A SHORT PROOF OF AN INDEX THEOREM

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Abstract. We give a $KK$-theoretical proof of an index theorem for Dirac-Schrödinger operators on a noncompact manifold.

1. Introduction

Index theorems, generally speaking, express an analytical index in terms of topological information. An analytical index is usually some generalization of the classical Fredholm index of a Fredholm operator, and the relevant topological information is usually given by the cohomological image of a $K$-theory/$K$-homology pairing, thus involving a Chern character map into a suitable cohomology theory. The best known example of an index theorem is the Atiyah-Singer index theorem [6]. A more complicated example is supplied by the Baum-Connes conjecture, where the analytical side is a $K$-group, and the other side is the topological $K$-homology (the elliptic operator group) of a suitable classifying space [8].

Anghel’s index theorem is an index formula for Dirac-Schrödinger operators. These operators are of the form $D + iA$, where $D$ is a (generalized) Dirac operator and $iA$ is a skew-adjoint order zero operator, both acting on $L^2$ sections of some bundle. The theorem involves warped cones, which are manifolds that have a collar at infinity. More precisely, they are isomorphic outside a compact set to $\mathbb{R} \times N$ with Riemannian metric $dr^2 + f(r)^2 \tilde{g}$, where $\tilde{g}$ is the Riemannian metric of the compact manifold $N$ and $f$ is a nondecreasing function $f : \mathbb{R} \to \mathbb{R}^+$.

We will give a short proof of the following theorem, the Euclidean space version of which was first proven by Callias [13], and then proven in greater generality by Anghel [11, 12].

Theorem 1.1. Let $D + iA$ be a Dirac-Schrödinger operator over a warped cone with compact even-dimensional base $N$. If $A^2$ becomes arbitrarily large outside a compact subset of $M$, and $[D, A]$ is bounded, then $D + iA$ is Fredholm, with index given by

$$
\int_N \hat{A}(TN) \wedge \text{ch} V^+ d(\text{vol}_N),
$$

where $\hat{A}$ denotes Atiyah’s $A$-genus and $V^+$ is the positive eigenbundle of $A$ over a copy of $N$ contained in a neighbourhood of infinity such that $A$ is invertible in that neighbourhood.

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Anghel’s original proof of this theorem used differential geometry and a “cutting and pasting” argument to reduce the problem to one that can be solved by a separation of variables technique.

2. OUTLINE OF THE PROOF

We give a short proof of Anghel’s theorem using $KK$-theory. The strategy of the proof is as follows:

First of all, we will show that the hypothesis of the theorem gives certain naturally defined $K$-theory or $K$-homology cycles over $N$ and $M$:

i) The Dirac operators over $N$ and over $M$ give cycles $[D_N] \in KK^0(C(N), \mathbb{C})$ and $[D_M] \in KK^1(C_0(M), \mathbb{C})$.

ii) The geometrical properties of $M$ give a cycle $[E] \in KK^1(C(N), C(M))$.

iii) The endomorphism $A$ defines cycles $[A] \in KK^1(C, C_0(M))$ and $[V^+] \in KK^0(C, C(N))$.

iv) The index of $D + iA$ defines a cycle $[D + iA] \in KK(C, \mathbb{C})$.

Then we show, by computing several Kasparov products, that

\begin{align}
\text{Ind}(D + iA) &= [D + iA] = [A] \otimes_{C_0(M)} [D_M] \\
&= ([V^+] \otimes_{C(N)} [E]) \otimes_{C_0(M)} [D_M] \\
&= [V^+] \otimes_{C(N)} [D_N].
\end{align}

The Kasparov product of $K$-theory and $K$-homology cycles over a compact manifold coincides with Atiyah’s [5] pairing of $K_*(X)$ and $\text{Ell}(X)$. We finish the proof by observing that the Atiyah-Singer index theorem, which expresses this pairing in terms of cohomology, gives exactly the result we are looking for:

