SHARP ESTIMATES FOR THE MAXIMUM OVER MINIMUM MODULUS OF RATIONAL FUNCTIONS

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Abstract. Let \( m, n \geq 0, \lambda > 1 \), and \( R \) be a rational function with numerator, denominator of degree \( \leq m, n \), respectively. In several applications, one needs to know the size of the set \( S \subset [0,1] \) such that for \( r \in S \),
\[
\max_{|z|=r} |R(z)|/\min_{|z|=r} |R(z)| \leq \lambda^{m+n}.
\]
In an earlier paper, we showed that
\[
\text{meas}(S) \geq \frac{1}{4} \exp \left( -\frac{13}{\log \lambda} \right),
\]
where \( \text{meas} \) denotes linear Lebesgue measure. Here we obtain, for each \( \lambda \), the sharp version of this inequality in terms of condenser capacity. In particular, we show that as \( \lambda \to 1^+ \),
\[
\text{meas}(S) \geq \frac{4}{21} \exp \left( -\frac{\pi^2}{2 \log \lambda} \right) (1 + o(1)).
\]

1. Introduction and results

In applications including rational approximation, and the theory of meromorphic functions, one needs estimates for the ratio of the maximum and minimum modulus of a rational function \[3\]. The classical way to obtain such estimates involves Cartan’s lemma on small values of polynomials. In \[3\], the author used a form of Cartan’s lemma in a metric space setting to establish the following result, and hence to investigate convergence of diagonal Padé approximants:

**Theorem 1.** Let \( \lambda > 1 \) and \( m, n \geq 0 \). Then for rational functions \( R \) with numerator, denominator of degree \( \leq m, n \) respectively,
\[
\max_{|z|=r} |R(z)|/\min_{|z|=r} |R(z)| \leq \lambda^{m+n}, \quad r \in S,
\]
where \( S \subset [0,1] \) has Lebesgue measure \( \text{meas}(S) \) satisfying
\[
\text{meas}(S) \geq \frac{1}{4} \exp \left( -\frac{13}{\log \lambda} \right).
\]

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This is sharp in form in the following sense: let $0 < \varepsilon < 1$. Then for $\lambda$ close enough to 1 and $m$ large enough, there exists a polynomial $R$ of degree $m$ for which the set $S \subset [0,1]$ on which (1.2) holds satisfies

$$\text{meas}(S) \leq \exp \left( -\frac{2 - \varepsilon}{\log \lambda} \right).$$

In this paper, we use potential theory to close the gap between 2 and 13. Let us recall some potential theoretic notions [4]. Let

$$\mathcal{H} := \{ z : \Re z > 0 \}$$

denote the open right-half plane. Its boundary is the imaginary axis $\partial \mathcal{H} = i\mathbb{R}$. The Green’s function for the right-half plane with pole at $\xi \in \mathcal{H}$ is

$$g(z, \xi) = \log \left| \frac{z + \bar{\xi}}{z - \bar{\xi}} \right|.$$ 

Moreover, given compact $E \subset \mathcal{H}$, its Green energy is

$$V_E^{\mathcal{H}} := \inf_{\mu(E)=1} \int_E \int_E g(x, t) \, d\mu(x) \, d\mu(t),$$

where the inf is taken over all non-negative Borel measures $\mu$ with support in $E$ and with $\mu(E) = 1$. It is known that there is a unique measure $\mu_E^{\mathcal{H}}$, called the Green equilibrium measure, attaining the infimum. The condenser capacity of the pair $(E, \partial \mathcal{H}) = (E, i\mathbb{R})$ is defined to be

$$C(E, i\mathbb{R}) = 1/V_E^{\mathcal{H}}.$$ 

It is easily seen from (1.4) that $V_E^{\mathcal{H}}$ decreases as the set $E$ increases, and hence $C(E, i\mathbb{R})$ is a monotone set function. For further orientation, see [4, p. 132 ff.]. We shall need to consider in detail the set $E = [b, 1]$, and, for that purpose, we need some notation for elliptic integrals. Given $b \in (0,1)$, the complete elliptic integrals of the first kind are

$$K(b) := \int_0^1 \frac{dx}{\sqrt{(1 - b^2 x^2) (1 - x^2)}}, \quad K'(b) := K\left(\sqrt{1 - b^2}\right).$$

