FINITE GROUPS AND THE FIXED POINTS OF COPRIME AUTOMORPHISMS

PAVEL SHUMYATSKY

(Communicated by Stephen D. Smith)

Abstract. Let \( p \) be a prime, and let \( G \) be a finite \( p' \)-group acted on by an elementary abelian \( p \)-group \( A \). The following results are proved:

1. If \( |A| \geq p^3 \) and \( C_G(a) \) is nilpotent of class at most \( c \) for any \( a \in A^\# \), then the group \( G \) is nilpotent of \( \{c,p\} \)-bounded class.

2. If \( |A| \geq p^4 \) and \( C_G(a)' \) is nilpotent of class at most \( c \) for any \( a \in A^\# \), then the derived group \( G' \) is nilpotent of \( \{c,p\} \)-bounded class.

1. Introduction

Let \( G \) be a group admitting an action of a group \( A \). We denote by \( C_G(A) \) the set \( C_G(A) = \{ x \in G | x^a = x \text{ for any } a \in A \} \), the centralizer of \( A \) in \( G \) (the fixed-point group). Throughout this paper we assume that \( A \) is a noncyclic elementary abelian \( p \)-group, and \( G \) is a finite \( p' \)-group. Let \( A^\# \) denote the set of non-identity elements of \( A \). It follows from the classification of finite simple groups that if \( C_G(a) \) is solvable for any \( a \in A^\# \), then so is the group \( G \) (see [3]). The case \( |A| \geq p^3 \) does not require the classification: the result follows from Glauberman’s theorem on solvable signalizer functors [1]. In certain specific situations much more can be said about the structure of \( G \).

Ward showed that if \( A \) has rank at least 3, and if \( C_G(a) \) is nilpotent for any \( a \in A^\# \), then the group \( G \) is nilpotent [7]. Another of Ward’s results is that if \( A \) has rank at least 4, and if \( C_G(a)' \) is nilpotent for any \( a \in A^\# \), then the derived group \( G' \) is nilpotent [8]. Later the author found that if, under these assumptions, \( C_G(a) \) is nilpotent of class at most \( c \) (respectively \( C_G(a)' \) is nilpotent of class at most \( c \)) for any \( a \in A^\# \), and if \( G \) has derived length \( d \), then the nilpotency class of \( G \) (respectively of \( G' \)) is \( \{c,d,p\} \)-bounded [6]. In the present paper we show that actually much stronger results are valid: the bounds on the class of \( G \) and \( G' \) can be chosen independent of \( d \).

Theorem 1.1. Let \( A \) be an elementary abelian group of order \( p^3 \) acting on a finite \( p' \)-group \( G \). Assume that \( C_G(a) \) is nilpotent of class at most \( c \) for any \( a \in A^\# \). Then \( G \) is nilpotent and the class of \( G \) is bounded by a function depending only on \( p \) and \( c \).
Theorem 1.2. Let $A$ be an elementary abelian group of order $p^4$ acting on a finite $p'$-group $G$. Assume $C_L(a)^{t}$ is nilpotent of class at most $c$ for any $a \in A^\#$. Then $G'$ is nilpotent and the class of $G'$ is bounded by a function depending only on $p$ and $c$.

We conjecture that these results can be generalized in the following way.

Conjecture 1.3. Let $A$ be an elementary abelian group of order $p^k$ with $k \geq 3$ acting on a finite $p'$-group $G$.

1. If $\gamma_{k-2}(C_G(a))$ is nilpotent of class at most $c$ for any $a \in A^\#$, then $\gamma_{k-2}(G)$ is nilpotent and has $(c, k, p)$-bounded class.

2. If, for some integer $d$ such that $2^d + 2 \leq k$, the $d$th derived group of $C_G(a)$ is nilpotent of class at most $c$ for any $a \in A^\#$, then the $d$th derived group $G^{(d)}$ is nilpotent and has $(c, k, p)$-bounded class.

