LARGE SETS OF ZERO ANALYTIC CAPACITY

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Abstract. We prove that certain Cantor sets with non-sigma-finite one-dimensional Hausdorff measure have zero analytic capacity.

1. Introduction

In this paper we consider a Cantor set $K$ similar to the $\frac{1}{4}$-Cantor set of [G70] and [IS]. Fix $p > 2$ and for $n > 0$ define

$$\sigma_n = 4^{-n}a_n = 4^{-n}[\log(n + 1)]^{1/p}.$$ 

Set $K_0 = [0, 1] \times [0, 1]$ and $K_1 = \bigcup_{j=1}^{4^n} K_{1,j}$, where $K_{1,j} \subset K_0$ is a square of sidelength $\sigma_1$ having sides parallel to the axis and containing one of the four corners of $K_0$. Next take $4^2$ squares $K_{2,j}$ of sidelength $\sigma_2$, one in each corner of each square $K_{1,j}$, and define $K_2 = \bigcup_{j=1}^{4^2} K_{2,j}$. Continuing we obtain $K_n = \bigcup_{j=1}^{4^n} K_{n,j}$, where $K_{n,j}$ is a square of sidelength $\sigma_n$. The Cantor set we study is

$$K = K(p) = \bigcap_{n=1}^{\infty} K_n.$$ 

If $E$ is a compact plane set define

$$A(E, 1) = \{ f : f \text{ analytic on } E^c, f(\infty) = 0, \| f \|_{L^\infty(E^c)} \leq 1 \}$$ 

and define the analytic capacity of $E$ by

$$\gamma(E) = \sup \{ | f'(\infty) | : f \in A(E, 1) \},$$ 

where

$$f'(\infty) = \lim_{z \to \infty} zf(z).$$ 

If $\gamma(E) = 0$, then the only function in $A(E, 1)$ is the constant $f \equiv 0$ and in this case $E$ is removable for bounded analytic functions. For more details about analytic capacity see [G72].

Theorem 1. Let $p > 2$, and let $K$ be the four-corner Cantor set $K(p)$. Then $\gamma(K) = 0$ but $K$ does not have $\sigma$-finite one-dimensional measure.

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The proof of Theorem 1 depends on a lemma of Jones [J89] used for a proof different from [G70] that the \( \frac{1}{4} \)-Cantor set has zero analytic capacity.

Let \( h(t) \) be an increasing continuous function on \( t \geq 0 \) with \( h(0) = 0 \), and write \( \Lambda_{h(t)}(E) \) for the Hausdorff \( h \)-measure of \( E \). Now define an increasing function \( h(t) \) so that \( h(0) = 0 \) and \( h(\sigma_{n}) = 4^{-n} \) for all \( n \). We say \( h(t) \) is a **measure function corresponding to the Cantor set** \( K \). For every \( n \) define a measure \( \mu_{n} \) on \( K_{n} \) by \( \mu_{n}(K_{n,j}) = 4^{-n} \) for all \( j \). Then \( \{\mu_{n}\} \) converges weak-star to a measure \( \mu \) supported on \( K \) and satisfying \( \Lambda_{h}(K_{n}) = 4^{-n} \). Suppose \( \frac{\sigma_{n}}{4} \leq r < \frac{\sigma_{n-1}}{4} \) and let \( D(z, r) \) be a disk of radius \( r \) and center \( z \in K \). Then \( D(z, r) \) can meet at most \( 4 \) squares of sidelength \( \sigma_{n} \). Hence

\[
\mu(D(z, r)) \leq 4\mu(K_{n,j}) = 4 \cdot 4^{-n} = 4h(\sigma_{n}) \leq 4h(r),
\]

so that \( \mu(D(z, r)) \leq 16h(r) \) for any disk \( D(z, r) \). Therefore \( \Lambda_{h}(K) > 0 \) by Frostman’s Theorem [G72]. Since

\[
\lim_{t \to 0} \frac{h(t)}{t} = 0,
\]

it follows that \( K \) has non-\( \sigma \)-finite 1-dimensional measure.