**Theorem 2.1** (Atiyah-Singer [6, 20]). If $N$ is a compact spin$^c$ manifold, $V$ is a vector bundle over $N$, and $D_N$ is a (generalized) Dirac operator over $N$, then

$$[V] \otimes_{C(N)} [D_N] = \int_N \hat{A}(TN) \wedge V^+d(\text{vol}_N).$$

3. CYCLES GIVEN BY DIRAC OPERATORS

Let us begin with a discussion of Clifford bundles, principally because the fact that the index of the target $KK$-group of a Dirac operator depends on the dimension of the base manifold can be understood in terms of an isomorphism of Clifford algebras. Let $\Gamma_0(S)$ denote the $C^*$-algebra of $C_0$-sections of a bundle $S$. Recall that a bundle $S$ with Riemannian metric and connection is a complex $\mathbb{Z}_2$-graded Dirac bundle [20] over $M$ if the sections of $S$ are left modules over the naturally $\mathbb{Z}_2$-graded bundle $\mathbb{C}l(M)$ which are compatible with the metric and connection on $\mathbb{C}l(M)$ in the sense that

i) $(ts_1, s_2) + (s_1, ts_2) = 0$ for all $t \in \Gamma_0(T^*M)$ and $s_1 \in \Gamma_0S$;

ii) $\nabla_t(\omega s) = (\nabla_t\omega)s + \omega\nabla_t s$ for all $t \in \Gamma_0(T^*M)$, $\omega \in \Gamma_0(\mathbb{C}l(M))$, $s \in \Gamma_0(S)$.

Such a bundle is said to be $\mathbb{Z}_2$-graded if there is a grading on $S$ that is preserved by covariant differentiation and is compatible with the grading on $\mathbb{C}l(M)$. The basic example of a complex $\mathbb{Z}_2$-graded Dirac bundle is $\mathbb{C}l(M) \otimes E$, where $E$ is some Riemannian vector bundle over $M$ with connection $\nabla^E$. If $M$ is odd-dimensional, the volume form $\omega$ in $\mathbb{C}l(M)$ can be regarded as the generator of the complex Clifford
algebra $C_1$, and $Cl(M) = Cl(M)^0 \otimes C_1$. Therefore, a Dirac bundle over an odd-dimensional manifold can be regarded as a $C_1 \otimes \Gamma_0(M)$-module. The Dirac operator $D$ on a $\mathbb{Z}_2$-graded complex Dirac bundle $S$ is always of odd degree, so we can define a $KK(\Gamma_0(Cl(M)), \mathbb{C})$ cycle by $[D] := (S, \phi, D)$, where $\phi$ is the standard Clifford action. If the dimension of $M$ is odd, this cycle is actually a $KK^1(\Gamma_0(Cl(M)^0), \mathbb{C})$ cycle, whereas if the dimension of $M$ is even, we have a $KK^0$ cycle. Because of the topological Thom isomorphism [10], we obtain the same $KK$-groups if we forget the Clifford structure on the $C^*$-algebras, and regard $(S, \phi, D)$ as a $KK^0(\Gamma_0(M), \mathbb{C})$ cycle if $M$ is even-dimensional, and as a $KK^0(\Gamma_0(M) \otimes C_1, \mathbb{C}) \cong KK^1(\Gamma_0(M), \mathbb{C})$ cycle if $M$ is odd-dimensional. It is more convenient to work with cycles over $\Gamma_0(M)$ if we wish to apply the Atiyah-Singer index theorem.

Note that if $D$ is a Dirac operator on $S$, we can tensor $S$ with any bundle $E$, obtaining another Dirac operator. This new Dirac operator is usually called the Dirac operator on $S$ with coefficients in $E$. For brevity we denote the $L^2$-spaces that the operators act on by $L^2(S)$ instead of $L^2(M, S)$, etc. Finally, we regard the graded complex Clifford algebra $C_1$ as two copies of $\mathbb{C}$. Elements of the form $(e, e)$ are then of degree 0 for the grading, and elements of the form $(e, -e)$ are of degree 1.

Lemma 3.1. Suppose $S$ is a Dirac bundle over $M$. Let $H$ and $C$ be the graded Hilbert spaces $L^2(S \otimes E) \oplus L^2(S \otimes E)$ and $L^2(S) \oplus L^2(S)$, respectively. Let $D$ be the Dirac operator on $S$, and let $D_E$ be the Dirac operator on $S$ with coefficients in $E$. Suppose $A$ is a smooth self-adjoint vector bundle endomorphism, such that $|D_E \pm iA|^2 - D_E^2$ is bounded below.