**Theorem 2.** Let $0 < b < 1$.

(a) $\mu_E^{\mathcal{H}}_{[b,1]}$ is absolutely continuous w.r.t. Lebesgue measure on $[b,1]$ and

$$\frac{d\mu_E^{\mathcal{H}}_{[b,1]}(x)}{dx} = \frac{\kappa(b)}{\sqrt{(x^2 - b^2)(1 - x^2)}}, \quad x \in (b,1),$$

where

$$\kappa(b) = 1/\int_b^1 \frac{dx}{\sqrt{(x^2 - b^2)(1 - x^2)}} = 1/K'(b).$$

(b) Let

$$F(b) := C([b,1], i\mathbb{R}), \quad 0 < b < 1.$$ 

Then

$$F(b) = \frac{K'(b)}{\pi K(b)}.$$
(c) $F$ is a strictly decreasing function of $b$, mapping $(0, 1)$ onto $(0, \infty)$, and satisfying

$$F(b) = \frac{2}{\pi^2} \left| \log \frac{b}{4} \right| + o(1), \quad b \to 0+;$$

$$F(b) = \frac{1}{\log(1 - b)} (1 + o(1)), \quad b \to 1-. $$

In the sequel, we let $F^{-1} : (0, \infty) \to (0, 1)$ denote the inverse of $F$, so that

$$F \left( F^{-1}(x) \right) = x, \quad x \in (0, \infty).$$

Following is our main result:

**Theorem 3.** Let $\lambda > 1$ and $m, n \geq 0$.

(a) Then for rational functions $R$ with numerator, denominator of degree $\leq m, n$ respectively,

$$\max_{|z|=r} |R(z)| / \min_{|z|=r} |R(z)| \leq \lambda^{m+n}, \quad r \in S,$$

where $S \subset [0, 1]$ satisfies

$$\text{meas}(S) \geq F^{-1} \left( \frac{1}{\log \lambda} \right).$$

(b) This is sharp in the sense that given $\varepsilon > 0$, there exists for large enough $m$, a polynomial $R$ of degree $m$, such that (with $n = 0$)

$$\text{meas}(S) \leq F^{-1} \left( \frac{1}{\log \lambda} \right) + \varepsilon.$$

(c) In particular,

$$F^{-1} \left( \frac{1}{\log \lambda} \right) = 4 \exp \left( -\frac{\pi^2}{2 \log \lambda} \right) (1 + o(1)), \quad \lambda \to 1+;$$

$$F^{-1} \left( \frac{1}{\log \lambda} \right) = 1 - \lambda^{-1+o(1)}, \quad \lambda \to \infty.$$

**Remarks.** (a) Let $\rho > 0$. By replacing $R(z)$ by $R(\rho z)$, we deduce that (1.12) holds on a set $S \subset [0, \rho]$ with

$$\text{meas}(S) \geq \rho F^{-1} \left( \frac{1}{\log \lambda} \right).$$

(b) One may formulate a generalisation of Theorem 3 for potentials (cf. [3, Theorem 6]).

(c) There is a (distant) connection between Theorem 3 and estimates for the minimum modulus of functions of slow growth [2, p. 376 ff.].