Our main evidence in favor of the above conjecture is Lie-theoretic: Theorem 2.7 obtained in Section 2 establishes the corresponding results for Lie algebras.

2. Action on Lie algebras

Throughout the paper the term Lie algebra means Lie algebra over an associative ring with unity. Let $L$ be a Lie algebra. If $X, Y, X_1, \ldots, X_s$ are subsets of $L$ we use $[X, Y]$ to denote the subspace of $L$ spanned by the set $\{[x, y]|x \in X, y \in Y\}$. If $t \geq 2$ we write $[X, tY]$ for $[[X, t-1Y], Y]$ and $[X_1, \ldots, X_t]$ for $[[X_1, \ldots, X_{t-1}], X_t]$. For any positive integer $w$, define commutator-spaces of weight $w$ in $X_1, \ldots, X_s$:

A subspace of $L$ is a commutator-space of weight 1 in $X_1, \ldots, X_s$ if and only if it is the linear span of $X_i$ for some $i \leq s$. A subspace $M$ of $L$ is a commutator-space of weight $w \geq 2$ in $X_1, \ldots, X_s$ if and only if $M = [M_1, M_2]$, where $M_1$ and $M_2$ are commutator-spaces of weights $w_1$ and $w_2$ respectively, such that $w_1 + w_2 = w$.

A well-known theorem of Kreknin [5] says that if a Lie ring $L$ admits a fixed-point-free automorphism of finite order $n$, then $L$ is solvable and the derived length of $L$ is bounded by a function of $n$. We will require the following extension of this result [3].

Theorem 2.1. Let a Lie ring $L$ admit an automorphism $\alpha$ of finite order $n$ such that $[L, tC_L(a)] = 0$. Assume that $nL = L$. Then $L$ is solvable with derived length at most $(t + 1)^{n-1} + \log_2 t$.

Lemma 2.2. Let $t \geq 1$. Let $L$ be a Lie algebra, and $K$ a nilpotent subalgebra of class $c$. Assume $K$ is generated by subspaces $X_1, \ldots, X_m$ such that for any commutator-space $Y$ in $X_1, \ldots, X_m$ we have $[L, tY] = 0$. Then there exists a $(c, m, t)$-bounded number $u$ such that $[L, uK] = 0$.

Proof. This is by induction on $c$. Since $K'$ is generated by commutator-spaces of weight $\geq 2$ in $X_1, \ldots, X_m$ and since the number of such spaces is $(c, m)$-bounded, the inductive hypothesis will be that there exists a $(c, m, t)$-bounded number $u_1$ such that $[L, u_1K'] = 0$. Now put $r = m(t - 1) + 1$ and consider the space $M = [L, Y_1, \ldots, Y_r]$ for some choice of $Y_1, \ldots, Y_r \in \{X_1, \ldots, X_m\}$. Obviously, for any permutation $\tau$ of the symbols $1, 2, \ldots, r$ we have $M \leq [L, Y_{\tau(1)}, \ldots, Y_{\tau(r)}] + [L, K']$. The number $r$ is big enough to ensure that some $X_i$ occurs in the list $Y_1, \ldots, Y_r$ at least $t$ times. Thus, we obtain $M \leq [L, tX_i, \ast, \ast] + [L, K']$, where the asterisks denote some spaces $Y_j$ which, in view of the fact that $[L, tX_i] = 0$, are of no consequence. Hence, $M \leq [L, K']$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Now take \( u = u_1 r \). Using the fact that \( K = K' + \sum X_j \) and \( M \leq [L, K'] \) for any choice of \( Y_1, \ldots, Y_r \in \{X_1, \ldots, X_m\} \), it is easy to see that \([L, u_1 K'] = 0\).

