POLYNOMIAL GROWTH SOLUTIONS
OF UNIFORMLY ELLIPTIC OPERATORS
OF NON-DIVERGENCE FORM

PETER LI AND JIAPING WANG

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Abstract. We give an explicit description of polynomial growth solutions to
a uniformly elliptic operator of non-divergence form with periodic coefficients
on the Euclidean spaces. We also show that the solutions are of one-to-one
correspondence to harmonic polynomials if the coefficients of the operator are
continuous.

§0. Introduction

In this article, we will study polynomial growth solutions to a uniformly elliptic
operator of non-divergence form defined on $\mathbb{R}^n$. In terms of rectangular coordinates
$\{x_1, \ldots, x_n\}$ of $\mathbb{R}^n$, let

\begin{equation}
L = a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}
\end{equation}

be second order differential operator. We assume that it is symmetric and uniformly
elliptic with the coefficients satisfying

\begin{equation}
a_{ij}(x) = a_{ji}(x)
\end{equation}

and

\begin{equation}
\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \mu |\xi|^2
\end{equation}

for some positive constants $0 < \lambda \leq \mu < \infty$. Moreover, we assume that the functions
$a_{ij}$ are measurable and periodic with respect to the lattice $\mathbb{Z}^n$ satisfying

\begin{equation}
a_{ij}(x + z) = a_{ij}(x)
\end{equation}

for all $x \in \mathbb{R}^n$ and $z \in \mathbb{Z}^n$. We say that a function $u \in W^{2,n}_{\text{loc}}(\mathbb{R}^n)$ is $L$-harmonic if
it is a weak solution of the equation

\begin{equation}
Lu = 0.
\end{equation}
Let $r(x)$ be the Euclidean distance function to the origin on $\mathbb{R}^n$. For any $d \geq 0$, let us define
\[ \mathcal{H}^d(L) = \{ u \in W^{2,n}_{\text{loc}}(\mathbb{R}^n) \mid Lu = 0, |u|(x) = O(r^d(x)) \} \]
to be the space of $L$-harmonic functions of polynomial growth of order at most $d$. Our main purpose is to prove the following theorem.

**Theorem.** If $L$ is a $\mathbb{Z}^n$-periodic, uniformly elliptic operator of non-divergence form, then the following properties hold:

(i) $\mathcal{H}^d(L) = \mathcal{H}^d(\Delta)$.

(ii) For any positive integer $N$ and $u \in \mathcal{H}^N(L)$,
\[ u(x) = \sum_{|
u| \leq N} \frac{p_\nu(x)}{\nu!} x^\nu, \]
where $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n_+$, $|
u| = \sum_{i=1}^n \nu_i$, $\nu! = \prod_{i=1}^n \nu_i!$, and $x^\nu = \prod_{i=1}^n x_i^{\nu_i}$. The functions $p_\nu(x)$ are $\mathbb{Z}^n$-periodic and Hölder continuous. Moreover, when $|
u| = N$ the functions $p_\nu(x)$ are constants.

(iii) If $a_{ij}$ are continuous, then the homogeneous polynomial
\[ u^{(N)}(x) = \sum_{|
u| = N} \frac{p_\nu(x)}{\nu!} x^\nu \]
solves
\[ Qu^{(N)} = \sum_{i,j=1}^n q_{ij} \frac{\partial^2 u^{(N)}}{\partial x_i \partial x_j} = 0, \]
where $(q_{ij})$ is a positive definite constant matrix. In particular,
\[ \dim \mathcal{H}^N(L) = \dim \mathcal{H}^N(\Delta). \]

Let us first point out some history relating to this particular problem. In 1989 [ALn], Avenalleda and Lin studied solutions to a uniformly elliptic operator of divergence form on $\mathbb{R}^n$. In their setting, they consider elliptic operators of the form
\[ P = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \]
defined on $\mathbb{R}^n$. They assumed that the coefficients $\{a_{ij}(x)\}$ of the operator are Lipschitz functions, satisfying (0.2), (0.3) and period under the $\mathbb{Z}^n$ action. They proved that the dimension of the space of polynomial growth solutions of degree at most $d$, $\mathcal{H}^d(P)$, must satisfy
\[ \dim \mathcal{H}^d(P) = \dim \mathcal{H}^d(\Delta), \]
where $\Delta$ is the Laplace operator on $\mathbb{R}^n$. Three years later, Moser and Struwe [MS] used a different argument and proved the same theorem without the Lipschitz assumption on the coefficients. Moreover, their argument applied to a certain class of non-linear operators as well.