This paper is organised as follows: in Section 2 we prove Theorem 3(a), in Section 3 we prove Theorems 3(b), (c), and in Section 4 we establish Theorem 2.
2. The proof of Theorem 3(a)

We shall do this in five steps:

**Step 1: Reduction to \( R \) with real poles and zeros.** Note first that if \( a, b \in \mathbb{C} \), then

\[
\frac{\max_{|z|=r} |z-a|}{\min_{|z|=r} |z-b|} \leq \left| \frac{r+|a|}{r-|b|} \right| \left| \frac{r+|b|}{r-|a|} \right|.
\]

It follows that it suffices to consider

\[ S := \left\{ r \in [0, 1] : \prod_{j=1}^{m+n} \left| \frac{r+\alpha_j}{r-\alpha_j} \right| \leq \lambda^{m+n} \right\}, \]

where all \( \alpha_j > 0 \). Indeed, this merely decreases the size of \( S \), and we are searching for a lower bound for that size. Next, note that we have also assumed that we have numerator and denominator of exact degree \( m \) and \( n \) respectively. This may be achieved by adding some \( j = 1 \), which again reduces the size of \( S \). Finally, we note that we may assume that all \( \alpha_j \leq 1 \); again, replacing any \( \alpha_j > 1 \) by 1 reduces the size of \( S \). So, in the sequel, we assume that all \( \alpha_j \in (0, 1] \).

Let us set \( \ell := m + n \) and

\[
S_0 := \left\{ r \in [0, 1] : \prod_{j=1}^{\ell} \left| \frac{r+\alpha_j}{r-\alpha_j} \right| < \lambda^\ell \right\};
\]

\[
E := \left\{ r \in [0, 1] : \prod_{j=1}^{\ell} \left| \frac{r+\alpha_j}{r-\alpha_j} \right| \geq \lambda^\ell \right\};
\]

\[
= \left\{ r \in [0, 1] : \prod_{j=1}^{\ell} \left| \frac{r-\alpha_j}{r+\alpha_j} \right| \leq \lambda^{-\ell} \right\}.
\]

Since the equation \( \left| \prod_{j=1}^{\ell} \left( \frac{r+\alpha_j}{r-\alpha_j} \right) \right| = \lambda^\ell \) has at most \( 2\ell \) solutions in \( r \), we see that

\[
\text{meas}(S) = \text{meas}(S_0) = 1 - \text{meas}(E).
\]

We must look for an upper bound for \( \text{meas}(E) \). It is clear that \( E \subset (0, 1] \) and consists of finitely many intervals, some of which may degenerate to a single point.

**Step 2: The basic inequality for \( E \).** We shall show that

\[
C(E, i\mathbb{R}) \leq \frac{1}{\log \lambda}.
\]

If firstly \( E \) consists of finitely many points, then \( V_E^M = \infty \) from (1.4), so \( C(E, i\mathbb{R}) = 0 \). Let us now assume that \( E \) contains at least one non-empty interval. Note that each \( \alpha_j \neq 1 \) lies inside such a non-empty interval; if \( \alpha_j = 1 \), it is the right-endpoint of a non-empty interval. Let \( \mu_E^M \) denote the Green equilibrium measure for \( E \). We shall need a property of the Green equilibrium potential:

\[
\int_E g(r, \alpha_j) \, d\mu_E^M(r) = V_E^M, \quad 1 \leq j \leq \ell.
\]
In [4, Thm. 5.11, p.132], it is shown that if we replace $\alpha_j$ by $x$, this identity holds for “quasi-every” $x \in E$. But the Green potential is continuous on each of the non-empty intervals of $E$, since these are regular with respect to the Dirichlet problem in the plane. (See, for example, [4, pp. 54-55].) Since, as we have noted, each such $\alpha_j$ is contained in such an interval, we have (2.4) as stated.

Next, from (2.1),

$$\lambda^{-\ell} \geq \int_E \left| \prod_{j=1}^{\ell} \frac{r - \alpha_j}{r + \alpha_j} \right| d\mu_E^H (r)$$

$$= \int_E \exp \left( - \sum_{j=1}^{\ell} g(r, \alpha_j) \right) d\mu_E^H (r)$$

$$\geq \exp \left( - \int_E \sum_{j=1}^{\ell} g(r, \alpha_j) d\mu_E^H (r) \right) \geq \exp \left( -\ell V_E^H \right).$$

Here we have used (2.4) and the arithmetic-geometric mean inequality. This last inequality is easily reformulated as (2.3).