If \( h(t) \) is a measure function corresponding to \( K \), then

\[
\int_{0}^{1} \frac{h(t)^2}{t^3} dt \sim \sum_{n=1}^{\infty} \frac{1}{(a_{n})^2} \quad \sum_{n=1}^{\infty} \frac{1}{(\log n)^p} = \infty.
\]

On the other hand, Mattila [M96] proved that \( \gamma(K) > 0 \) if \( K \) is a Cantor set built with squares of side \( \sigma_{n} \) and if

\[
\int_{0}^{1} \frac{h(t)^2}{t^3} dt < \infty,
\]

where \( h \) is any measure function for corresponding to \( K \). Mattila’s proof used Menger curvature (see [Me94] and [MMV94]). However, if the Cantor set \( K \) has corresponding measure function \( h \) satisfying

\[
\int_{0}^{1} \frac{h(t)^2}{t^3} dt = \infty,
\]

then Eiderman [E98] proved that \( \gamma^{+}(K) = 0 \), where

\[
\gamma^{+}(E) = \sup \left\{ \int_{E} d\mu : \left| \int_{E} \frac{d\mu(z)}{z - \zeta} \right| < 1, \forall \zeta \in \mathbb{C} \setminus E, \mu > 0, \text{spt}(\mu) \subset E \right\}.
\]

Since \( \gamma^{+}(E) \leq \gamma(E) \), our result is a partial improvement of Eiderman’s theorem. Mattila [M96] has conjectured that for Cantor sets of this type \( \gamma(K) = 0 \) if and only if

\[
\int_{0}^{1} \frac{h(t)^2}{t^3} dt = \infty,
\]

when \( h \) corresponds to \( K \). This latter condition holds if and only if

\[
\sum_{n=1}^{\infty} \frac{1}{(a_{n})^2} = \infty.
\]

If Mattila’s conjecture is true, then together with Eiderman’s theorem it gives Cantor set evidence supporting the more ambitious conjecture that \( \gamma(E) > 0 \) implies \( \gamma^{+}(E) > 0 \).
In [G72] it was incorrectly claimed that $\gamma(K) > 0$ if and only if
$$\int_0^1 \frac{h(t)}{t^2} dt < \infty.$$ 

Eiderman, however, found a mistake in the proof. In fact the result in [M96] shows that the claim is false. See L.D. Ivanov [I84] for the first example of a Cantor set of non-$\sigma$-finite linear measure and zero analytic capacity.

2. Two lemmas of Peter Jones

We need the following two lemmas from [J89]. The proofs we give are small variations on [J89] and [C90].

Define $n_j = \partial cK_n;j$, where $cK_n;j$ is the square concentric to $K_n;j$ with sidelength $c\sigma_n$ and where $c > 1$ is chosen so that the $\gamma_{n;j}$ do not overlap. We refer to $\gamma_{n;j}$ as a square, although it is only the boundary of a square. Notice that
$$\lambda(\gamma_{n;j}) = c\lambda(\partial K_{m;k})$$
for the same constant $c$. We associate to each $\gamma_{n;j}$ a “square annulus”
$$A_{n;j} = \{ w : \text{dist}(w, \gamma_{n;j}) \leq \varepsilon_0 \sigma_m \}$$
and we choose $\varepsilon_0 > 0$ so small that the annuli $A_{n;j}$ are pairwise disjoint.

Define $\Omega = \mathbb{C}\setminus K$. Since $K$ has positive logarithmic capacity, Green’s function $G(z, \zeta)$ exists for $\zeta, z \notin K$, and harmonic measure $\omega(\zeta, E)$ exists for $\zeta \notin K$ and $E \subset K$. We write $\omega(\zeta, K_{m;k})$ for $\omega(\zeta, K_{m;k} \cap K)$.

We also define the slightly larger “squares”
$$S_{m,k} = \{ w : \text{dist}(w, K_{m;k}) \leq \varepsilon_1 \sigma_m \}$$
and set
$$S_m = \bigcup_{k=1}^{4^m} S_{m,k},$$
where $\varepsilon_1 > 0$ is so small that $S_{m,k} \cap A_{n;j} = \emptyset$. Then $K = \bigcap_{m=1}^{\infty} S_m$. Green’s function and harmonic measure also exist for the domain $\Omega_m = \mathbb{C}\setminus S_m$. Denote these by $G_m(z, \zeta)$ and $\omega_m(\zeta, w)$ respectively.

Lemma 2. Let $z \in A_{k,m}^n$.