**Hypothesis 2.3.** Let \( \omega \) be a primitive \( p \)-th root of unity, and let \( L \) be a Lie algebra over \( \mathbb{Z}[\omega] \) such that \( L = pL \). Let \( A \) be an elementary abelian group of order \( p^k \) acting by automorphisms on \( L \). Let \( \hat{A} \) be the character group of \( A \). For any \( \alpha \in \hat{A} \) we set \( L_\alpha = \{x \in K | x^a = \alpha(a)x \text{ for each } a \in A\} \).

It is well-known that \( A \) and \( \hat{A} \) are isomorphic, \([L_\alpha, L_\beta] \leq L_{\alpha \beta} \) for all \( \alpha, \beta \in \hat{A} \) and \( L = \bigoplus_\alpha L_\alpha \). For any positive integer \( n \) and any \( \alpha_1, \ldots, \alpha_{2^n} \in \hat{A} \) define inductively

\[
\gamma(\alpha_1) = L_{\alpha_1} \text{ and } \gamma(\alpha_1, \ldots, \alpha_n) = [\gamma(\alpha_1, \ldots, \alpha_{n-1}), L_{\alpha_n}],
\]

\[
\delta(\alpha_1) = L_{\alpha_1} \text{ and } \delta(\alpha_1, \ldots, \alpha_{2^n}) = [\delta(\alpha_1, \ldots, \alpha_{2^{n-1}}), \delta(\alpha_{2^{n-1}+1}, \ldots, \alpha_{2^n})].
\]

As usual, \( \gamma_n(L) \) and \( L^{(n)} \) denote the \( n \)-th term of the lower central series and the \( n \)-th term of the derived series of \( L \), respectively.

**Lemma 2.4.** Under Hypothesis 2.3 we have \( \gamma_n(L) = \sum \gamma(\alpha_1, \ldots, \alpha_n) \) and \( L^{(n)} = \sum \delta(\alpha_1, \ldots, \alpha_{2^n}) \), where \( \alpha_1, \ldots, \alpha_{2^n} \) range independently through \( \hat{A} \).

**Proof.** Set \( Q = \sum \gamma(\alpha_1, \ldots, \alpha_n) \). For any \( \beta \in \hat{A} \) we have

\[
[\gamma(\alpha_1, \ldots, \alpha_n), L_\beta] \leq \gamma(\alpha_1 \alpha_2, \alpha_3, \ldots, \alpha_n, \beta) \leq Q,
\]

which shows that \( Q \) is normalized by \( L_\beta \) and therefore is an ideal of \( L \). It is easy to see that \( L/Q \) is nilpotent of class at most \( n - 1 \) and so \( \gamma_n(L) \leq Q \). The opposite inclusion is obvious, whence \( \gamma_n(L) = Q \).

To prove the other claim we set \( R_n = \sum \delta(\alpha_1, \ldots, \alpha_{2^n}) \) and, arguing by induction on \( n \), assume that \( R_{n-1} = L^{(n-1)} \). We now need to show that \( R_n = R_n' \), where \( R_n = R_n' \). For any \( \beta_1, \ldots, \beta_{2^{n-1}} \in \hat{A} \) we see that

\[
[\delta(\alpha_1, \ldots, \alpha_{2^n}), \delta(\beta_1, \ldots, \beta_{2^{n-1}})] \leq R_n,
\]

which shows that \( \delta(\beta_1, \ldots, \beta_{2^{n-1}}) \) normalizes \( R_n \). Therefore \( R_n \) is an ideal in \( R_{n-1} \) and it follows that \( R_n = R_n' \).

**Corollary 2.5.** Assume Hypothesis 2.3. Then, for any \( \beta \in \hat{A} \), we have \( L_\beta \cap \gamma_n(L) = \sum \gamma(\alpha_1, \ldots, \alpha_n) \), where the summation is taken over those \( \alpha_1, \ldots, \alpha_n \in \hat{A} \) for which \( \alpha_1 \ldots \alpha_n = \beta \). Similarly, \( L_\beta \cap L^{(n)} = \sum \delta(\alpha_1, \ldots, \alpha_{2^n}) \), where \( \alpha_1 \ldots \alpha_{2^n} = \beta \).

