A SHORT PROOF OF ERGODICITY OF BABILLOT-LEDRAPPIER MEASURES

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Abstract. Let $M$ be a compact manifold, and let $\phi_t$ be a transitive homologically full Anosov flow on $M$. Let $\hat{M}$ be a $\mathbb{Z}^d$-cover for $M$, and let $\hat{\phi}_t$ be the lift of $\phi_t$ to $\hat{M}$. Babillot and Ledrappier exhibited a family of measures on $\hat{M}$, which are invariant and ergodic with respect to the strong stable foliation of $\hat{\phi}_t$. We provide a new short proof of ergodicity.

Let $M$ be the unit tangent bundle of a compact Riemann surface, and let $\phi_t$ be the geodesic flow on $M$. Let $\hat{M}$ be a $\mathbb{Z}^d$-cover of $M$, and let $\hat{\phi}_t$ be the lift of $\phi_t$ to $\hat{M}$. Babillot and Ledrappier [BL96] constructed a family of measures on $\hat{M}$, which are preserved by the horocycle flow on $\hat{M}$, and proved that they are ergodic. Kaimanovich [K98] gave an alternative proof of ergodicity in case of Liouville measure ($=\mu_0$).

The construction of Babillot and Ledrappier generalizes to a $\mathbb{Z}^d$-extension of an arbitrary Anosov flow, where it gives a family of measures $\{\mu_v\}_{v \in \mathbb{R}^d}$ invariant w.r.t. the strong stable foliation for $\hat{\phi}_t$. The lift of the measure of maximal entropy coincides with $\mu_0$. In this setting Pollicott [P98] proved ergodicity of $\mu_0$ by a different method, and Coudene [C99] obtained a generalization to all $\{\mu_v\}_{v \in \mathbb{R}^d}$. We give a new proof of ergodicity of $\mu_v$ for all $v \in \mathbb{R}^d$ using a theorem of Guivarc’h [GS90].

The flow $\phi_t : M \to M$ is said to be homologically full if there exists a closed orbit of $\phi_t$ in each homology class; see [S93].

Theorem (Babillot and Ledrappier [BL96], Pollicott [P98], Coudene [C99]). Let $(\hat{M}, \hat{\phi}_t)$ be a $\mathbb{Z}^d$-extension of a transitive Anosov flow, and let $\{\mu_v\}_{v \in \mathbb{R}^d}$ be the family of Babillot-Ledrappier measures on $\hat{M}$. If $\phi_t$ is homologically full, then for any $v \in \mathbb{R}^d$ the strong stable foliation of $\hat{\phi}_t$ is ergodic w.r.t. $\mu_v$.

Proof. Using a method of [BM77], the problem can be reformulated in terms of symbolic dynamics; see [P98]. Let $(\Sigma, T)$ be a topologically mixing one-sided shift of finite type (see e.g. [B75] for the definitions). Let $r : \Sigma \to \mathbb{R}^+$ and $\psi : \Sigma \to \mathbb{Z}^d$ be Hölder continuous functions, and assume that $\psi$ depends only on the first two coordinates, i.e. $\psi(x) = \psi(x_1, x_2)$. Consider the set

$$\Lambda = \{(x, t, i) : 0 \leq t \leq r(x)\} \subset \Sigma \times \mathbb{R}^+ \times \mathbb{Z}^d$$

with identifications $(x, r(x), j) = (Tx, 0, j + \psi(x))$.

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Define an equivalence relation \( R_\Lambda \subset \Lambda \times \Lambda \) by setting \((x,t,i) \sim (y,s,j)\) if there exist \(m,n \geq 0\) such that

\[
\begin{align*}
T^m(x) &= T^n(y), \\
r_m(x) - t &= r_n(y) - s, \\
i + \psi_m(x) &= j + \psi_n(y),
\end{align*}
\]

where \(r_n\) and \(\psi_n\) denote the corresponding ergodic sums: \(r_n(x) = r(x) + r(Tx) + \ldots + r(T^{n-1}x)\) and \(\psi_n(x) = \psi(x) + \psi(Tx) + \ldots + \psi(T^{n-1}x)\).