In 1996, Lin [Ln] considered uniformly elliptic operators of divergence form which are asymptotic to a conic operator at infinity. This result can be viewed as a generalization of the previous results since a periodic operator is asymptotic to an operator with constant coefficients. Recently, Zhang [Z] considered a more general class, namely operators that are asymptotic to a compact 1-parameter family of
conic operators. In this setting, he only proved an upper bound on the dimension of $\mathcal{H}^d(P)$.

The most general theorem on uniformly elliptic operators of divergence form was included in the work [L1] of the first author. He showed that for any uniformly elliptic operator of divergence form with measurable coefficients the dimension of $\mathcal{H}^d(P)$ must satisfy

$$\dim \mathcal{H}^d(P) \leq C d^{n-1}$$

for all $d \geq 1$. In fact, his argument also works for uniformly elliptic operators, $L$, of non-divergence form (0.1) with measurable coefficients satisfying (0.2) and (0.3). In that case, the upper bound

$$\dim \mathcal{H}^d(L) \leq C d^{n-1}$$

is obtained.

In a recent paper [LW], the authors found a sharp asymptotic upper bound for the dimension of $\mathcal{H}^d(P)$ as $d \to \infty$, for uniformly elliptic operators of divergence form with measurable coefficients. They showed that

$$\sum_{i=1}^d \dim \mathcal{H}_i(P) \leq \left( \frac{\mu_\infty}{\lambda_\infty} \right)^{n-1} \frac{2}{n!} (d + 2n)^n$$

where $\lambda_\infty$ and $\mu_\infty$ are the ellipticity constants at infinity. This estimate is sharp in the sense that for the Laplace operator, we have

$$\lim_{d \to \infty} d^{-n} \sum_{i=1}^d \dim \mathcal{H}_i(\Delta) = \frac{2}{n!}.$$

An obvious question is to ask if the above theories apply to uniformly elliptic operators, $L$, of non-divergence form. Other than the result in [L1], there are no known results for this class of operators. The first step in our attempt to obtain some form of sharp estimates on $\dim \mathcal{H}^d(L)$ is to see if similar results such as [ALn] and [MS] are valid. Our program follows pretty much that of Moser-Struwe with the exception that different treatments are used in various crucial steps to deal with the non-divergence case.

We would like to remark that the continuity assumption of the coefficients is only required in discussing the solvability of the equation

$$Lu = f.$$
§1. Proof of the Theorem

We will now give a proof of the Theorem.

Proof. Denote $E_i$ to be the translation operator defined on functions given by

$$(E_i u)(x) = u(x + e_i) - u(x),$$

where $e_i = (0, \ldots, 1, \ldots, 0)$ is the $i$-th Cartesian coordinate vector. We also denote $E^\nu = \prod_{i=1}^n E_i^{\nu_i}$ for $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}_+^n$.

As a general fact, we will first show that

$$u(x) = \sum_{|\nu| < N} p_\nu(x) x^\nu$$

if $E^\nu u = 0$ for all $\nu$ with $|\nu| = N$, where each $p_\nu(x)$ is periodic. We will prove this by induction on $N$. For $N = 1$, if $E_i u = 0$ for all $i = 1, \ldots, n$, then $u$ is $\mathbb{Z}^n$-periodic and the result is trivially true. Suppose now that this is true for $N = k$ and we would like to prove it for $N = k + 1$. Note that for any $\nu \in \mathbb{Z}_+^n$ with $|\nu| = k$, the assumption asserts that $E_i (E^\nu u) = 0$ for all $i = 1, \ldots, n$. This implies that $E^\nu u = p_\nu(x)$ for some periodic function $p_\nu$. Now for a fixed $\nu_0 \in \mathbb{Z}_+^n$ with $|\nu_0| = k$, one checks directly that

$$E^{\nu_0} u(x) - \sum_{|\nu| = k} \frac{p_\nu(x)}{\nu!} x^\nu = E^{\nu_0} u - p_{\nu_0}(x) = 0.$$

Hence the induction hypothesis implies that

$$u(x) - \sum_{|\nu| = k} \frac{p_\nu(x)}{\nu!} x^\nu = \sum_{|\nu| < k} \frac{p_\nu(x)}{\nu!} x^\nu.$$

Rewrite this as

$$u(x) = \sum_{|\nu| \leq k} \frac{p_\nu(x)}{\nu!} x^\nu;$$

the induction argument is complete.