**Lemma 2.6.** Assume Hypothesis 2.3 with \( k \geq 2 \). Suppose there exists an integer \( u \) such that \([L, uC_L(a)] = 0\) for any \( a \in A^\# \). Then \( L \) is nilpotent of \( \{p, u\} \)-bounded class.

**Proof.** By Theorem 2.4 \( L \) is solvable and the derived length \( d \) of \( L \) is at most \((u + 1)^{p - 1} + \log_2 u\). We will prove the lemma by induction on \( d \). Applying the inductive hypothesis to \( L' \) assume that \( L' \) is nilpotent of \( \{p, u\} \)-bounded class, say.
Let $B$ be any subgroup of $A$ of order $p^2$, and let $B_1, \ldots, B_{p+1}$ be the cyclic subgroups of $B$. We set $C_i = C_L(B_i)$, $1 \leq i \leq p + 1$. Then $L = \sum_i C_i$. Let $r = (u-1)(p+1) + 1$. If $Z = Z(L)$ we obviously have $[Z, X, Y] = [Z, Y, X]$ for any subsets $X, Y$ of $L$. Having this in mind we write

$$[Z, rL] = [Z, r \sum_i C_i] = \sum [Z, u_1 C_1, \ldots, u_{p+1} C_{p+1}],$$

where $u_1 + u_2 + \ldots + u_{p+1} = r$. The number $r$ is big enough to ensure that $u_i \geq u$ for some $i$, so it follows that $[Z, u_1 C_1, \ldots, u_{p+1} C_{p+1}] = 0$ since $L, u C_i = 0$. Thus, $[Z, rL] = 0$ and therefore $Z \leq Z_r(L)$, where $Z_r(L)$ is the $r$th term of the upper central series of $L$. Applying this argument repeatedly to $L/Z, L/Z_2(K)$ and so on, we conclude that $L' \leq Z_{er}(L)$ and therefore $L$ is of nilpotency class at most $er + 1$.

**Theorem 2.7.** Assume Hypothesis 2.3 with $k \geq 3$.

1. If $\gamma_{k-2}(C_L(a))$ is nilpotent of class at most $c$ for any $a \in A^\#$, then $\gamma_{k-2}(L)$ is nilpotent and has $(c, k, p)$-bounded class.

2. If, for some integer $d$ such that $2^d + 2 \leq k$, the $d$th derived group of $C_L(a)$ is nilpotent of class at most $c$ for any $a \in A^\#$, then $L^{(d)}$ is nilpotent and has $(c, k, p)$-bounded class.

**Proof.** 1. Obviously, for any $\beta, \alpha_1, \ldots, \alpha_{k-2} \in \hat{A}$ there exists $a \in A^\#$ such that $L_\beta, L_{\alpha_1}, \ldots, L_{\alpha_{k-2}} \leq C_L(a)$. Since $\gamma_{k-2}(C_L(a))$ is nilpotent of class at most $c$, it follows that $[L_{\beta c+2} \gamma_{1, \alpha_1, \ldots, \alpha_{k-2}}] = 0$. Now, using that $L = \bigoplus \beta L_\beta$, we derive that $[L, c+2 \gamma_{1, \alpha_1, \ldots, \alpha_{k-2}}] = 0$. Corollary 2.5 shows that $C_L(a) \cap \gamma_{k-2}(L) = \gamma_{1, \alpha_1, \ldots, \alpha_{k-2}}$, where the summation is taken over all those $\alpha_1, \ldots, \alpha_{k-2}$ for which $\alpha_1 \ldots \alpha_{k-2} = 1$. We now apply Lemma 2.2 with $K = C_L(a) \cap \gamma_{k-2}(L)$ and the spaces $\gamma_{1, \alpha_1, \ldots, \alpha_{k-2}} \leq C_L(a)$ in place of $X_i$ to deduce that there exists a $(c, k, p)$-bounded number $u$ such that $L, u K = 0$. But then it follows from Lemma 2.6 that $\gamma_{k-2}(L)$ is nilpotent of $(p, u)$-bounded class.

