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BUMP FUNCTIONS AND DIFFERENTIABILITY IN BANACH SPACES

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Abstract. We show that if a Banach space E admits a continuous symmetrically Fréchet subdifferentiable bump function, then E is an Asplund space.

1. Introduction

Our result strengthens that of R. Deville, G. Godefroy, and V. Zizler [DGZ, see Lemma III.6, p. 315] who proved that every Banach space admitting a pointwise Lipschitz Fréchet smooth bump function is Asplund; we weaken the assumption of pointwise Lipschitz Fréchet smooth to upper semicontinuous symmetrically Fréchet subdifferentiable, but it is easy to see that an upper semicontinuous symmetrically Fréchet subdifferentiable bump function is continuous. It is also easy to see that there are bump functions on Hilbert space that are continuous and symmetrically Fréchet subdifferentiable but neither pointwise Lipschitz nor bounded.

To obtain differentiability points of a continuous convex function on a Banach space one method is to apply a variational principle (the function is supported from above by a function of a given type) such as Ekeland's variational principle, which can be used to show that a Banach space that admits a Fréchet differentiable bump function is an Asplund space [EL]. (For the converse it is not known whether an Asplund space admits a lower Fréchet smooth bump function; indeed Richard Haydon [Ha] exhibits an Asplund space that admits no Gâteaux smooth (differentiable at non-zero vectors) equivalent norm.) The usual use of these principles involves operations requiring additional conditions (use of Lipschitzness). Here we also use a variational principle (in fact the basic Ekeland's variational principle) but in a somewhat dual way. The principle is used on a rough norm restricted to a set defined using the norm and the bump function. This approach allows us to avoid additional conditions like the pointwise Lipschitz condition. A straightforward application of Ekeland's variational principle would work in our case assuming also that the bump function is bounded.

By our method we can show that if E admits an upper semicontinuous symmetrically Gâteax subdifferentiable bump function, then for any $\epsilon > 0$ the space E is an ϵ-GDS, that is, for any continuous convex function f on E there is a dense

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subset D of E such that for all $x \in D$ and all y in the unit sphere of E the limit

$$\lim_{t\to 0} \frac{f(x+ty)+f(x+ty)-2f(x)}{t}$$

is less than ϵ.

We recall some definitions. Let E be a Banach space. A *bump function* on E is a function $b : E \to \mathbf{R}$ that has bounded non-empty support and attains a positive value. We say that a function $\phi : E \to \mathbf{R}$ is *Fréchet differentiable* at $x \in E$ if there is a continuous linear functional $\phi'(x)$, called the *Fréchet derivative* of ϕ at x, such that

$$\lim_{\|h\|\to 0} \frac{\phi(x+h)-\phi(x)-\langle \phi'(x), h\rangle}{\|h\|} = 0.$$

A function $\phi : E \to \mathbf{R}$ is *symmetrically Fréchet subdifferentiable* at x if

$$\liminf_{h\to 0} \frac{\phi(x+h)+\phi(x-h)-2\phi(x)}{\|h\|} \geq 0.$$

Then a function $\phi : E \to \mathbf{R}$ is said to be *ϵ-rough* at x if

$$\limsup_{h\to 0} \left| \frac{\phi(x+h)+\phi(x-h)-2\phi(x)}{\|h\|} \right| \geq \epsilon.$$

A $\phi : E \to \mathbf{R}$ is said to be *ϵ-rough* if it is ϵ-rough at all points. We note by the triangle inequality that if a norm $\|\cdot\|$ is ϵ-rough at z, then the value of ϵ cannot exceed 2.

A set E is *residual* if it is the complement of a first category set in E. A Banach space E is an *Asplund space* if every continuous convex function on E is Fréchet differentiable on a residual set.

2. Lemmata

Lemma 2.1 is due to E. B. Leach and J. H. M. Whitfield [LW].

Lemma 2.1. *A Banach space E is not an Asplund space if and only if for some $\epsilon > 0$, E admits an equivalent ϵ-rough norm.*

In Lemma 2.2 we show that if a norm is ϵ-rough at z, then for any $\delta > 0$ we can find $\|\hat{h}\| < 2\delta$ for which $\frac{\|z+\hat{h}\|-\|z\|}{\|\hat{h}\|} \geq \epsilon/8$ and $\frac{\|z-\hat{h}\|-\|z\|}{\|\hat{h}\|} \geq \epsilon/8$.