To continue with the proof of the theorem, let us denote $\omega_{B_{1}(R)}(f)$ to be the oscillation of a function over the ball $B_{1}(R)$. The Harnack inequality of Krylov-Safanov [KS1], [KS2] implies that there exists a positive constant $C$ depending only on $\lambda$, $\mu$ and $n$, such that, if $f \in W^{2,n}_{loc}(\mathbb{R}^n)$ is an $L$-harmonic function, then

$$\omega_{B_{1}(R)}(f) \leq C \omega_{B_{2}(2R)}(f) \tag{1.1}$$

for all $x \in \mathbb{R}^n$ and $R > 0$. We claim that for any $u \in \mathcal{H}^d(L)$, $E_i u \in \mathcal{H}^{d-\delta}(L)$ for all $i = 1, \ldots, n$, where $\delta = \log_2 \frac{1}{\tau} > 0$. In fact, it is clear that if we define

$$v(x) = E_i u(x) = u(x + e_i) - u(x),$$

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then $Lv = 0$ and $v \in W^{2,n}_{\text{loc}}(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$, let $k$ be the minimum positive integer such that $r(x) \leq 2^k$. Using (1.1), we get

$$
|v|(x) \leq \omega_{B_{r}(1)}(u)
\leq C^k \omega_{B_{r}(2^k)}(u)
\leq C^k A (r(x) + 2^k)^d
\leq B r^{d-\delta}(x).
$$

This implies that $v \in \mathcal{H}^{d-\delta}(L)$ and the claim follows.

A standard argument using the Harnack inequality also implies that there exists $\alpha > 0$ such that $\mathcal{H}^{\alpha}(L)$ only consists of constant functions. For a fixed $d > 0$, there exists a positive integer $m$ such that $d - m\delta \leq \alpha$. Therefore if $u \in \mathcal{H}^d(L)$ and $\nu \in \mathbb{Z}^n_+$ with $|\nu| = m$, then $E^{\nu}u \in \mathcal{H}^{d-\delta}(L)$ must be a constant. In particular, for $\nu \in \mathbb{Z}^n_+$ with $|\nu| = m + 1$, $E^{\nu}u = 0$. Hence we conclude that

$$
(1.2) \quad u(x) = \sum_{|\nu| \leq m} \frac{p_\nu(x)}{\nu!} x^\nu,
$$

where the functions $p_\nu(x)$ are periodic. However, the growth order of $u$ is of at most $d$. The non-zero terms in (1.2) must satisfy $|\nu| \leq [d]$. Thus, $\mathcal{H}^{d}(L) = \mathcal{H}^{[d]}(L)$ and (i) is proved.

Using (1.2), we conclude that if $u \in \mathcal{H}^N(L)$ with $N$ being a positive integer, then

$$
(1.3) \quad u(x) = \sum_{|\nu| \leq N} \frac{p_\nu(x)}{\nu!} x^\nu.
$$

Direct computation then shows that $E^{\nu}u = p_\nu(x) \in \mathcal{H}^0(L)$ for $|\nu| = N$. Due to the fact that the space $\mathcal{H}^0(L)$ consists of constant functions only, we conclude that $p_\nu$ must be constant for $|\nu| = N$. The Hölder regularity of $L$-harmonic functions implies that all the $p_\nu(x)$ are Hölder continuous.

To show (iii), we first remark that we may view $L$ as an operator on the torus $T^n$. The standard $L^p$ estimate implies that the image of $H^2(T^n)$ under $L$ is a closed subspace of $L^2(T^n)$. Note that by the maximum principle, positive functions are not in the image space. This implies that the kernel of the dual operator $L^* : L^2(T^n) \rightarrow H^2(T^n)$ is non-trivial. Let $f \in L^2(T^n)$ be a non-zero function in the kernel of $L^*$. Without loss of generality, we may assume that

$$
\int_{T^n} f \geq 0.
$$

We now claim that $f$ must be non-negative on $T^n$. In fact, for any non-negative smooth function $h$ on $T^n$, let $w(x,t)$ be the solution of