(a) There are constants $0 < c_1 < c_2 < 1$, independent of $k$ and $m$, such that
$$c_1 \leq \omega_m(z, \partial S_{m,k}) \leq c_2.$$ 

(b) If $\zeta \in \Omega$ and $1 \geq \text{dist}(\zeta, K) \geq 2 \text{dist}(z, K)$, then
$$G_m(z, \zeta) \sim \omega_m(\zeta, \partial S_{m,k}).$$

Proof. For (a) note that there is $c' > 0$ such that there exists a second square annulus $B_{k,m}^n$ so that $A_{k,m}^n \subset B_{k,m}^n \subset \Omega_m$ and $\text{dist}(z, \partial B_{k,m}^n) \geq c' \sigma_m$. The lower bound then follows by a comparison with $B_{k,m}^n$. There is $S_{m,j}$ with $j \neq k$ such that $\text{dist}(S_{m,j}, S_{m,k}) \leq c_4 \sigma_m$ and the upper bound follows by a comparison with $\mathbb{C}\setminus(S_{m,k} \cup S_{m,j})$, using symmetry and Harnack’s inequality.

To prove (b) note first that as in the proof of (a) there are constants $C_1$ and $C_2$ such that by Harnack’s inequality and a comparison
$$C_1 \leq G_m(z, w) \leq C_2.$$
for $w \in \partial B^m_k$. Then using the symmetry of Green’s function and (a) for a larger square we obtain

$$C_1 \omega_m(\zeta, \partial S_{m,k}) \leq G_m(\zeta, z) \leq C_2 \omega_m(\zeta, \partial S_{m,k}).$$

We write $\gamma^m_k \prec \gamma^n_j$ and say $\gamma^m_k$ is subordinate to $\gamma^n_j$ if $\gamma^n_j$ has winding number one about $\gamma^m_k$. If the winding number is zero, we write $\gamma^m_k \not\prec \gamma^n_j$. For any $f \in A(K, 1)$ and $\gamma^m_k$ define

$$D(\gamma^m_k) = \sup_{w \in \gamma^m_k} |f'(w)| \sigma_n.$$ 

We say a square $\gamma^m_k$ has condition J if

$$D(\gamma^m_k) \leq \delta$$

for some previously defined $f$ and $\delta > 0$.

**Lemma 3.** Let $f \in A(K, 1)$. For every $\delta > 0$ there exists a $C_0 > 0$ such that for every $\gamma^n_j$ there exists $\gamma^m_k \prec \gamma^n_j$ such that $m \leq n + C_0 \delta^{-2}$ and such that $\gamma^m_k$ has condition J.

**Proof.** Observe that by Harnack’s inequality

$$\sup_{\gamma_{n,j}} |f'(z)|^2 \sim \int \int_{A^n_j} |f'(z)|^2 \frac{dxdy}{\sigma_n^2}.$$ 

Suppose the lemma is false. Choose $\zeta$ with $\text{dist}(\zeta, K) = 1$. Then by Green’s theorem and the above observation

$$4 \geq \int_{\partial \Omega_n} |f(z) - f(\zeta)|^2 d\omega_n(\zeta, z)$$

$$= \int_{\Omega_n} |f'(z)|^2 G_n(z, \zeta) dxdy$$

$$\geq \sum_{t=m+1}^{n} \sum_{j} \int_{A^n_j} |f'(z)|^2 G_n(z, \zeta) dxdy$$

$$\geq C \delta^2 \sum_{t=m+1}^{n} \sum_{j} \omega(\zeta, S_{t,j} \cap K)$$

$$\geq C'(n - m) \delta^2$$

and we have a contradiction.

3. A STOPPING-TIME ARGUMENT

We choose $n_\delta = 4^M q$ where $q > 1$ and $M = \left[1 + \frac{C_0}{\delta^2}\right]$. Then because $p > 2$ in the definition of $a_n = (\log(n + 1))^{1/p}$ we have

$$\lim_{\delta \to 0^+} \delta \cdot a_{n_\delta M} = 0$$

and

$$\lim_{\delta \to 0^+} \left(1 - 4^{-M}\right)^{n_\delta} a_{n_\delta M} = 0.$$ 

By construction, either $\gamma^m_k \prec \gamma^n_j$, $\gamma^n_j \prec \gamma^m_k$, or neither is subordinate to the other. We also write $\gamma^m_k \not\prec \gamma^n_j$ if $\gamma^m_k \not\prec \gamma^n_j$ for all $\gamma^n_j \in F$ where $F$ is some family of $\gamma^n_j$.
Lemma 4. For every $\varepsilon > 0$, there exists $\delta > 0$, integer $m$ and two families of sets $F_1$ and $F_2$, such that for some constant $c$:

(a) $F_1 = \{ \gamma^n_j : \gamma^n_j \text{ has condition J} \}$,
(b) $\delta \Lambda_1(\bigcup F_1 \gamma^n_j) < c \varepsilon$,
(c) $F_2 = \{ \gamma^m_k : \gamma^m_k \not\in F_1 \}$,
(d) $\Lambda_1(\bigcup F_2 \gamma^m_k) < c \varepsilon$,
(e) $F_1 \bigcup F_2$ has winding number 1 about $K$.