Lemma 2.2. *Let $\|\cdot\|$ be a norm on a Banach space E that is ϵ-rough at z. Then for all δ for which $\frac{\epsilon\|z\|}{4} > \delta > 0$ and all h such that $0 < \|h\| < \delta$ and $\frac{\|z+h\|+\|z-h\|-2\|z\|}{\|h\|} \geq \epsilon$ there is a $\hat{h} \in \operatorname{span}\{z, h\}$ such that*

(i) $\|z+\hat{h}\| = \|z-\hat{h}\|$,
(ii) $0 < \|\hat{h}\| < 2\delta$, *and*
(iii) $\frac{\|z+\hat{h}\|-\|z\|}{\|\hat{h}\|} \geq \frac{\epsilon}{8}$.

Proof. The point $\hat{h}$ is obtained from h by subtracting a small z component. We may suppose that $\|z+h\| > \|z-h\|$. Define $\mathbf{a} : \mathbf{R} \to E$ by $\mathbf{a}(t) = h - tz$ and $a : \mathbf{R} \to \mathbf{R}$ by $a(t) = \|z+\mathbf{a}(t)\| - \|z-\mathbf{a}(t)\|$, and, since $a(0) > 0$, and $a(1) < 0$, then by the intermediate value theorem we obtain a $t_0 \in (0,1)$ such that $a(t_0) = 0$.

Let $\hat{h} = \mathbf{a}(t_0)$ so that $\|z - \hat{h}\| = \|z + \hat{h}\|$. Applying the triangle inequality we find that

$$t_0\|z\| \leq \|h\|. \tag{1}$$

Hence

$$\|\hat{h}\| = \|h - t_0 z\| < 2\delta$$

which is property (ii).

For (iii) we define a function $f : E \to \mathbf{R}$ as follows. By the Hahn-Banach theorem find $x^\star \in E^\star$ such that $\|x^\star\| = 1$ and $\langle x^\star, z\rangle = \|z\|$; then define $f : E \to \mathbf{R}$ by $f(x) = \|x\| - \langle x^\star, x\rangle$. Then clearly f is convex and non-negative, Lip $(f) \leq 2$, $f(\lambda z) = 0$ for all $\lambda \geq 0$, and $f(\lambda x) = \lambda f(x)$ for all $\lambda \geq 0$. We claim that $f(z+h) \leq f(z + \hat{h})$, and $\frac{f(z+\hat{h})+f(z-\hat{h})}{\|\hat{h}\|} \geq \epsilon/4$. The claim that $f(z + h) \leq f(z + \hat{h})$ follows, since

$$\begin{aligned} f(z + \hat{h}) - f(z + h) &= \|z + \hat{h}\| - \|z + h\| + \langle x^\star, h - \hat{h}\rangle \\ &= \|z + \hat{h}\| - \|z + h\| + \|h - \hat{h}\| \\ &\geq 0. \end{aligned}$$

To see that $\frac{f(z+\hat{h})+f(z-\hat{h})}{\|\hat{h}\|} \geq \epsilon/4$, let $\beta = 1/(1 + t_0)$ so that $0 < \beta \leq 1$. Then $z - \beta h = \beta(z - \hat{h})$ so that $f(z - \beta h) = \beta f(z - \hat{h}) \leq f(z - \hat{h})$. We estimate that

$$\begin{aligned} f(z + h) + f(z - h) &\leq f(z + \hat{h}) + f(z - h) \\ &\leq f(z + \hat{h}) + f(z - \hat{h}) + (f(z - h) - f(z - \beta h)) \\ &\leq f(z + \hat{h}) + f(z - \hat{h}) + 2(1 - \beta)\|h\|. \end{aligned} \tag{2}$$

From (1) we have $1 - \beta = t_0/(1 + t_0) \leq t_0 \leq \|h\|/\|z\|$ and this with $\|\hat{h}\| \leq 2\|h\|$ in (2) gives

$$\begin{aligned} \frac{f(z + \hat{h}) + f(z - \hat{h})}{\|\hat{h}\|} &\geq \frac{f(z + \hat{h}) + f(z - \hat{h})}{\|2h\|} \\ &\geq \frac{f(z + h) + f(z - h)}{\|2h\|} - (1 - \beta) \\ &\geq \frac{f(z + h) + f(z - h)}{\|2h\|} - \frac{\|h\|}{\|z\|}. \end{aligned} \tag{3}$$

