LARGE SETS OF ZERO ANALYTIC CAPACITY

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Abstract. We prove that certain Cantor sets with non-sigma-finite one-dimensional Hausdorff measure have zero analytic capacity.

1. Introduction

In this paper we consider a Cantor set $K$ similar to the $\frac{1}{4}$-Cantor set of [G70] and [IS4]. Fix $p > 2$ and for $n > 0$ define

$$\sigma_n = 4^{-n}a_n = 4^{-n}[\log (n+1)]^{1/p}.$$ 

Set $K_0 = [0,1] \times [0,1]$ and $K_1 = \bigcup_{j=1}^4 K_{1,j}$, where $K_{1,j} \subset K_0$ is a square of sidelength $\sigma_1$ having sides parallel to the axis and containing one of the four corners of $K_0$. Next take $4^2$ squares $K_{2,j}$ of sidelength $\sigma_2$, one in each corner of each square $K_{1,j}$, and define $K_2 = \bigcup_{j=1}^{4^2} K_{2,j}$. Continuing we obtain $K_n = \bigcup_{j=1}^{4^n} K_{n,j}$, where $K_{n,j}$ is a square of sidelength $\sigma_n$. The Cantor set we study is

$$K = K(p) = \bigcap_{n=1}^\infty K_n.$$ 

If $E$ is a compact plane set define

$$A(E,1) = \{ f : f \text{ analytic on } E^c, \ f(\infty) = 0, \ \| f \|_{L^\infty(E^c)} \leq 1 \}$$

and define the analytic capacity of $E$ by

$$\gamma(E) = \sup \{ |f'(\infty)| : f \in A(E,1) \},$$

where

$$f'(\infty) = \lim_{z \to \infty} zf(z).$$

If $\gamma(E) = 0$, then the only function in $A(E,1)$ is the constant $f \equiv 0$ and in this case $E$ is removable for bounded analytic functions. For more details about analytic capacity see [G72].

Theorem 1. Let $p > 2$, and let $K$ be the four-corner Cantor set $K(p)$. Then $\gamma(K) = 0$ but $K$ does not have $\sigma$-finite one-dimensional measure.
The proof of Theorem 1 depends on a lemma of Jones [189] used for a proof different from [179] that the $1/4$-Cantor set has zero analytic capacity.

Let $h(t)$ be an increasing continuous function on $t \geq 0$ with $h(0) = 0$, and write $\Lambda_{h(t)}(E)$ for the Hausdorff $h$-measure of $E$. Now define an increasing function $h(t)$ so that $h(0) = 0$ and $h(\sigma_n) = 4^{-n}$ for all $n$. We say $h(t)$ is a measure function corresponding to the Cantor set $K$. For every $n$ define a measure $\mu_n$ on $K_n$ by $\mu_n(K_{n,j}) = 4^{-n}$ for all $j$. Then $\{\mu_n\}$ converges weak-star to a measure $\mu$ supported on $K$ and satisfying $\mu(K_{n,j}) = 4^{-n}$. Suppose $\frac{2^n}{2^n} \sigma_n < r < \frac{2^n}{2^n} \sigma_{n-1}$ and let $D(z, r)$ be a disk of radius $r$ and center $z \in K$. Then $D(z, r)$ can meet at most $4$ squares of sidelength $\sigma_n$. Hence

$$\mu(D(z, r)) \leq 4 \mu(K_{n,j}) = 4 \cdot 4^{-n} = 4h(\sigma_n) \leq 4h(r),$$

so that $\mu(D(z, r)) \leq 16h(r)$ for any disk $D(z, r)$. Therefore $\Lambda_{h}(K) > 0$ by Frostman’s Theorem [179]. Since

$$\lim_{t \to 0} \frac{h(t)}{t} = 0,$$

it follows that $K$ has non-$\sigma$-finite 1-dimensional measure.

If $h(t)$ is a measure function corresponding to $K$, then

$$\int_0^1 \frac{h(t)^2}{t^3} dt \sim \sum_{n=1}^{\infty} \frac{1}{(a_n)^2} = \sum_{n=1}^{\infty} \frac{1}{(\log n)^{\beta}} = \infty.$$