Then, if the following three cycles are well-defined, the third one is the Kasparov product of the first two:

$$[D] := \left( C, \phi, \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \right) \in KK(C_0(M) \otimes C_1, \mathbb{C}),$$

$$[A] := \left( \Gamma_0(E) \otimes C_1, 1, \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \right) \in KK(\mathbb{C}, C_0(M) \otimes C_1),$$

and

$$[D + iA] := \left( H, 1, \begin{pmatrix} 0 & D_E - iA \\ D_E + iA & 0 \end{pmatrix} \right) \in KK(\mathbb{C}, \mathbb{C}).$$

Remark. The main case of interest is one where the commutator of $D_E$ and $1 \otimes \mathbb{C} A$ is bounded. This immediately implies the semiboundedness of $|D_E \pm iA|^2 - D_E^2$.

Proof. The action of $\phi : C_0(M) \otimes C_1 \to \mathcal{L}(C)$ is given by

$$\phi : b \oplus b \mapsto \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \quad \phi : b \oplus -b \mapsto \begin{pmatrix} 0 & -ib \\ ib & 0 \end{pmatrix},$$

and a calculation with gradings shows that there is a Hilbert module isomorphism that identifies $H$ with the inner tensor product of $B := \Gamma_0(E) \otimes C_1$ and $C = L^2(S) \oplus L^2(S)$ over $\phi$.

In order to apply the criterion for an unbounded cycle to be the Kasparov product of two given cycles [24], we have to verify a semiboundedness condition and a connection condition. Define

$$\bar{L} := \begin{pmatrix} 0 & D_E - iA \\ D_E + iA & 0 \end{pmatrix} : H \to H$$

and

$$\bar{D} := \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} : C \to C.$$
We denote the image of \([D + iA]\) under the module isomorphism \(\omega : H \to B \circ \phi C\) by \((B \circ \phi C, 1, G)\). We now verify the connection condition. We have to show that \(T_b \tilde{D} - (-1)^{\partial b} G T_b : C \to B \circ \phi C\) is bounded for all homogeneous \(b\) in some dense subset of \(B\), where \(T_b : c \mapsto b \circ c\). We take \(b\) to be smooth and compactly supported in order to satisfy the appropriate range and domain conditions.

If we take the case of \(b = g \oplus -g\), we get

\[
(\omega^{-1} T_{g \oplus -g} \tilde{D} + \tilde{L} \omega^{-1} T_{g \oplus -g}) \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} -ig \otimes Dl_1 \\ ig \otimes Dl_2 \end{pmatrix} + \begin{pmatrix} D_E (ig \otimes l_1) \\ -D_E (ig \otimes l_2) \end{pmatrix} + \begin{pmatrix} Ag \otimes l_1 \\ Ag \otimes l_2 \end{pmatrix}.
\]

Since \(g\) is compactly supported, \(Ag\) is bounded, and the term on the right can be neglected. The compactly supported first-order differential operators \(l \mapsto g \otimes Dl\) and \(l \mapsto D_E (g \otimes l)\) have the same symbols, and hence differ by a bounded operator. The case of \(b\) with even degree is similar.

We come to the semiboundedness condition. We are to show that \([G, (A \oplus -A) \circ \phi Id]\) is semibounded below. As an operator on \(H\), this commutator is

\[
[L, A] = 2 \begin{pmatrix} A^2 & 0 \\ 0 & A^2 \end{pmatrix} + i \begin{pmatrix} DA - AD & 0 \\ 0 & AD - DA \end{pmatrix},
\]

which is semibounded below by hypothesis. \(\square\)