Proof. Given $\varepsilon > 0$, choose $\delta > 0$ so that $\delta a_{\alpha M} < \varepsilon$ and $(1 - 4^{-M})^n a_{\alpha M} < \varepsilon$. Fix $m = n \delta M$.

Now define $F_1$ to be the set of $\gamma^n_k$ such that $n \leq m$, $\gamma^n_k$ has condition J, and $\gamma^n_k$ is maximal, i.e. if $K_{n,k} \subset K_{t,j}$ with $t < n$, then $\gamma^n_j$ does not have condition J. Then (a), (c) and (e) hold for $F_1$ and $F_2$.

To prove (b) consider $\gamma^n_j \in F_1$. Since $0 \leq n \leq m$ we may replace $\gamma^n_j$ by $4^{m-n}$ squares of the form $\gamma^m_k$. Consequently,

$$\Lambda_1(\gamma^n_j) \leq 4^{m-n} \cdot \sigma_m = 4^{-n} a_m.$$ 

Since the $\gamma^n_j \in F_1$ have pairwise disjoint $K_{n,j}$, $\bigcup F_1 \gamma^n_j$ has smaller $\Lambda_1$ measure than $\bigcup_{k=1}^{4^m} \gamma^m_k$ and therefore

$$\delta \Lambda_1(\bigcup F_1 \gamma^n_j) \leq \delta \Lambda_1(\bigcup_{k=1}^{4^m} \gamma^m_k) \leq c \delta \cdot 4^m \cdot 4^{-m} a_m = c \delta a_{\alpha M} \leq c \varepsilon,$$

where $c$ is a universal constant.

To prove (d) we use Lemma 3 to obtain

$$\Lambda_1(\bigcup F_2 \gamma^m_k) \leq c (4^M - 1)^m 4^{-m} a_m \leq c (1 - 4^{-M})^n a_{\alpha M} \leq c \varepsilon.$$

4. Proof of Theorem 1

Suppose $f \in A(K,1)$ and $\varepsilon > 0$ are arbitrary. Let $F_1$ and $F_2$ be the two families provided by Lemma 4. Let $z^m_k$ be an arbitrary point in $\gamma^m_k$. Then

$$2\pi |f'(\infty)| = \sum_{\gamma^m_k \in F_1} \int_{\gamma^m_k} f(z)dz + \sum_{\gamma^m_k \in F_2} \int_{\gamma^m_k} f(z)dz \leq \sum_{\gamma^m_k \in F_1} \int_{\gamma^m_k} |f(z) - f(z^m_k)|dz + \Lambda_1(\bigcup_{\gamma^m_k} F_2)$$
\[ \begin{align*}
&\leq c \sum_{\gamma_m \in F_1} \int_{\gamma_m} \sup_{w \in \gamma_m} |f'(w)| 4^{-m} a_m dz + \varepsilon \\
&= c \sum_{\gamma_m \in F_1} D(\gamma_m) A(\gamma_m^m) + \varepsilon \\
&\leq c \delta \sum_{\gamma_m \in F_1} A(\gamma_m) + \varepsilon \\
&\leq c \delta a_{n_1} M + \varepsilon \\
&\leq c \varepsilon.
\end{align*} \]

Since \( \varepsilon \) was chosen arbitrarily and \( c \) is a universal constant, \( f'(\infty) = 0 \). Therefore, \( \gamma(K) = 0 \).

\[ \square \]

5. Remark

We could obtain a better result if we could improve the estimate in Jones’ lemma (Lemma 3). For example, if we could only replace \( M = \frac{C_0}{n} \) by \( \frac{C_0}{n^q} \) for \( q < 2 \), then in the theorem \( a_n \) could grow like \( (\log n)^{\frac{1}{q}} \). As noted above, Mattila conjectured that \( \gamma(K) = 0 \) if the Cantor set \( K \) has \( \sum (a_n)^{\frac{1}{n}} = +\infty \). Matilla’s conjecture would follow from the method here if the Jones’ lemma could be established with \( M = c \log \left( \frac{1}{n} \right) \) with \( c \) constant.

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