**Step 3: Show that $\text{meas}(E)$ is maximal if $E$ is of the form $[b, 1]$.** Set

$$b := F[\log \lambda] \left( \frac{1}{\log \lambda} \right) \Leftrightarrow F(b) = \frac{1}{\log \lambda}.$$ 

The existence and uniqueness of $b$ follows from Theorem 2(c). Then

$$V_{[b, 1]}^H = \frac{1}{C([b, 1], \mathbb{R})} = \log \lambda.$$ 

We shall assume that $E$ of (2.1) satisfies

$$\text{meas} (E) > \text{meas} ([b, 1])$$

and derive a contradiction. Now

$$\lim_{y \to 1^-} \text{meas} (E \cap [0, y]) = \text{meas} (E),$$

so we may choose $y_0 < 1$ such that

$$E_0 := E \cap [0, y_0] \text{ has } \text{meas} (E_0) = \text{meas} ([b, 1]).$$

We shall “shift left” the Green equilibrium measure from $[b, 1]$ to $E_0$, and then derive a contradiction to (2.3). The basic idea is that

$$g(x + c, y + c) > g(x, y) \text{ if } x, y, c > 0.$$ 

We may omit the discrete points from $E_0$ and assume that $E_0$ is a union of $k$ disjoint intervals

$$E_0 = \bigcup_{j=1}^{k} I_j,$$

where

$$I_j = [\alpha_j, \beta_j] \text{ and each } \beta_j < \alpha_{j+1}.$$
Define a strictly increasing piecewise linear map \( h \) from \( E_0 \) onto \( [b, 1] \) by
\[
h(x) := x + b - \alpha_j + \sum_{i=1}^{j-1} (\beta_i - \alpha_i) =: x + A_j, \quad x \in [\alpha_j, \beta_j],
\]
\( 1 \leq j \leq k. \) (The empty sum is interpreted as 0.) Now define an absolutely continuous measure \( \nu \) on \( E_0 \) by
\[
\nu'(x) := (\mu_{[b,1]}(h(x)))' = \left( \mu_{[b,1]}^H \right)'(h(x)) h'(x), \quad x \in E_0.
\]
Then \( \nu \) has total mass 1. Next, as \([b, 1]\) is regular with respect to the Dirichlet problem in the plane, we have
\[
\int_b^1 g(x, t) \, d\mu_{[b,1]}^H(t) = V_{[b,1]}^H = \log \lambda, \quad x \in [b, 1].
\]
Hence
\[
\int_{E_0} g(h(y), h(s)) \, d\nu(s) = \log \lambda, \quad y \in E_0.
\]
We shall show that there exists \( \eta > 0 \) such that
\[
(2.5) \quad g(h(y), h(s)) \geq g(y, s) + \eta \forall s, \quad y \in E_0,
\]
and then
\[
\log \lambda \geq \int_{E_0} \int_{E_0} g(y, s) \, d\nu(s) \, d\nu(y) + \eta
\geq V_{E_0}^H + \eta.
\]
This implies that
\[
C(E, i\mathbb{R}) \geq C(E_0, i\mathbb{R}) \geq \frac{1}{\log \lambda - \eta},
\]
so we obtain the desired contradiction to (2.3).

**Step 4: Proof of (2.5).** Let us suppose that \( y \in I_i, s \in I_j \), where, for example, \( i \leq j \), so that
\[
(2.6) \quad g(h(y), h(s)) = \log \left| \frac{(y + A_i) + (s + A_j)}{(y + A_i) - (s + A_j)} \right|
= \log \left| \frac{y + s}{y - s} + \log \left( 1 + \frac{A_i + A_j}{y + s} \right) - \log \left( 1 - \frac{A_j - A_i}{y - s} \right) \right|.
\]
Note that for each \( m \),
\[
A_m - A_{m-1} = \beta_{m-1} - \alpha_m < 0
\]
so \( A_j - A_i \leq 0 \), while \( y - s \leq 0 \). Also
\[
\frac{A_j - A_i}{y - s} \leq \frac{A_i - A_j}{\alpha_j - \beta_i} \leq 1.
\]
Then as \( A_k \leq A_i, A_j \),
\[
g(h(y), h(s)) \geq g(y, s) + \log (1 + A_k) + 0,
\]
so we may take \( \eta := \log(1 + A_k) \). Note here that \( h(\beta_k) = 1 \Rightarrow A_k = 1 - \beta_k > 0. \)
Step 5: Completion of the proof. We have shown that 

\[ \text{meas}(E) \leq \text{meas}([b, 1]) = 1 - b, \]

so (2.2) gives 

\[ \text{meas}(S) \geq b = F([-1] \left( \frac{1}{\log \lambda} \right)). \quad \square \]