In order to apply this lemma, one must verify that the 3 cycles mentioned are in fact cycles, if \(A\) goes to infinity at infinity as in the statement of Anghel’s theorem. For the cycle denoted \([A]\), the only thing to check is that \((1 + A^2)^{-1}\) is compact in the Hilbert module sense, which (for noncompact \(M\)) is true if and only if \(A^2 - \lambda\) is positive, for any real \(\lambda\), outside a compact subset of \(M\). For the cycles given by Dirac operators, more work is required. The principal question is if the operator \(D_E - iA\) from the above proof has an \(L^2\)-index at all. This corresponds to proving that the resolvent of \(L := \begin{pmatrix} D_E & 0 \\ 0 & D_E + iA \end{pmatrix} : H \to H\) is compact, or, equivalently, to proving that \(L\) has discrete spectrum. There is already a lot of literature on the subject of Fredholmness for first-order operators. In the case of operators of this particular type, Anghel has studied this question \([2, 3, 4]\). We give a different proof, based on a general technique adapted from Bochner’s method as used by Gromov and Lawson \([18]\). First let us obtain a substitute for Lichnerowicz’s identity by trivially rewriting our hypothesis in terms of quadratic forms.

**Lemma 3.2.** If \(D\) and \(A\) have bounded commutator and \(A^2\) goes to infinity, there is a semibounded endomorphism \(R\) that becomes arbitrarily large outside a compact set, such that

\[
\langle Ls, Ls \rangle = \langle Ds, Ds \rangle + \langle Rs, s \rangle
\]

for all \(s\) in the domain of \(L\) and \(D\).

**Proof.** This is a consequence of the already used fact that \(L\) and \(D\) are closed self-adjoint operators (for a proof of this, see Chernoff \([16]\)). \(\square\)

Next we prove a lemma about approximation of \(L\) on a subspace by bounded operators.

**Lemma 3.3.** If \(H_c\) is a vector space such that \(\|Ls\| \leq c\|s\|\) for all \(s \in H_c\), then \(H_c\) is finite-dimensional.
Proof. Define \( \chi(A) \) to be the characteristic function of the measurable set \( A \subset M \), and let \( \|s\|_A := \|\chi(A)s\| \). Since the norm is an integral norm, \( \|s\| = \|s\|_A + \|s\|_{M \setminus A} \).

Let \( -b \) be a lower bound for the endomorphism \( R \) in the lemma, and choose \( c_0 > -b \) so large that \( \frac{c_0 - c^2}{c_0 + b} > 1/2 \). There is a compact set \( K \) such that \( R \geq c_0 \text{Id} \) outside \( K \), and of course \( R \geq -b \text{Id} \) on \( K \). If \( s \in H_c \), then

\[
\|Ds\|^2 + \langle Rs, s \rangle \leq c_0^2 \|s\|^2
\]

so that

\[
\|Ds\|^2 + c_0\|s\|^2_{M \setminus K} \leq c_0^2 \|s\|^2 + b\|s\|^2_K.
\]

We conclude that

\[
(c_0 - c^2)\|s\|^2 \leq (c_0 + b)\|s\|^2_K.
\]

Now let \( Q := (i + L/2c)^{-1} \) and \( S := \chi(K)(1 - QL/2c) \), so that

\[
\|Ss\| + 1/2\|\chi(K)Q\| \|s\| \geq \|s\|_K \geq \frac{c_0 - c^2}{c_0 + b} \|s\|
\]

and because of the way that \( c_0 \) was chosen, there is a \( c' > 0 \) such that \( \|Ss\| \geq c' \|s\| \)
for all \( s \) in \( H_c \), implying that \( S : H_c \to H \) is injective and has closed range. By the Rellich lemma, the operator \( S^* = -iQ^* \chi(K) \) is compact, so \( S \) is of finite rank, and \( H_c \) is finite-dimensional.

Finally, we apply the well-known Glazman variational lemma [27, p. 233].

**Proposition 3.4 (Glazman).** Let \( A \) be a self-adjoint Hilbert space operator that is semibounded from below. Let \( N_h(\lambda) \) denote the number of eigenvalues in \((-\infty, \lambda]\), with multiplicity, and counting points of the continuous spectrum as points with infinite multiplicity. Then

\[
N_h(\lambda) = \sup_{H \in \mathcal{H}} \dim H,
\]

where the supremum is taken over all subspaces \( H \) which are such that \( \langle Ah, h \rangle \leq \lambda \langle h, h \rangle \) for all \( h \in H \).