$$
\left( L - \frac{\partial}{\partial t} \right) w = 0
$$

with

$$
w(x,0) = h(x).
$$

Note that by the $L^p$ estimate, $w(x,t) \in H^2(T^n)$ for all $t > 0$ and

$$
\lim_{t \to \infty} w(x,t) = c
$$

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for some constant $c \geq 0$. Since $L^*(f) = 0$ and $w(x, t) \in H^2(T^n)$ for $t > 0$, 
\[
\int_{T^n} f(x) Lw(x, t) \, dx = 0
\]
for all $t > 0$. Thus,
\[
\frac{d}{dt} \int_{T^n} f(x) w(x, t) \, dx = 0
\]
and
\[
\int_{T^n} f(x) h(x) \, dx = \lim_{t \to \infty} \int_{T^n} f(x) w(x, t) \, dx
\]
\[
= c \int_{T^n} f(x) \, dx \geq 0.
\]
Since $h$ is arbitrary, we conclude that $f(x) \geq 0$ on $T^n$. Let us now normalize $f$ such that
\[
\int_{T^n} f(x) dx = 1
\]
and define
\[
q_{ij} = \int_{T^n} a_{ij}(x) f(x) dx.
\]
Since $f \geq 0$ and is not identically zero, $q_{ij}$ is positive definite. We will now verify that $Qu^{(N)} = 0$ with this choice of $q_{ij}$. This is obvious for $N = 1$ as $u^{(1)}(x) = c_i x_i$ for some constants $c_i$. For $N = 2$ and $u \in \mathcal{H}^2(L)$, (ii) implies that there exists a periodic function $p$ such that
\[
u(x) = c_{ij} x_i x_j + c_i x_i + p(x).
\]
Hence $u^{(2)}(x) = c_{ij} x_i x_j$, where $c_{ij}$ are constants. Therefore, we have
\[
Q(u^{(2)}) = q_{ij} c_{ij}
\]
\[
= \int_{T^n} f(x) a_{ij}(x) c_{ij} \, dx
\]
\[
= \int_{T^n} f(x) L(u^{(2)})(x) \, dx
\]
\[
= - \int_{T^n} f(x) L(p)(x) \, dx
\]
\[
= - \int_{T^n} L^*(f)(x) p(x) \, dx
\]
\[
= 0.
\]
We will now proceed by induction for $N \geq 3$. Suppose for all $u \in \mathcal{H}^{(N-1)}(L)$ we have $Qu^{(N-1)} = 0$. Given $u \in \mathcal{H}^N(L)$, we know that $E_i u \in \mathcal{H}^{(N-1)}(L)$ for each $i$. Note that
\[
\frac{\partial u^{(N)}}{\partial x_i} = (E_i u)^{(N-1)},
\]
so by the induction hypothesis, we have
\[ \frac{\partial (Q u^{(N)})}{\partial x_i} = Q \left( \frac{\partial u^{(N)}}{\partial x_i} \right) = 0. \]
This implies that the function \( Q u^{(N)} \) is constant. On the other hand, \( Q u^{(N)} \) is a homogeneous polynomial of degree \( N - 2 > 0 \), hence it is identically zero. This implies the first part of (iii).