3. The proof of Theorem 3(b), (c)

The proof of Theorem 3(b). We shall use a crude discretisation procedure, of
the type used in the theory of orthogonal polynomials in the 1980’s. The finer
method of Totik [4] would yield sharper estimates, but those are not needed here.
Fix \( \lambda > 1, \varepsilon > 0 \), and choose \( \lambda' > \lambda \) such that

\[ b' := F([-1] \left( \frac{1}{\log \lambda'} \right)) < F([-1] \left( \frac{1}{\log \lambda} \right) + \frac{\varepsilon}{4}. \tag{3.1} \]

Recall that

\[ \int_{b'}^{1} g(x, t) \, d\mu_{[b', 1]}(t) = V_{[b', 1]} = \log \lambda', \quad x \in [b', 1]. \tag{3.2} \]

Let us choose 

\[ b' = t_0 < t_1 < t_2 < \cdots < t_m = 1 \]

such that if \( J_j := [t_j, t_{j+1}) \), then

\[ \int_{J_j} d\mu_{[b', 1]}(t) = \frac{1}{m}, \quad 0 \leq j \leq m - 1. \tag{3.3} \]

It is easily seen from the explicit formula (1.6) for \( \mu_{[b', 1]} \) that \( \exists C_i \neq C_i \) \( \lambda' > 0, i = 1, 2 \), such that

\[ t_{j+1} - t_j \geq \frac{C_1}{m} \text{ if } J_j \subset \left[ b' + \frac{\varepsilon}{8}, 1 - \frac{\varepsilon}{8} \right], \tag{3.4} \]

and

\[ t_{j+1} - t_j \leq \frac{C_2}{m}, \quad 0 \leq j \leq m - 1. \tag{3.5} \]

As our polynomial, we choose

\[ R(x) := \prod_{j=1}^{m} (x - t_j) \]

so that for \( r \in [0, 1], \)

\[ \frac{\max_{z=r} |R(z)|}{\min_{z=r} |R(z)|} = \left| \prod_{j=1}^{m} \frac{r + t_j}{r - t_j} \right| =: |U(r)|, \tag{3.6} \]

say. Next, for \( r \in [b', 1], \) (3.2) implies that

\[ \frac{1}{m} \log |U(r)| - \log \lambda' = \sum_{j=0}^{m-1} \int_{J_j} [g(r, t_j) - g(r, t)] \, d\mu_{[b', 1]}(t) \]

\[ =: \sum_{j=0}^{m-1} \Delta_j, \tag{3.7} \]

\[ =: \sum_{j=0}^{m-1} \Delta_j, \]

\[ \text{say.} \]
say. We shall find a lower bound for this difference for large enough $m$ and all $r \in [b' + \frac{x}{4}, 1 - \frac{x}{4}]$. For such an $r$, choose $k = k(r)$ such that $r \in J_k$. Since $g(r, t_j) \geq 0$, we see that for $|j - k| \leq 2$,

$$
\Delta_j \geq -\int_{t_j}^r g(r, t) \, d\mu_{\psi,1}^k(t)
\geq -C_3 \int_{t_j}^r \log \left| \frac{2}{r - t} \right| \, dt
\geq -C_4 \frac{\log m}{m},
$$

where $C_3, C_4 > 0$ are independent of $m, j, r$. Here we have used the fact that $(\mu_{\psi,1}^k)'$ is bounded in $[b' + \frac{x}{8}, 1 - \frac{x}{8}]$, as well as (3.5). Next, if $|j - k| \geq 2$ and $t \in J_j$, we see that for some $s$ between $t_j$ and $t$,

$$
|g(r, t_j) - g(r, t)| = \left| \frac{\partial g}{\partial s} (x, s) (t - t_j) \right|
\leq \frac{2}{|r - s|} (t_{j+1} - t_j)
\leq \frac{C_5}{|r - t|} (t_{j+1} - t_j).
$$

