in\mathcal{O}(Z)$ only depend upon P. Here we use the usual multi-index notation $\partial_y^\beta=\partial_{y_1}^{\beta_1}\cdots\partial_{y_n}^{\beta_n}$. Since $P(1\otimes s)$ and the $\varpi^*(\partial_y^\beta(s))$ are G-invariant, we may replace the f_β by their projections to the invariants (Reynolds operator). Then $f_\beta=\varpi^*\bar f_\beta$, $\bar f_\beta\in\mathcal{O}(Z/\!\!/G)$, and $P(1\otimes s)=\varpi^*(\bar P s)$ where $\bar P=\sum_\beta \bar f_\beta\partial_y^\beta$. From the case $\mathcal{L}=\mathcal{O}_{Z/\!\!/G}$ it is clear that $\bar P$ is the image of P under ϖ_*. □

Proof of Theorem 0.1. Recall diagram $(*)$ of the introduction. Set $M=\mathcal{O}(\mathfrak{h}')$ and let $\pi:\mathfrak{h}'\to\mathfrak{h}/W\simeq\mathfrak{g}/\!\!/G$ be the natural étale morphism with Jacobian σ. Let $Q\in\mathcal{D}(\mathfrak{g}/\!\!/G)$ and $f\in\mathcal{O}(\mathfrak{h}')$. The Corollary in 2.4 shows that Q sends $f':=1\otimes1\otimes f\in\pi_+\mathcal{O}(\mathfrak{h}')$ to $1\otimes1\otimes(m_{\sigma^{-1}}\circ Q\circ m_\sigma)f$. The construction of δ (see 1.2) and Lemma 3.2 show that $P\in\mathcal{D}(\mathfrak{g})^G$ sends χ^*f' to $\chi^*(1\otimes1\otimes(m_{\sigma^{-1}}\circ\delta'(P)\circ m_\sigma)f)=\chi^*(1\otimes1\otimes\delta(P)f)$. Thus $\mathcal{D}(\mathfrak{g})^G$ acts on $\mathcal{O}(\mathfrak{h}')\simeq\Gamma(\mathfrak{g},\chi^*\pi_+\mathcal{O}'_{\mathfrak{h}})^G$ via δ. Since $\mathcal{O}(\mathfrak{h})\to\mathcal{O}(\mathfrak{h}')$ is injective, we obtain the Theorem in 0.1. □

Note that we have not shown that $\mathcal{D}(\mathfrak{g})^G$ leaves $\mathcal{O}(\mathfrak{h})\subset\mathcal{O}(\mathfrak{h}')$ stable. This still requires a separate proof (see [12, §3], for example).

3.3. For the rest of this paper we assume that $\mathcal{D}(\mathfrak{g})^G$ leaves $\mathcal{O}(\mathfrak{h})$ stable. We show how to replace $\mathcal{M}=\chi^*\pi_+\mathcal{O}(\mathfrak{h})$ by another module whose G-invariants are $\mathcal{O}(\mathfrak{h})$ on the nose.

4. The homorphism δ and the Grothendieck-Springer resolution

4.1. In [8], Wallach and Hunziker gave a different construction of δ. Starting with a triangular decomposition $\mathfrak{g}=\bar{\mathfrak{n}}\oplus\mathfrak{h}\oplus\mathfrak{n}$ they define a homomorphism $\gamma:\mathcal{D}(\mathfrak{g})^H\to\mathcal{D}(\mathfrak{h})$ such that $\mathcal{D}(\mathfrak{g})^G$ lands inside $\mathcal{D}(\mathfrak{h})^W$. From the construction of γ it is immediate that properties ($\delta1$), ($\delta2$) and ($\delta3$) hold. The hard part is to show that ($\delta4$) holds, i.e., that γ vanishes on the ideal $(\mathcal{D}(\mathfrak{g})\tau(\mathfrak{g}))^G$. In the proof of this fact in [8], a certain $(\mathcal{D}_{\mathfrak{g}},G)$-module was introduced, which we will denote by $\mathcal{N}_0$. The construction of $\mathcal{N}_0$ was such that the space of G-invariant global sections, looked upon as a $\mathcal{D}(\mathfrak{g})^G$-module, identifies canonically with the polynomial functions on $\mathfrak{h}$:

$$\Gamma(\mathfrak{g}, \mathcal{N}_0)^G = \mathcal{O}(\mathfrak{h}).$$

Here the $\mathcal{D}(\mathfrak{g})^G$-action on $\mathcal{O}(\mathfrak{h})$ is given via γ.

4.2. In [3], Evens related the module $\mathcal{N}_0$ to constructions of Hotta and Kashiwara [7]. Recall diagram $(**)$ of 0.2 (also reproduced in 4.3 below).

Theorem (Evens [3, §1]). *The $(\mathcal{D}_{\mathfrak{g}}, G)$-module $\mathcal{N}_0$ is naturally isomorphic to the direct image $\mathcal{N}$ of the regular functions on $\tilde{\mathfrak{g}}$ by φ:*

$$\mathcal{N}_0 \simeq \mathcal{N} = \varphi_+ \mathcal{O}_{\tilde{\mathfrak{g}}} \ .$$

Remark. A priori, $\varphi_+ \mathcal{O}_{\tilde{\mathfrak{g}}}$ is an object in the derived category. However, by a result due to Hotta and Kashiwara [7, Corollary 4.23], $h^i(\varphi_+ \mathcal{O}_{\tilde{\mathfrak{g}}}) = 0$ for $i \neq 0$ and hence we may regard $\varphi_+ \mathcal{O}_{\tilde{\mathfrak{g}}}$ as a $\mathcal{D}_{\mathfrak{g}}$-module.

4.3. We now want to relate the module $\mathcal{N} = \varphi_+ \mathcal{O}_{\tilde{\mathfrak{g}}}$ to the compatible $(\mathcal{D}_{\mathfrak{g}}, G)$-module $\mathcal{M} = \chi^* \pi_+ \mathcal{O}_{\mathfrak{h}}$ that we studied in §3. Recall the Grothendieck-Springer resolution diagram:

$$(**) \qquad \begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \mathfrak{g} \arrow[d, "\chi"] \\ \mathfrak{h} \arrow[r, "\pi"] & \mathfrak{h}/W \end{tikzcd}$$

The morphism $\varphi \times \theta : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \times \mathfrak{h}$ is a birational map onto the fiber product $\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h} = \{(x,t) \mid u(x) = u(t) \text{ for all } u \in \mathcal{O}(\mathfrak{g})^G\}$. Let $\mathfrak{g}'$ be the set of regular semisimple elements in $\mathfrak{g}$ and let $\tilde{\mathfrak{g}}' = \theta^{-1}(\mathfrak{h}')$. Then $(\varphi \times \theta)|_{\tilde{\mathfrak{g}}'} : \tilde{\mathfrak{g}}' \xrightarrow{\sim} \mathfrak{g}' \times_{\mathfrak{h}'/W} \mathfrak{h}'$ is an isomorphism. This isomorphism is the key to the following lemma that is proved by base change.

Lemma. *We have a natural isomorphism of compatible $(\mathcal{D}_{\mathfrak{g}'}, G)$-modules:*

$$\mathcal{N}|_{\mathfrak{g}'} \simeq \mathcal{M}|_{\mathfrak{g}'} \ .$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} & \tilde{\mathfrak{g}}' \arrow[rr, "\varphi"] \arrow[dl, "\theta"'] \arrow[dd] & & \mathfrak{g}' \arrow[dl, "\chi"'] \arrow[dd, "j"] \\ \mathfrak{h}' \arrow[rr, "\pi"] \arrow[dd, "j"'] & & \mathfrak{h}'/W \arrow[dd] & \\ & \tilde{\mathfrak{g}} \arrow[rr] \arrow[dl, "\theta"'] & & \mathfrak{g} \arrow[dl, "\chi"'] \\ \mathfrak{h} \arrow[rr, "\pi"] & & \mathfrak{h}/W & \end{tikzcd}$$