To prove the second part of (iii), note that the correspondence of \( u \mapsto u^{(N)} \) gives a one-to-one map from \( H^N(L)/H^{(N-1)}(L) \) to \( H^N(Q)H^{(N-1)}(Q) \). We will now show that this map is onto, that is, for any \( h(x) = \sum_{|\nu|=N} c_\nu x^\nu \in H^N(Q) \), there exists \( u \in H^N(L) \) such that \( u^{(N)} = h \). Let us write
\[ u(x) = \sum_{|\nu|\leq N} p_\nu(x) x^\nu, \]
where \( p_\nu(x) = c_\nu \) for \( |\nu| = N \). For \( \epsilon > 0 \), we have
\[ \epsilon^N u \left( \frac{x}{\epsilon} \right) = \sum_{|\nu|\leq N} \epsilon^{N-|\nu|} x^\nu p_\nu \left( \frac{x}{\epsilon} \right) = U(x, y, \epsilon), \]
after substituting the new variable \( y = \frac{x}{\epsilon} \). Direct calculation shows that \( Lu = 0 \) if and only if
\[ (L_0 + \epsilon L_1 + \epsilon^2 L_2) U(x, y, \epsilon) = 0, \]
where
\[ L_0 = a_{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j}, \]
\[ L_1 = a_{ij}(y) \left( \frac{\partial^2}{\partial x_i \partial y_j} + \frac{\partial^2}{\partial y_i \partial x_j} \right) \]
and
\[ L_2 = a_{ij}(y) \frac{\partial^2}{\partial x_i \partial x_j}. \]
We now claim that there exists a sequence of functions \( \psi_\nu(y) \) such that \( \psi_0(y) = 1 \) and
\[ (L_0 + \epsilon L_1 + \epsilon^2 L_2) \sum_{|\nu|=0}^{\infty} \epsilon^{N-|\nu|} \psi_\nu(y) \frac{\partial^{|\nu|}}{\partial x^{\nu}} \]
(1.4)
\[ = \epsilon^2 \sum_{|\nu|=2} \alpha_\nu \frac{\partial^{|\nu|}}{\partial x^{\nu}} + \epsilon^3 \sum_{|\nu|=3} \alpha_\nu \frac{\partial^{|\nu|}}{\partial x^{\nu}} + \ldots, \]
where \( \alpha_\nu \) are constants and \( \alpha_{ij} = q_{ij} \). In fact, comparing the coefficients of \( \epsilon^s \) and the corresponding terms of \( \frac{\partial^{|\nu|}}{\partial x^{\nu}} \) of (1.4), we get
\[ L_0 \psi_0(y) = 0 \quad \text{for } s = 0 \]
and
\[ L_0 \psi_\nu(y) = 0 \quad \text{for } |\nu| = s = 1. \]
Let us set \( \psi_{\nu}(y) = 0 \) for all \( \nu \) with \( |\nu| = 1 \). For \( s = 2 \), we have
\[
(1.5) \quad L_0 \psi_{ij}(y) + a_{ij}(y) = \alpha_{ij}
\]
for all \( i, j = 1, \ldots, n \). Setting \( \alpha_{ij} = q_{ij} \) and using the normalization on \( f \), we check that
\[
\int_{T^n} f(y) \left( \alpha_{ij} - a_{ij}(y) \right) dy = 0.
\]
Hence there exist solutions \( \psi_{ij}(y) \) to (1.5). We will now use an induction argument on \( s \geq 3 \). Let us assume that \( \psi_{\nu}(y) \) exists for all \( \nu \) with \( |\nu| \leq s - 1 \). Using (1.5), we know that for each \( \nu \) with \( |\nu| = s \),
\[
(1.6) \quad L_0 \psi_{\nu}(y) = F(\psi_{\nu_1}, \psi_{\nu_2})(y) + \alpha_{\nu},
\]
where \( |\nu_1| = s - 1 \) and \( |\nu_2| = s - 2 \). So we may choose \( \alpha_{\nu} \) such that the right-hand side of (1.6) is perpendicular to \( f(y) \) in \( L^2(T^n) \). Thus, solutions \( \psi_{\nu} \) exist for (1.6) and the claim is proved.

Note that the operator \( Q \) maps the set of homogeneous polynomials of degree \( s \) onto the set of homogeneous polynomials of degree \( s - 2 \). There exists a linear operator \( R \) mapping polynomials to polynomials such that \( QR = I \). Let us define formally
\[
A = I + \epsilon R \sum_{|\nu| = 3} \alpha_{\nu} \frac{\partial^{|
u|}}{\partial x^\nu} + \epsilon^2 R \sum_{|\nu| = 4} \alpha_{\nu} \frac{\partial^{|
u|}}{\partial x^\nu} + \ldots.
\]
We form the unique formal inverse
\[
A^{-1} = I - \epsilon R \sum_{|\nu| = 3} \alpha_{\nu} \frac{\partial^{|
u|}}{\partial x^\nu} + \ldots.
\]
Finally, let \( V(x) = A^{-1} h(x) \) and
\[
u(x) = \sum_{\nu} \epsilon^{|
u|} \psi_{\nu}(y) \frac{\partial^{|
u|}}{\partial x^\nu} V(x).
\]
Note that all the infinite series appearing here and in the following only have finitely many non-zero terms. The convergence is trivially true. Then by the claim (1.4),
\[
Lu(x) = (L_0 + \epsilon L_1 + \epsilon^2 L_2) \sum_{|\nu|=0}^\infty \epsilon^{|
u|} \psi_{\nu}(y) \frac{\partial^{|
u|}}{\partial x^\nu} V(x)
\]
\[
= \left( \epsilon^2 \sum_{|\nu|=2} \alpha_{\nu} \frac{\partial^{|
u|}}{\partial x^\nu} + \epsilon^3 \sum_{|\nu|=3} \alpha_{\nu} \frac{\partial^{|
u|}}{\partial x^\nu} + \ldots \right) V(x)
\]
\[
= \epsilon^2 \sum_{|\nu|=2} \alpha_{\nu} \frac{\partial^{|
u|}}{\partial x^\nu} A(A^{-1} h(x))
\]
\[
= \epsilon^2 Q h(x)
\]
\[
= 0.
\]
Since \( \psi_0(y) = 1 \), one easily verifies that \( u^{(N)}(x) = h(x) \). Thus the solution \( u(x) \) exists and the proof is complete.
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, IRVINE, CALIFORNIA 92697-3875
E-mail address: pli@math.uci.edu

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455
E-mail address: jiaping@math.umn.edu