Again, $C_5$ is independent of $m, j, r, t$, and we have used (3.4). Then, using (3.5), we obtain for some $C_6, \ldots, C_9 > 0$, independent of $m, j, r$,

$$
\sum_{0 \leq j \leq m - 1 : |j - k| \geq 2} |\Delta_j| \leq \frac{C_6}{m} \int \left\{ t \in [b' + \frac{x}{8}, 1 - \frac{x}{8}] : |t - r| \geq C_7 / m \right\} \frac{dt}{|x - t|} + \frac{C_8}{m} \log m.
$$

Thus for $r \in [b' + \frac{x}{4}, 1 - \frac{x}{4}]$, (3.7) shows that

$$
\log |U(r)| \geq m \log \lambda' - C_{10} \log m > m \log \lambda,
$$

for $m$ large enough. Then it follows from (3.6) that

$$
\frac{\max_{|z|=r} |R(z)|}{\min_{|z|=r} |R(z)|} > \lambda^m, \quad r \in \left[ b' + \frac{\varepsilon}{4}, 1 - \frac{\varepsilon}{4} \right],
$$

so the set $S \subset [0, 1]$ satisfying (1.12) (with $n = 0$) has

$$
S \subset [0, b' + \frac{\varepsilon}{4}) \cup (1 - \frac{\varepsilon}{4}, 1]
\Rightarrow \text{meas} (S) \leq b' + \frac{\varepsilon}{2} < F[-1] \left( \frac{1}{\log \lambda} \right) + \varepsilon,
$$

by (3.1).

**The proof of Theorem 3(c).** We note that (1.15) and (1.16) follow easily from (1.10) and (1.11) by inverting the asymptotic relations. □
4. The proof of Theorem 2

The proof of Theorem 2(a), (b). We shall use a well-known example [4, p.133]: let \( 0 < a < 1 \) and \( \mathcal{G} := \{ z : |z| < 1 \} \). Then

\[
\frac{d\mu^G_{[-a,a]}(x)}{dx} = \frac{\tau}{\sqrt{(a^2 - x^2)(1 - a^2x^2)}}, \quad x \in [-a,a],
\]

where \( \tau > 0 \) is chosen so that \( \mu^G_{[-a,a]} \) has total mass 1. The Green’s function for \( \mathcal{G} \) with pole at \( t \) is

\[
g^G_G(z,t) = \log \frac{1 - \overline{tz}}{|z - t|}.
\]

The properties of the Green equilibrium potential then give [4, p. 132.]

\[
\int_{a}^{1} g^G_G(x,t) d\mu^G_{[-a,a]}(t) = V^G_{[-a,a]}, \quad x \in [-a,a].
\]

We now map \( \mathcal{H} \) conformally onto \( \mathcal{G} \) in such a way that \([b,1]\) is mapped onto \([-a,a]\) for some \( a > 0 \). Let us set, for the given \( b \),

\[
\phi(z) := \frac{z - \sqrt{b}}{z + \sqrt{b}}, \quad a := \frac{1 - \sqrt{b}}{1 + \sqrt{b}}.
\]

Then \( \phi \) maps \( \mathcal{H} \) conformally onto \( \mathcal{G} \), with \( \phi([b,1]) = [-a,a] \). Now let us set

\[
x = \phi(y); \quad t = \phi(s).
\]

Straightforward (but lengthy) calculations show that

\[
g^G_G(x,t) = g^G_G(\phi(y), \phi(s)) = \log \left| \frac{y + s}{y - s} \right| = g(y, s),
\]

and for some constant \( \kappa > 0 \),

\[
\left( \mu^G_{[-a,a]} \right)'(\phi(s)) \phi'(s) = \frac{\kappa}{\sqrt{(s^2 - a^2)(1 - s^2)}}, \quad s \in (b,1).
\]