Setting \( A = L^2 \) and using Lemma 3.3, we find that \( L^2 \) has only a point spectrum, consisting of isolated points with no finite point of accumulation.

We now know that \((H, 1, L)\) is an unbounded Kasparov cycle, corresponding to the element of \( Z \) given by \( \text{Ind}(1 + L^+L^-)^{-1/2}L^+ \).

### 4. A geometrically defined \( KK^1(C(N), C_0(M)) \) cycle

We have now established all the claims made in section 2 except those involving the cycle \([E] \in KK^1(C(N), C_0(M)) \). We need to show that taking Kasparov products with \([E]\) takes a Dirac operator cycle to a Dirac operator cycle, and takes \([V^+] \) to \([A]\). The existence of the cycle \([E]\) and the fact that it maps a Dirac operator cycle to a Dirac operator cycle is a \( KK \)-theoretical reformulation of one of the basic results of Baum-Douglas-Taylor’s topological \( K \)-homology [9, 19].

**Lemma 4.1.** The cycle \([E] \in KK^1(C(N), C_0(M)) \) is given by

\[
\left( C_0(W) \otimes C_1, \phi, \begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix} \right),
\]

where \( W \cong (-\infty, \infty) \times N \) is the collar of \( M \), \( \phi \) is induced by the obvious map from \( W \) to \( N \), and \( h : (-\infty, \infty) \to \mathbb{R} \) is any continuous function that goes to infinity at
both endpoints. As an element of \( \text{Ext}(C(N), C_0(M)) \), this cycle becomes the exact sequence

\[
0 \to C_0(M) \to C_v(M) \to C(N) \to 0,
\]

where \( C_v(M) \) is the subalgebra of \( C_0(M) \) given by functions having a radial limit at infinity.

Proof. Under the isomorphism of \( KK^1(C(N), C_0(M)) \) and \( \text{Ext}(C(N), C_0(M)) \) given by “cutting down” the representation, the given \( KK^1 \) cycle corresponds to the extension with Busby map \( b : C(N) \to C_0(M)/C_0(M) \) given by \( f \mapsto \hat{f} \mod C_0(M) \), where \( \hat{f} \in C_0(M) \) is any function which has \( f \) as a (uniform) radial limit at infinity. Functions \( \hat{f} \) having this property necessarily form a \( C^* \)-algebra, denoted \( C_v(M) \), and hence the cycle \([E]\) is represented by the short exact sequence

\[
0 \to C_0(M) \to C_v(M) \to C(N) \to 0. \quad \square
\]

As mentioned above, it is known that \([E] \otimes_{C_v(M)} [D_M] = [D_N]\). However, this fact can be obtained from our previous calculations. If we observe that we can take \( h \) to be a potential function, in the sense of Anghel’s theorem, then the proof of Lemma 3.1 shows that the Kasparov product of \((C_0(W) \otimes C_1, \phi \otimes 1, (\begin{smallmatrix} b & 0 \\ 0 & -b \end{smallmatrix})) \in KK(C(N), C_0(M)) \) and \((L^2(M,S) \otimes C_1, \phi_2, (\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix})) \) is

\[
\left( L^2(W,S) \oplus L^2(W,S), \phi \oplus \phi_2, \begin{pmatrix} 0 & D_W + ih \\ D_W - ih & 0 \end{pmatrix} \right) \in KK(C(N), \mathbb{C}).
\]

Now we write \( D_W + ih \) as a sum of longitudinal and transverse parts, \( D_N + (e_r \partial_r + ih) \). The transverse part commutes exactly with \( \phi \), since \( \phi \) is constant in the transverse direction. We can use a homotopy to get rid of the transverse part and to simultaneously replace \( L^2(W,S) \) by the module of radially constant functions \( i^* L^2(N,S) \).

Next we proceed to show that \([A] = [V^+] \otimes_{C(N)} [E]\), where \([A] \in KK^1(\mathbb{C}, C_0(M))\) is \((\Gamma_0(E) \otimes C_1, 1, (\begin{smallmatrix} 0 & b \\ b & -0 \end{smallmatrix})) \) and \([V^+] \in K_0(N)\) is the cycle corresponding to the positive eigenbundle of \( A \) over some leaf \( N \) which is in the component of infinity of the subset \( M \) where \( r \) is invertible.