For any \(v \in \mathbb{R}^d\), consider the potential \(\phi_v : \Sigma \to \mathbb{R}\), given by \(\phi_v = \lambda_v r(x) - \langle \psi(x) | v \rangle\), where \(\langle \cdot | \cdot \rangle\) stands for the scalar product and \(\lambda_v\) is the unique real number such that topological pressure \(P(\phi_v) = 0\). Let \(\nu_v\) be the eigenmeasure for the corresponding Perron-Frobenius-Ruelle operator (the reader is referred to e.g. [B75] for the definitions). Set

\[
\mu_v = \nu_v \times e^{-\lambda_v t} dt \times e^{\langle z | v \rangle} dz.
\]

Ergodicity of the strong stable foliation w.r.t. Babillot-Ledrappier measures amounts to ergodicity of \(R_\Lambda\) w.r.t. \(\mu_v\); see [BL96].

It is convenient to consider the space \(Y = \Sigma \times \mathbb{R} \times \mathbb{Z}^d\) rather than \(\Lambda\), where the equivalence relation \(R_Y\) and measures \(\mu_v\) are defined by (1)-(3) and (4), respectively. It suffices to prove ergodicity of \(R_Y\) w.r.t. \(\mu_v\); or, equivalently, w.r.t. \(\nu_v \times dt \times dz\), where \(dz\) is Haar measure on \(\mathbb{Z}^d\).

Recall that a Hölder continuous function \(f : \Sigma \to G\), where \(G\) is a locally compact abelian polish group, is called periodic (see [GS99]) if there exist a nonconstant measurable \(g : \Sigma \to S^1\), a constant \(z \in S^1\) and a character \(\gamma \in \hat{G}\) such that \(\gamma \circ f(x) = zg(Tx)\overline{g}(x)\). If \(f\) is not periodic, it is called aperiodic. We set \(G = \mathbb{R} \times \mathbb{Z}^d\) and \(f(x) = (-r(x), \psi(x))\).

**Lemma.** If \(\phi_t\) is homologically full, then \(f\) is aperiodic.

(The proof is given below.)

Consider the skew-product transformation \(T_f : \Sigma \times G \to \Sigma \times G\) defined by

\[
T_f(x,y) = (Tx, y + f(x)).
\]

By a theorem of Guivarc’h [GS99] (see also [AD99]), \(T_f\) is exact w.r.t. \(p \times m\), where \(p\) is any Gibbs measure and \(m\) is Haar measure on \(G\). Now, let \(R'\) be the tail equivalence relation for \(T_f\), i.e. \((x,y) \sim (x',y')\) iff for some \(n\)

\[
\begin{align*}
T^n(x) &= T^n(x'), \\
y + f_n(x) &= y' + f_n(x').
\end{align*}
\]

Clearly, ergodicity of \(R'\) amounts to the exactness of \(T_f\), so \(R'\) is ergodic. But \(R'\) is smaller than \(R_Y\), so \(R_Y\) is ergodic.

**Proof of the Lemma.** If \(f\) is periodic, then there exist \(\gamma \in \hat{G}\) and \(z \in S^1\) such that \(\gamma \circ f_n(x) = z^n\) for any periodic \(x \in \Sigma\) with period \(n\). Fix \(\alpha \in \mathbb{Z}^d\) and \(\delta > 0\). By a result of Sharp [S92], the number of periodic \(x\) with \(\psi_n(x) = \alpha\) and \(r_n(x) \in (T - \delta, T]\), where \(n\) is the period of \(x\), is exponential in \(T\). Since the roof function \(r\) is strictly positive, the number of possible periods \(n\) is linear in \(T\). So, one can find \(x, y\) with the same period \(n\), such that

\[
0 < |r_n(x) - r_n(y)| < 2\delta, \quad \psi_n(x) = \psi_n(y) = \alpha.
\]
This shows that $\gamma(\cdot, \alpha)$ is constant in the first coordinate. Similarly, for any $\alpha, \beta \in \mathbb{Z}^d$ one can find $x, y$ of the same period $n$ with $\psi_n(x) = \alpha$, $\psi_n(y) = \beta$, which shows that $\gamma$ is constant. Thus, $f$ is aperiodic.

Remark. The theorem of Guivarc'h [G89] was applied in the same way in [ANSS].

References


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