Then (4.1) shows that for some constant \( A \),

\[
\int_{b}^{1} g(y,s) \left( \mu^G_{[-a,a]} \right)'(\phi(s)) \phi'(s)ds = A, \quad y \in [b,1].
\]

The uniqueness property of the Green equilibrium potential [4, Thm. 5.12, p.132] then shows that

\[
\left( \mu^H_{[b,1]} \right)'(s) = \left( \mu^G_{[-a,a]} \right)'(\phi(s)) \phi'(s), \quad s \in [b,1];
\]

\[
A = V^H_{[b,1]}.
\]
We then obtain (1.6) and the first equality in (1.7). Next, the property (4.2) with \( y = 1 \) gives

\[
(4.3)
\]

\[
F(b) = \frac{1}{H_{[b,1]}} = \int_b^1 \frac{dx}{\sqrt{(x^2 - b^2)(1 - x^2)}} / \int_b^1 \log \left| \frac{1 + x}{1 - x} \right| \frac{dx}{\sqrt{(x^2 - b^2)(1 - x^2)}}.
\]

Then (1.9) follows from [1, p.564, 4.297, no.9] and [1, p.246, 3.152, no.9]. This also gives the second equality in (1.7).

The proof of Theorem 2(c). We have already noted that \( C(E, i\mathbb{R}) \) increases as \( E \) increases, and hence \( F(b) \) is a decreasing function. To show that it is strictly increasing one assumes that \( F(b') = F(b) \), for some \( b' < b \), and “shifts left” \( \mu_{[b,1]} \) to a unit measure on \( [b', b' + 1 - b] \), thereby obtaining a contradiction as in Step 3 in the proof of Theorem 3(a). We proceed with the proof of (1.10). For \( b \in (0,1) \), let

\[
q := \exp \left( -\pi \frac{K'(b)}{K(b)} \right) = \exp \left( -\frac{\pi^2}{2} F(b) \right).
\]

Then there is the identity [1, p.924, 8.197, no. 3]

\[
4 \sqrt{q} \prod_{n=1}^{\infty} \left( \frac{1 + q^{2n}}{1 - q^{2n}} \right) = b.
\]

We see then that as \( b \to 0+ \),

\[
b = 4 \sqrt{q} (1 + o(1))
\]

\[
= 4 \exp \left( -\frac{\pi^2}{2} F(b) \right) (1 + o(1))
\]

and (1.10) follows.

Finally, for (1.11), we note that since \( b \to 1- \), we may introduce an extra factor of \( 2x \) in the numerator and denominator of (4.3). Then a substitution \( t = x^2 \) and standard integrals give the result. Indeed,

\[
F(b) = (1 + o(1)) \int_b^1 \frac{2x}{\sqrt{(x^2 - b^2)(1 - x^2)}} / \int_b^1 \log \left| \frac{1 + x}{1 - x} \right| \frac{2xdx}{\sqrt{(x^2 - b^2)(1 - x^2)}}
\]

\[
= (1 + o(1)) \int_{b^2}^1 \frac{dt}{\sqrt{(t - b^2)(1 - t)}} / \left[ \int_{b^2}^1 \log \frac{4}{1 - t} \frac{dt}{\sqrt{(t - b^2)(1 - t)}} + o(1) \right]
\]

\[
= (1 + o(1))/ \log \left( 1 - b^2 \right).
\]

Here we have used standard integrals in potential theory [4, pp.45-46]

\[
\int_{b^2}^1 \frac{dt}{\pi \sqrt{(t - b^2)(1 - t)}} = 1;
\]

\[
\int_{b^2}^1 \log |1 - t| \frac{dt}{\pi \sqrt{(t - b^2)(1 - t)}} = \log \left( \frac{1 - b^2}{4} \right).
\]

Then (1.11) follows.
REFERENCES


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