By hypothesis $\|h\| < \delta < \frac{\epsilon\|z\|}{4}$ so that

$$\frac{f(z + h) + f(z - h)}{\|2h\|} = \frac{\|z + h\| + \|z - h\| - 2\|z\|}{\|2h\|} \geq \frac{\epsilon}{2}.$$

This is used in (3) so that $\frac{f(z+\hat{h})+f(z-\hat{h})}{\|\hat{h}\|} \geq \frac{\epsilon}{4}$ and since $\|z+\hat{h}\| = \|z-\hat{h}\|$ we deduce that

$$\frac{\|z + \hat{h}\| - \|z\|}{\|\hat{h}\|} \geq \frac{\epsilon}{8}.$$

□

3. An improvement on the Ekeland-Lebourg Theorem

The following Theorem improves on the result of Ekeland and Lebourg that a Banach space that admits a Fréchet differentiable bump function is an Asplund space.

Theorem 3.1. *Let E be a Banach space which admits an upper semicontinuous symmetrically Fréchet subdifferentiable bump function. Then E is an Asplund space.*

Proof. We argue by contradiction. Suppose that E is not an Asplund space. Then, by Lemma 2.1, E admits an equivalent ϵ-rough norm, say $\|\cdot\|$, for some $\epsilon > 0$. Recall that a bump function b on E has bounded support and attains a positive value. So we may assume that $b(0) = 1$ and supp $(b) \subset B(0,1)$. We let $S = \{\, x : 4b(x)+\|x\| \geq 3\}$ where b is upper semicontinuous and symmetrically Fréchet subdifferentiable. We apply Ekeland's variational principle to the norm on $T = S \cap B(0,2)$ to find a point at which the perturbed norm has a maximum. We get a contradiction by showing that the bump function b is not symmetrically Fréchet subdifferentiable at this point.

The set T is non-empty since $0 \in T$. We show that $T \subset B(0,1)$. Indeed if $x \in T \setminus \text{supp }(b)$, then, since $b(x) = 0$ and $x \in S$, we must have $\|x\| \geq 3$. But $T \subset B(0,2)$. Therefore $T \setminus \text{supp }(b) = \emptyset$. Hence $T \subset \text{supp }(b) \subset B(0,1)$.

Define a metric on T by $d(x,y) = \|x - y\|$. Since b is upper semicontinuous, S, and so T, is closed; so T with $d(x,y)$ is complete and Ekeland's variational principle is applicable in T to the continuous function $\|\cdot\|$, which is bounded above, so that we obtain a point $z \in T$ such that

$$\|x\| \leq \|z\| + \epsilon\frac{\|x-z\|}{16} \tag{4}$$

for all $x \in T$. As b is lower Fréchet smooth at 0 we have

$$\frac{b(h) + b(-h) - 2}{\|h\|} > -1 \tag{5}$$

for all sufficiently small $\|h\|$. It follows that there is some non-zero $h \in T$, and that in particular $T \neq \{0\}$. If this were not the case, then we would get $4b(h)+\|h\| < 3$ and $4b(-h) + \|h\| < 3$ for all sufficiently small $\|h\|$ so that $\frac{b(h)+b(-h)-2}{\|h\|} < -1/2 - \frac{1}{2\|h\|}$ which clearly contradicts (5). Since $T \neq \{0\}$ we must also have $z \neq 0$ since $z = 0$ in (4) would imply that $\|x\| \leq \epsilon\|x\|/16$ for all non-zero $x \in T$.

Let $0 < \delta < \frac{\epsilon\|z\|}{4}$ be such that

$$\frac{b(z+h) + b(z-h) - 2b(z)}{\|h\|} > -\frac{\epsilon}{16}$$

for $0 < \|h\| < 2\delta$. Since the norm is ϵ-rough at z we may apply Lemma 2.2 to obtain a point $\hat{h} \in E$ with the following properties:

(i) $\|z + \hat{h}\| = \|z - \hat{h}\|$,
(ii) $\|\hat{h}\| < 2\delta$, and
(iii) $\frac{\|z+\hat{h}\|-\|z\|}{\|\hat{h}\|} \geq \frac{\epsilon}{8}$.