On the other hand, Mattila [199] proved that $\gamma(K) > 0$ if $K$ is a Cantor set built with squares of side $\sigma_n$ and if

$$\int_0^1 \frac{h(t)^2}{t^3} dt < \infty,$$

where $h$ is any measure function for corresponding to $K$. Mattila’s proof used Menger curvature (see [190] and [191]). However, if the Cantor set $K$ has corresponding measure function $h$ satisfying

$$\int_0^1 \frac{h(t)^2}{t^3} dt = \infty,$$

then Eiderman [198] proved that $\gamma^+(K) = 0$, where

$$\gamma^+(E) = \sup \left\{ \int_E \, d\mu : \left| \int_E \frac{d\mu(\zeta)}{\zeta - z} \right| < 1, \forall z \in \mathbb{C} \setminus E, \mu > 0, spt(\mu) \subset E \right\}.$$

Since $\gamma^+(E) \leq \gamma^+(E)$, our result is a partial improvement of Eiderman’s theorem. Mattila [199] has conjectured that for Cantor sets of this type $\gamma(K) = 0$ if and only if

$$\int_0^1 \frac{h(t)^2}{t^3} dt = \infty,$$

when $h$ corresponds to $K$. This latter condition holds if and only if

$$\sum_{n=1}^{\infty} \frac{1}{(a_n)^2} = \infty.$$

If Matilla’s conjecture is true, then together with Eiderman’s theorem it gives Cantor set evidence supporting the more ambitious conjecture that $\gamma^+(E) > 0$ implies $\gamma^+(E) > 0$. 

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In [G72] it was incorrectly claimed that \( \gamma(K) > 0 \) if and only if
\[
\int_0^1 \frac{h(t)}{t^2} \, dt < \infty.
\]
eiderman, however, found a mistake in the proof. In fact the result in [M96] shows that the claim is false. See L.D. Ivanov [I84] for the first example of a Cantor set of non-\( \sigma \)-finite linear measure and zero analytic capacity.

2. Two lemmas of Peter Jones

We need the following two lemmas from [J89]. The proofs we give are small variations on [J89] and [C90].

Define \( \gamma_{n,k} = \partial_cK_{n,j} \), where \( cK_{n,j} \) is the square concentric to \( K_{n,j} \) with sidelength \( c\sigma_n \) and where \( c > 1 \) is chosen so that the \( \gamma_{n,j} \) do not overlap. We refer to \( \gamma_{n,k} \) as a square, although it is only the boundary of a square. Notice that
\[
\Lambda_1(\gamma_{n,k}) = c\Lambda_1(\partial K_{m,k})
\]
for the same constant \( c \). We associate to each \( \gamma_{n,k} \) a “square annulus”
\[
A_{n,k} = \{ w : \text{dist}(w, \gamma_{n,k}) \leq \varepsilon_0\sigma_m \}
\]
and we choose \( \varepsilon_0 > 0 \) so small that the annuli \( A_{n,k} \) are pairwise disjoint.

Define \( \Omega = \mathbb{C} \setminus K \). Since \( K \) has positive logarithmic capacity, Green’s function \( G(z, \zeta) \) exists for \( \zeta, z \notin K \), and harmonic measure \( \omega(\zeta, E) \) exists for \( \zeta \notin K \) and \( E \subset K \). We write \( \omega(\zeta, K_{m,k}) \) for \( \omega(\zeta, K \cap K_{m,k}) \).

We also define the slightly larger “squares”
\[
S_{m,k} = \{ w : \text{dist}(w, K_{m,k}) \leq \varepsilon_1\sigma_m \}
\]
and set
\[
S_m = \bigcup_{k=1}^{4^m} S_{m,k},
\]
where \( \varepsilon_1 > 0 \) is so small that \( S_{m,k} \cap A_{m,k} = \emptyset \). Then \( K = \bigcap_{m=1}^{\infty} S_m \). Green’s function and harmonic measure also exist for the domain \( \Omega_m = \mathbb{C} \setminus S_m \). Denote these by \( G_m(z, \zeta) \) and \( \omega_m(\zeta, \partial S_{m,k}) \) respectively.

**Lemma 2.** Let \( z \in A_{m,k} \).

(a) There are constants \( 0 < c_1 < c_2 < 1 \), independent of \( k \) and \( m \), such that
\[
c_1 \leq \omega_m(z, \partial S_{m,k}) \leq c_2.
\]

(b) If \( \zeta \in \Omega \) and \( 1 \geq \text{dist}(\zeta, K) \geq 2 \text{dist}(z, K) \), then
\[
G_m(z, \zeta) \sim \omega_m(\zeta, \partial S_{m,k}).
\]

**Proof.** For (a) note that there is \( c' > 0 \) such that there exists a second square annulus \( B_{k,m} \) so that \( A_{k,m} \subset B_{k,m} \subset \Omega_m \) and \( \text{dist}(z, \partial B_{k,m}) \geq c'\sigma_m \). The lower bound then follows by a comparison with \( B_{k,m} \). There is \( S_{m,j} \) with \( j \neq k \) such that \( \text{dist}(S_{m,j}, S_{m,k}) \leq c_4\sigma_m \) and the upper bound follows by a comparison with \( \mathbb{C} \setminus (S_{m,k} \cup S_{m,j}) \), using symmetry and Harnack’s inequality.