2. The proof of the second claim is not really much different from what we have done above. We establish first that $[L, c+2 \delta_{1, \alpha_1, \alpha_2}] = 0$ for all $\alpha_1, \alpha_2 \in \hat{A}$. Next, we apply Lemma 2.2 to deduce that there exists a $(c, k, p)$-bounded number $u$ such that $[L, u C_{L \alpha_1}(a)] = 0$ for all $a \in A$. Finally, we observe that the required assertion follows from Lemma 2.6.

3. **Main results**

The next lemma is well-known (see [2] 6.2.2, 6.2.4 for the proof).

**Lemma 3.1.** Let $A$ be a finite $p$-group acting on a finite $p'$-group $G$.

1. If $N$ is an $A$-invariant normal subgroup of $G$, then $C_{G/N}(A) = C_G(A)N/N$.

2. If $A$ is an elementary abelian group, and if $A_1, \ldots, A_s$ are the maximal subgroups of $A$, then $G = \{ C_G(A_i) | 1 \leq i \leq s \}$.

**Lemma 3.2.** Let $p$ be a prime, and $G$ a finite $p'$-group acted on by an elementary abelian $p$-group $A$ of rank at least 3. Let $A_1, \ldots, A_s$ be the maximal subgroups of $A$. Then

$$G' = \langle C_G(A_i), C_G(A_j) | 1 \leq i, j \leq s \rangle.$$
Proof. By Lemma 3.1, \( G = \langle C_G(A_1), \ldots, C_G(A_s) \rangle \). Consider the subgroup \( R = \langle [C_G(A_i), C_G(A_j)] | 1 \leq i, j \leq s \rangle \). Obviously, \( R \) is \( A \)-invariant so \( R = \langle C_R(A_1), \ldots, C_R(A_s) \rangle \). To show that \( R \) is normal it is sufficient to establish that \( y^r \in R \) for any \( y \in C_R(A_i) \) and \( x \in C_G(A_j) \). We have \( y^r = y^r y^{-1} y \) and obviously both \( y^r y^{-1} \) and \( y \) belong to \( R \). Hence \( y^r \in R \) and \( R \) is normal. Using that \( G = \langle C_G(A_1), \ldots, C_G(A_s) \rangle \), it is now easy to see that \( G/R \) is abelian, as required. 

We are now ready to prove the main results.

**Theorem 1.1.** Let \( A \) be an elementary abelian group of order \( p^3 \) acting on a finite \( p' \)-group \( G \). Assume that \( C_G(a) \) is nilpotent of class at most \( c \) for any \( a \in A^\# \). Then \( G \) is nilpotent and the class of \( G \) is bounded by a function depending only on \( p \) and \( c \).

**Proof.** We know from Ward’s result cited in the Introduction that \( G \) is nilpotent. Let \( L(G) \) be the Lie ring corresponding to the lower central series of \( G \). The construction associating the Lie ring with \( G \) is well-known. Let \( \gamma_i \) denote the \( i \)th term of the lower central series of \( G \). Set \( L_i = \gamma_i/\gamma_{i+1} \) and view \( L_i \) as an additive abelian group. Then \( L(G) = \bigoplus L_i \). If \( x \in \gamma_i \), \( y \in \gamma_j \), then, for corresponding elements \( x\gamma_{i+1}, y\gamma_{j+1} \) of \( L(G) \), we set \([x\gamma_{i+1}, y\gamma_{j+1}] = [x, y]\gamma_{i+j+1} \). This operation can be uniquely extended by linearity on the additive abelian group \( L(G) \) and, equipped with the product, \( L(G) \) becomes a Lie ring. The Lie ring has the same nilpotency class as \( G \). In our situation the group \( A \) acts naturally on each quotient \( \gamma_i/\gamma_{i+1} \) and this action extends uniquely to an action by automorphisms on the Lie ring \( L(G) \). Lemma 3.1 shows that if \( a \in A \), then \( C_{L(G)}(a) \) is the direct sum of the quotients \( \gamma_i(a)\gamma_{i+1}/\gamma_{i+1} \). It follows that \( C_{L(G)}(a) \) is nilpotent of class at most \( c \) for any \( a \in A^\# \). Finally, we note that \( L(G) \) is finite and has the same order as \( G \). Therefore \( pL(G) = L(G) \). Set \( L = L(G) \otimes \mathbb{Z}[\omega] \). We can view \( L \) as a Lie algebra over \( \mathbb{Z}[\omega] \) and \( A \) as a group acting on \( L \). By Theorem 2.7 \( L \) is nilpotent of \( \{c, p\}\)-bounded class and so is \( G \).