We claim that $z + \hat{h} \notin T$, and $z - \hat{h} \notin T$. If $z + \hat{h} \in T$, then from (4), $\|z + \hat{h}\| \leq \|z\| + \frac{\epsilon\|\hat{h}\|}{16}$, which contradicts (iii). Similarly for $z - \hat{h}$.

Since $\|z\| \le 1$ and as $\|\hat{h}\| \le 1$ we have $z+\hat{h} \in B(0,2)$ and $z-\hat{h} \in B(0,2)$. From the definition of T we deduce that $z+\hat{h} \notin S$ and $z-\hat{h} \notin S$.

From the definition of the set S we obtain $4b(z+\hat{h}) + \|z+\hat{h}\| < 3$ and $4b(z-\hat{h}) + \|z-\hat{h}\| < 3$. Adding these together with $-8b(z) - 2\|z\| \le -6$, it follows that

$$0 < 4\frac{b(z+\hat{h}) + b(z-\hat{h}) - 2b(z)}{\|\hat{h}\|} + \frac{\|z+\hat{h}\| + \|z-\hat{h}\| - 2\|z\|}{\|\hat{h}\|} < 0$$

which is a contradiction. Hence E is an Asplund space. □

In Theorem 3.1, it is enough to assume that for every $\epsilon > 0$ there is a bump function $b_\epsilon : E \to \mathbf{R}$, with $b_\epsilon(0) = 1$, for which supp $(b_\epsilon) \subset B(0,1)$, which is u.s.c., and is symmetrically $\epsilon-$Fréchet subdifferentiable, that is, for $x \in E$,

$$\liminf_{h \to 0} \frac{b_\epsilon(z+h) + b_\epsilon(z-h) - 2b_\epsilon(z)}{\|h\|} > -\epsilon.$$

A partial analogy of the above result also holds. For this we introduce some definitions. We say that $\phi : E \to \mathbf{R}$ is *symmetrically Gâteaux subdifferentiable* at x if for each $y \in E$

$$\liminf_{t \to 0^+} \frac{\phi(x+ty) + \phi(x-ty) - 2\phi(x)}{t} \ge 0.$$

Let $\epsilon > 0$. We say that ϕ is *ϵ-Gâteaux rough* at x if we can find $y \in E$ such that $\|y\| = 1$ and

$$\liminf_{t \to 0^+} \left| \frac{\phi(x+ty) + \phi(x-ty) - 2\phi(x)}{t} \right| \ge \epsilon.$$

If ϕ is ϵ-Gâteaux rough at x, then it is ϵ-rough at x. A Banach space E is an *ϵ-Gâteaux differentiability space* (ϵ-GDS) if for any continuous convex function f on E there is a dense subset D of E such that for all $x \in D$ and all y in the unit sphere of E the limit

$$\lim_{t \to 0} \frac{f(x+ty) + f(x+ty) - 2f(x)}{t}$$

is less than ϵ.

With these definitions we get, by the same method, that a Banach space which admits an upper semicontinuous symmetrically Gâteaux subdifferentiable bump function is an ϵ-GDS for all $\epsilon > 0$.

The main part of the proof is the analogy of Lemma 2.2; that if E is not an ϵ-GDS, then E admits an ϵ_0-Gâteaux rough norm for some $\epsilon_0 > 0$. This can be shown as follows. By a modification of a result of Fabian (see [Ph, p. 97]), we show that if a continuous convex function f on $E \times \mathbf{R}$ is ϵ-Gâteaux rough on an open set $B \times I$ where B is an open ball and I is an open interval, then there is a continuous convex function h on E that is ϵ-Gâteaux rough on B. We also adjust a result of of Larman and Phelps (see [Ph, p. 96]), who show that, given a Banach space E, if every continuous convex Minkowski guage on $E \times \mathbf{R}$ is Gâteaux differentiable at the points of a dense set, then E is a GDS. From this result we show that if a continuous convex function is ϵ-Gâteaux rough on an open set U in a Banach space E, then $E \times \mathbf{R}$ admits an ϵ_0-Gâteaux rough norm for some $\epsilon_0 > 0$. Combining these two statements it is easy to see that if E is not an ϵ-GDS, then E admits an ϵ_0-Gâteaux rough norm for some $\epsilon_0 > 0$.

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