To prove (b) note first that as in the proof of (a) there are constants \( C_1 \) and \( C_2 \) such that by Harnack’s inequality and a comparison
\[
C_1 \leq G_m(z, w) \leq C_2
\]
for \( w \in \partial B^m_k \). Then using the symmetry of Green’s function and (a) for a larger square we obtain
\[
C_1 \omega_m(\zeta, \partial S_{m,k}) \leq G_m(\zeta, z) \leq C_2 \omega_m(\zeta, \partial S_{m,k}).
\]

We write \( \gamma^m_k \prec \gamma^n_j \) and say \( \gamma^m_k \) is \textbf{subordinate} to \( \gamma^n_j \) if \( \gamma^n_j \) has winding number one about \( \gamma^m_k \). If the winding number is zero, we write \( \gamma^m_k \not\prec \gamma^n_j \). For any \( f \in A(K, 1) \) and \( \gamma^m_k \) define
\[
D(\gamma^m_k) = \sup_{w \in \gamma^m_k} |f'(w)| \sigma_n.
\]
We say a square \( \gamma^m_k \) has \textbf{condition J} if
\[
D(\gamma^m_k) \leq \delta
\]
for some previously defined \( f \) and \( \delta > 0 \).

**Lemma 3.** Let \( f \in A(K, 1) \). For every \( \delta > 0 \) there exists a \( C_0 > 0 \) such that for every \( \gamma^n_j \) there exists \( \gamma^m_k \prec \gamma^n_j \) such that \( m \leq n + C_0 \delta^{-2} \) and such that \( \gamma^m_k \) has condition J.

**Proof.** Observe that by Harnack’s inequality
\[
\sup_{\gamma^m_k} |f'(z)|^2 \sim \int \int_{A^2_n} |f'|^2 \frac{dx dy}{\sigma_n^2}.
\]
Suppose the lemma is false. Choose \( \zeta \) with \( \text{dist}(\zeta, K) = 1 \). Then by Green’s theorem and the above observation
\[
4 \geq \int_{\partial \Omega_n} |f(z) - f(\zeta)|^2 d\omega_n(\zeta, z)
= \int_{\Omega_n} |f'(z)|^2 G_n(z, \zeta) dx dy
\geq \sum_{t=m+1}^{n} \sum_{j} \int_{A^2_j} |f'(z)|^2 G_n(z, \zeta) dx dy
\geq C \delta^2 \sum_{t=m+1}^{n} \sum_{j} \omega(\zeta, S_{t,j} \cap K)
\geq C'(n - m) \delta^2
\]
and we have a contradiction.

3. A STOPPING-TIME ARGUMENT

We choose \( n_\delta = 4^M q \) where \( q > 1 \) and \( M = \left( 1 + \frac{C_0}{2} \right)^p \). Then because \( p > 2 \) in the definition of \( a_n = (\log(n + 1))^{\frac{1}{p}} \) we have
\[
\lim_{\delta \to 0^+} \delta \cdot a_{n_\delta M} = 0
\]
and
\[
\lim_{\delta \to 0^+} (1 - 4^{-M})^{n_\delta} a_{n_\delta M} = 0.
\]

By construction, either \( \gamma^m_k \prec \gamma^n_j \), \( \gamma^n_j \prec \gamma^m_k \), or neither is subordinate to the other. We also write \( \gamma^m_k \not\prec \gamma^n_j \) if \( \gamma^m_k \not\prec \gamma^n_j \) for all \( \gamma^n_j \in F \) where \( F \) is some family of \( \gamma^n_j \).
Lemma 4. For every \( \varepsilon > 0 \), there exists \( \delta > 0 \), integer \( m \) and two families of sets \( F_1 \) and \( F_2 \), such that for some constant \( c \):

(a) \( F_1 = \{ \gamma_j^n : \gamma_j^n \text{ has condition } J \} \),

(b) \( \delta \Lambda_1(\bigcup F_1, \gamma_j^n) < c\varepsilon \),

(c) \( F_2 = \{ \gamma_k^m : \gamma_k^m \not\in F_1 \} \),

(d) \( \Lambda_1(\bigcup F_2, \gamma_k^m) < c\varepsilon \),

(e) \( F_1 \cup F_2 \) has winding number 1 about \( K \).