The vertical arrows j are the inclusion maps. Every face except the bottom face of this cube is a Cartesian diagram. Note also that every map in this diagram is G-equivariant if we define trivial G-actions on $\mathfrak{h}$ and $\mathfrak{h}/W$, respectively. We then have the following natural isomorphisms of functors (in the equivariant derived category):

$$
\begin{aligned}
j^! \circ \chi^! \circ \pi_+ &= \chi^! \circ j^! \circ \pi_+ && \text{(composition law, right face)}\\
&= \chi^! \circ \pi_+ \circ j^! && \text{(base change, front face)}\\
&= \varphi_+ \circ \theta^! \circ j^! && \text{(base change, top face)}\\
&= \varphi_+ \circ j^! \circ \theta^! && \text{(composition law, left face)}\\
&= j^! \circ \varphi_+ \circ \theta^! && \text{(base change, back face).}
\end{aligned}
$$

(We refer the reader to the appendix below for notation and basic background on operations on $\mathcal{D}$-modules.) Thus we obtain a natural isomorphism

$$(\chi^! \pi_+ \mathcal{O}_{\mathfrak{h}})|_{\mathfrak{g}'} \simeq (\varphi_+ \theta^! \mathcal{O}_{\mathfrak{h}})|_{\mathfrak{g}'}.$$

Since $\theta^! \mathcal{O}_{\mathfrak{h}} = \theta^* \mathcal{O}_{\mathfrak{h}}[d_{\mathfrak{g}/\mathfrak{h}}] = \mathcal{O}_{\tilde{\mathfrak{g}}}[d_{\mathfrak{g}/\mathfrak{h}}]$, the lemma follows. □

An interpretation of Lemma 4.3. Recall that we have a natural injective homomorphism $\mathcal{O}(\mathfrak{h}) \to \Gamma(\mathfrak{g}, \mathcal{M})^G$ of $\mathcal{D}(\mathfrak{g})^G$-modules, where $\mathcal{D}(\mathfrak{g})^G$ acts on $\mathcal{O}(\mathfrak{h})$ via δ. On the other hand, we have a natural isomorphism $\mathcal{O}(\mathfrak{h}) \xrightarrow{\sim} \Gamma(\mathfrak{g}, \mathcal{N}_0)^G$, where $\mathcal{D}(\mathfrak{g})^G$ acts on $\mathcal{O}(\mathfrak{h})$ via γ. The isomorphism $\mathcal{N}|_{\mathfrak{g}'} \simeq \mathcal{M}|_{\mathfrak{g}'}$ identifies $\Gamma(\mathfrak{g}', \mathcal{M})^G$ with $\Gamma(\mathfrak{g}', \mathcal{N})^G$. We therefore have a geometric explanation that $\gamma = \delta$. More precisely, in light of the proof of the Lemma in 4.3 we may say that $\gamma = \delta$ via base change.

Proof of Theorem 0.2. We have part (1) of the theorem, i.e., that $\Gamma(\mathfrak{g}, \mathcal{N})^G \simeq \mathcal{O}(\mathfrak{h})$. It was shown by Hotta and Kashiwara that $\mathcal{N} \simeq \varphi_+ \mathcal{O}_{\mathfrak{h}}$ is a regular holonomic $\mathcal{D}_{\mathfrak{g}}$-module equal to the minimal extension of $(\varphi_+ \mathcal{O}_{\tilde{\mathfrak{g}}})|_{\mathfrak{g}'}$. Thus the restriction map $\Gamma(\mathfrak{g}, \mathcal{N}) \to \Gamma(\mathfrak{g}', \mathcal{N})$ is injective. It also follows from their work that $\Gamma(\mathfrak{g}, \mathcal{N})$ is the quotient of $\mathcal{D}(\mathfrak{g})$ by a G-stable left ideal, in particular, $\mathcal{N}$ is generated by $\Gamma(\mathfrak{g}, \mathcal{N})^G$. Thus the canonical inclusion $\mathcal{O}(\mathfrak{h}) \to \Gamma(\mathfrak{g}, \mathcal{M})^G$ induces an inclusion $\mathcal{N} \to \mathcal{M}$ of $(\mathcal{D}_{\mathfrak{g}}, G)$-modules, and we have part (2). □