**Theorem 1.2.** Let \( A \) be an elementary abelian group of order \( p^4 \) acting on a finite \( p' \)-group \( G \). Assume \( C_L(a) \) is nilpotent of class at most \( c \) for any \( a \in A^\# \). Then \( G' \) is nilpotent and the class of \( G' \) is bounded by a function depending only on \( p \) and \( c \).

**Proof.** Let \( A_1, \ldots, A_s \) be the maximal subgroups of \( A \). Then, by Lemma 4.2 \( G' = \langle [C_G(A_i), C_G(A_j)] | 1 \leq i, j \leq s \rangle \). We know that \( G' \) is nilpotent. Let \( L(G') \) be the Lie ring corresponding to the lower central series of \( G' \). Set \( L = L(G') \otimes \mathbb{Z}[\omega] \). We will view \( L \) as a Lie algebra over \( \mathbb{Z}[\omega] \) and \( A \) as a group acting on \( L \). By Theorem 2.7 \( L' \) is nilpotent of \( \{c, p\}\)-bounded class, say \( e \). Let \( X_1, \ldots, X_e \) be the images of various subgroups of the form \( [C_G(A_i), C_G(A_j)] \) in \( G'/G'' \). So \( L \) is generated by the sets \( X_1, \ldots, X_e \). For any \( i, j, k \leq s \) we observe that there exists some \( a \in A^\# \) such that the centralizers \( C_G(A_i), C_G(A_j), C_G(A_k) \) are all contained in \( C_G(a) \). Therefore \( [C_G(A_k), (c+2)C_G(A_i), C_G(A_j)] = 1 \). Now, if \( X_i \) is the image of \( [C_G(A_i), C_G(A_j)] \) in \( G'/G'' \), it follows that \( [C_G(A_k), (c+2)X_i] = 0 \), whence \( [L, (c+2)X_i] = 0 \).

Set \( r = (c + 1) + 1 \). If \( Z = Z(L') \), we obviously have \( [Z, X, Y] = [Z, Y, X] \) for any subsets \( X, Y \) of \( L \). Having this in mind, and taking into account that \( L \) is generated by the sets \( X_i \), we write

\[
[Z, L] = \sum [Z, u_1 X_1, \ldots, u_e X_e],
\]
where \( u_1 + u_2 + \cdots + u_t = r \). The number \( r \) is big enough to ensure that \( u_j \geq c + 2 \) for some \( j \), so it follows that \( [Z, u_1 X_1, \ldots, u_t X_t] = 0 \) since \( [L, c+2 X_j] = 0 \). Thus, \( [Z, L] = 0 \) and therefore \( Z \leq Z_r(L) \), where \( Z_r(L) \) is the \( r \)th term of the upper central series of \( L \). Applying this argument repeatedly to \( L/Z, L/Z_2(K') \) and so on, we conclude that \( L' \leq Z_{cr}(L) \) and therefore \( L \) is of nilpotency class at most \( er + 1 \). \( \square \)

References