Proof. Given \( \varepsilon > 0 \), choose \( \delta > 0 \) so that \( \delta a_{n\delta M} < \varepsilon \) and \( (1 - 4^{-M})^{n\delta} a_{n\delta M} < \varepsilon \). Fix \( m = n\delta M \).

Now define \( F_1 \) to be the set of \( \gamma_k^n \) such that \( n \leq m, \gamma_k^n \) has condition \( J \), and \( \gamma_k^n \) is maximal, i.e. if \( K_{n,k} \subset K_{t,j} \) with \( t < n \), then \( \gamma_j^t \) does not have condition \( J \).

To prove (b) consider \( \gamma_j^n \in F_1 \). Since \( 0 \leq n \leq m \) we may replace \( \gamma_j^n \) by \( 4^{m-n} \) squares of the form \( \gamma_m^m \).

Consequently,

\[
\Lambda_1(\gamma_j^n) \leq 4^{m-n} \cdot a_m = 4^{-n} a_m.
\]

Since the \( \gamma_j^n \in F_1 \) have pairwise disjoint \( K_{n,j} \), \( \bigcup F_1 \gamma_j^n \) has smaller \( \Lambda_1 \) measure than \( \bigcup_{k=1}^{4^m} \gamma_k^m \) and therefore

\[
\delta \Lambda_1(\bigcup_{F_1} \gamma_j^n) \leq \delta \Lambda_1(\bigcup_{k=1}^{4^m} \gamma_k^m) \leq c\delta \cdot 4^m \cdot 4^{-m} a_m = c\delta a_{n\delta M} \leq c\varepsilon,
\]

where \( c \) is a universal constant.

To prove (d) we use Lemma 3 to obtain

\[
\Lambda_1(\bigcup_{F_2} \gamma_k^m) \leq c(4^M - 1)^m 4^{-m} a_m \leq c(1 - 4^{-M})^{n\delta} a_{n\delta M} \leq c\varepsilon.
\]

4. Proof of Theorem 1

Suppose \( f \in A(K,1) \) and \( \varepsilon > 0 \) are arbitrary. Let \( F_1 \) and \( F_2 \) be the two families provided by Lemma 4. Let \( z_k^m \) be an arbitrary point in \( \gamma_k^m \). Then

\[
2\pi |f'(\infty)| = \left| \sum_{\gamma_k^m \in F_1} \int_{\gamma_k^m} f(z)dz + \sum_{\gamma_k^m \in F_2} \int_{\gamma_k^m} f(z)dz \right| \\
\leq \left| \sum_{\gamma_k^m \in F_1} \int_{\gamma_k^m} f(z)dz \right| + \left| \sum_{\gamma_k^m \in F_2} \int_{\gamma_k^m} f(z)dz \right| \\
\leq \sum_{\gamma_k^m \in F_1} \int_{\gamma_k^m} |f(z) - f(z_k^m)|dz + \Lambda_1(\bigcup_{F_2} \gamma_k^m)
\]

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\[
\begin{align*}
\leq c & \sum_{\gamma^m_k \in F_1} \int_{\gamma^m_k} \sup_{w \in \gamma^m_k} |f'(w)| 4^{-m} a_m \, dz + \varepsilon \\
= c & \sum_{\gamma^m_k \in F_1} D(\gamma^m_k) A_1(\gamma^m_k) + \varepsilon \\
\leq c& \delta \sum_{\gamma^m_k \in F_1} A_1(\gamma^m_k) + \varepsilon \\
\leq c& \delta a_{n,1} M + \varepsilon \\
\leq c\varepsilon.
\end{align*}
\]

Since \( \varepsilon \) was chosen arbitrarily and \( c \) is a universal constant, \( f'(\infty) = 0 \). Therefore, \( \gamma(K) = 0 \).

\section{Remark}

We could obtain a better result if we could improve the estimate in Jones’ lemma (Lemma 3). For example, if we could only replace \( M = \frac{C_0}{n^q} \) by \( \frac{C_0}{n^{q+1}} \) for \( q < 2 \), then in the theorem \( a_n \) could grow like \( (\log n)^{\frac{1}{q}} \). As noted above, Mattila [M96] conjectured that \( \gamma(K) = 0 \) if the Cantor set \( K \) has \( A_n^2 + 1 = +\infty \). Mattila’s conjecture would follow from the method here if the Jones’ lemma could be established with \( M = c \log(\frac{1}{n}) \) with \( c \) constant.

\section*{References}