Appendix: Operations on $\mathcal{D}$-modules

A.1. If X is a smooth variety we denote by $D^b(\mathcal{D}_X)$ the derived category of bounded complexes of $\mathcal{D}_X$-modules. If $\mathcal{M}^\cdot \in D^b(\mathcal{D}_X)$ we write $h^i(\mathcal{M}^\cdot)$ for the i-th cohomology module of the complex. We will identify the category of $\mathcal{D}_X$-modules with the full category of complexes $\mathcal{M}^\cdot \in D^b(\mathcal{D}_X)$ such that $h^i(\mathcal{M}^\cdot) = 0$ for $i \neq 0$.

If $\pi : X \to Y$ is a morphism between smooth varieties, then there are functors

$$\pi_+ : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y), \qquad \pi^! : D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X).$$

If $\psi : Y \to Z$ is another morphism between smooth varieties we have the following composition laws:

$$(\psi \circ \pi)_+ = \psi_+ \circ \pi_+, \qquad (\psi \circ \pi)^! = \pi^! \circ \psi^!.$$

A.2. Now suppose that X is a smooth G-variety, where G is some linear algebraic group. Bernstein defined an equivariant derived category $D^b_G(\mathcal{D}_X)$ together with a forgetful functor $D^b_G(\mathcal{D}_X) \to D^b(\mathcal{D}_X)$. If X is an affine G-variety and if $\mathcal{M}^\cdot$ is an object in $D^b_G(\mathcal{D}_X)$, then the $\mathcal{D}_X$-modules $h^i(\mathcal{M}^\cdot)$ inherit a structure of a compatible

$(\mathcal{D}_X, G)$-module. If $\pi : X \to Y$ is a G-equivariant morphism between smooth G-varieties, then there are functors $\pi_+ : D^b_G(\mathcal{D}_X) \to D^b_G(\mathcal{D}_Y)$ and $\pi^! : D^b_G(\mathcal{D}_Y) \to D^b_G(\mathcal{D}_X)$.

A.3. We recall the principle of base change for operations on $\mathcal{D}$-modules (see [6, Chapter 1]). Consider a Cartesian square of smooth varieties:

$$\begin{tikzcd} \tilde{Z} \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{\varphi}"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "\varphi"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

Cartesian means that

$$\tilde{Z} = X \times_Y Z = \{(x, z) \in X \times Z \mid \pi(x) = \varphi(z)\}.$$

Then we have a natural isomorphism of functors from $D^b(\mathcal{D}_X)$ to $D^b(\mathcal{D}_Z)$:

$$\varphi^! \circ \pi_+ = \tilde{\pi}_+ \circ \tilde{\varphi}^! \ .$$

For equivariant maps we have natural isomorphisms of functors between equivariant derived categories.

A.4. Let $\varphi : Z \to Y$ be a smooth morphism and let $d_{Z/Y} = \dim(Z) - \dim(Y)$. Then $h^i \circ \chi^! = 0$ for $i \neq d_{Z/Y}$ and for every $\mathcal{D}_Y$-module $\mathcal{L}$, $\chi^! \mathcal{L} = \chi^* \mathcal{L}\,[d_{Z/Y}]$. Here, as usual, if $\mathcal{C}^{\cdot}$ is a complex, then $\mathcal{C}^{\cdot}[d]$ denotes the shift of the complex by d.

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