The quickest way to do this calculation is probably via the isomorphism with \( K \)-theory, interpreting \( \cdot \otimes_{C(N)} E \) as the exponential map \( \delta \) in the 6-term exact sequence:

\[
\begin{array}{c}
K_0(C_0(M)) \longrightarrow K_0(C_v) \longrightarrow K_0(C_0(M)) \\
\delta \downarrow \quad \quad \quad \quad \quad \downarrow \phi
\end{array}
\]

Supposing that \( p \) is a projection onto the given \( n \)-dimensional positive eigenbundle of \( A \), we recall that the \( K \)-theory exponential map takes \([p] - [I_n]\) to \([\exp(2\pi i P)] \in i^* K_1(C_0(M))\), where \( P \) is a self-adjoint lifting of \( p \in M_\infty(C(N)) \) to \( M_\infty(C_v(M)) \). An obvious quotient exact sequence gives another exponential map, \( \delta' : K_0(Q) \to K_1(C_0(M)) \), where \( Q \) denotes the stable outer multiplier algebra \( Q := C_{b,s}(M, \mathcal{L})/C_0(M, \mathcal{K}) \).
Combining these maps with the Paschke-Valette-Skandalis duality map \[29, 28, 19, 26\] from \(K_0(Q)\) to \(\text{Ext}(C, C_0(M))\), we have the diagram

\[
\begin{array}{ccc}
K_1(C_0(M)) & \xrightarrow{\delta} & K_1(C_0(M)) \\
\| & & \downarrow \delta' \\
[p] \in K_0(C(N)) & \xrightarrow{i} & K_0(Q) \\
\| & & \downarrow \text{PV} \\
& & \text{Ext}(C, C_0(M))
\end{array}
\]

where all the maps except \(\delta\) and \(i\) are isomorphisms. Comparing the definition of the two exponential maps \(\delta\) and \(\delta'\), we see that the map \(i\) takes the projection \(p\) to the class of \(P\) in the \(K_0\) group of \(Q\), if \(P\) is regarded as a matrix with coefficients in \(C_b(M)\). Since the Busby map of an extension in \(\text{Ext}(C, C_0(M))\) is given by a projection in \(C_b(M, L)/C_0(M, K)\), the duality map just maps \(P\) to itself, under the above identification. The Busby map of the \(KK^1(C, C_0(M))\) cycle \([A] := (\Gamma_0(E) \otimes C_1, 1, (A^* 0, 0))\) is the image of \(l(A)\) in \(C_b(M, L)/C_0(M, K)\), where \(l : \mathbb{R} \to \mathbb{R}\) is any bounded function with limit 1 at infinity to the right and limit 0 at infinity to the left. But if we restrict \(b(A)\) to a leaf \(N_r\), then the condition on the endomorphism \(A\) implies that \(b(A)\) approaches a projection in the class of \(p\) as \(r\) approaches infinity.

This completes the proof of the theorem.

Finally, let us point out that the standard differential geometry construction of a tubular neighbourhood gives a large supply of manifolds with a collar at infinity. Hence one has theorems such as the following known result:

**Corollary 4.2 ([1] [21]).** Let \(D + iA\) be a Dirac-Schrödinger operator which is bounded strictly away from zero outside a compact set \(K \subset M\). Then the index of \(D + iA\) is given by

\[
\int_N \hat{A}(TN) \wedge \text{ch} V^+ d(\text{vol}_N),
\]

where \(V^+\) is the positive eigenbundle of \(A\) on \(N\), the boundary of any compact set containing \(K\).

In terms of \(KK\)-theory, this theorem reflects the fact that Fredholm cycles are stable under coning operations, as shown in the work of J. Cheeger [15]. We only outline a proof.

**Proof.** Choose a tubular neighbourhood of \(N\), obtaining a warped product. Blow up the metric at \(N\), so that the commutator \([D, A]\) goes to zero at \(N\), and rescale the potential (outside a compact set) by \(1/||[D, A]||\). This puts us in the situation of the previous theorem.

**References**

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