PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 12, Pages 3501–3510
S 0002-9939(01)06273-6
Article electronically published on June 27, 2001

INVARIANT SUBSPACES AND REPRESENTATIONS OF CERTAIN VON NEUMANN ALGEBRAS

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(Communicated by David R. Larson)

Abstract. Let (N, α, G) be a covariant system and let (π, U) be a covariant representation of (N, α, G) on a Hilbert space $\mathcal{H}$. In this note, we investigate the representation of the covariance algebra M and the σ-weakly closed subalgebra $\mathfrak{A}$ generated by $\pi(N)$ and $\{U_g\}_{g\geq 0}$ in the case of $G = \mathbb{Z}$ or $\mathbb{R}$ when there exists a pure, full, $\mathfrak{A}$-invariant subspace of $\mathcal{H}$.

1. Introduction

If G is a locally compact group and $\alpha : G \to \mathrm{Aut}(N)$ is a continuous homomorphism of G into the group of $*$-automorphisms of a von Neumann algebra N, then the triple (N, α, G) is called a covariant system. This notation was introduced by Doplicher, Kastler and Robinson in [2]. Covariant systems have turned out to be very interesting objects, both in theoretical physics and in mathematics. The covariant representation of (N, α, G) means a pair (π, U) consisting of a unitary representation U of G and a $*$-representation π of N with π and U operating over the same Hilbert space $\mathcal{H}$ such that

$$\pi(\alpha_g(x)) = U_g \pi(x) U_g^* \qquad (\forall x \in N, \forall g \in G).$$

The covariance algebra M of (N, α, G) is a von Neumann algebra generated by $\pi(N)$ and $\{U_g\}_{g\in G}$. When G has an order $\geq$, we consider the σ-weakly closed subalgebra $\mathfrak{A}$ of M which is generated by $\pi(N)$ and $\{U_g\}_{g\geq 0}$. The representation theory of M has been extensively studied by M. Takesaki in [16], M. Landstad in [5] and I. Raeburn in [13], among others. The covariance algebras provide us with a rich variety of examples of operator algebras. In this note, we consider the representation theory of M and $\mathfrak{A}$ in the particular case of $G = \mathbb{R}$ or $\mathbb{Z}$. By inspiring the scattering theory of Lax and Phillips in [6], we study the representation of M and $\mathfrak{A}$ to a crossed product and an analytic crossed product, respectively, using the theory of invariant subspace for $\mathfrak{A}$. Therefore, our setting is as follows.

Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ generated by a von Neumann algebra N and a unitary operator v satisfying $vNv^* = N$, and let $\mathfrak{A}$ be a σ-weakly closed subalgebra of M generated by N and the non-negative powers of v. At first, we prove that if there is a pure, full, $\mathfrak{A}$-invariant subspace $\mathfrak{M}$

Received by the editors September 16, 1999.
2000 *Mathematics Subject Classification.* Primary 46L10, 47L65; Secondary 46L40.
This work was supported in part by a Grant-in-Aid for Scientific Research, Japan Society for Promotion of Science.

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of $\mathcal{H}$, then M is $*$-isomorphic to a (discrete) crossed product $N \rtimes_\alpha \mathbb{Z}$ of N by a $*$-automorphism $\alpha = \mathrm{ad}v$, and that $\mathfrak{A}$ is simultaneously isomorphic to the analytic crossed product $N \rtimes_\alpha \mathbb{Z}_+$.

Similarly, we also consider the representation of a von Neumann algebra M_0 generated by a von Neumann algebra N and a strongly continuous one-parameter unitary group $\{u_t\}_{t\in\mathbb{R}}$ satisfying $u_t N u_t^* = N$ for every t in $\mathbb{R}$. Let $\mathfrak{B}$ be the σ-weakly closed subalgebra of M_0 generated by N and $\{u_t\}_{t>0}$. We prove that if there is a pure, full, $\mathfrak{B}$-invariant subspace of $\mathcal{H}$, then M_0 is $*$-isomorphic to a continuous crossed product, and that $\mathfrak{B}$ is simultaneously isomorphic to the related analytic crossed product.

Next, in §3, for a strongly continuous one-parameter unitary group $\{u_t\}_{t\in\mathbb{R}}$, we construct the unitary operator v by the Cayley transform of the infinitesimal generator of $\{u_t\}_{t\in\mathbb{R}}$. From this unitary v and $\{u_t\}_{t\in\mathbb{R}}$, we define the von Neumann algebras M, M_0 and the subalgebras $\mathfrak{A}$, $\mathfrak{B}$, respectively, as in §2, and we shall show that a closed subspace of $\mathcal{H}$ is pure, full, $\mathfrak{A}$-invariant if and only if it is pure, full, $\mathfrak{B}$-invariant (Proposition 3.2). Finally, we shall discuss the relation between a discrete crossed product and continuous crossed product.

2. Representation of certain von Neumann algebras to a crossed product

At first, we consider the representation of the covariance subalgebra in the case of $G = \mathbb{Z}$. Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ generated by a von Neumann algebra N and a unitary operator v satisfying $vNv^* = N$ and let $\mathfrak{A}$ be the σ-weakly closed subalgebra of M generated by N and non-negative powers of v. We now define the notion of invariant subspaces of $\mathcal{H}$ with respect to $\mathfrak{A}$ as in [8]–[10].

Definition 2.1. Let $\mathfrak{M}$ be a closed subspace of $\mathcal{H}$. We shall say that $\mathfrak{M}$ is: $\mathfrak{A}$-invariant, if $\mathfrak{A}\mathfrak{M} \subset \mathfrak{M}$; reducing, if $M\mathfrak{M} \subset \mathfrak{M}$; pure, if $\mathfrak{M}$ contains no non-trivial reducing subspace; and full, if the smallest reducing subspace containing $\mathfrak{M}$ is all of $\mathcal{H}$.

Since $vNv^* = N$, we put $\alpha(x) = vxv^*$ ($\forall x \in N$). We recall that the crossed product $N \rtimes_\alpha \mathbb{Z}$ of N by the $*$-automorphism group $\{\alpha^n\}_{n\in\mathbb{Z}}$ is the von Neumann algebra acting on the Hilbert space $\ell^2(\mathbb{Z}, \mathcal{H})$ generated by the operators $\pi_\alpha(x)$ ($\forall x \in N$) and S defined by the equations

$$\{\pi_\alpha(x)\xi\}(n) = \alpha^{-n}(x)\xi(n) \quad (\forall \xi \in \ell^2(\mathbb{Z}, \mathcal{H}),\ \forall n \in Z)$$

and

$$(S\xi)(n) = \xi(n-1) \quad (\forall \xi \in \ell^2(\mathbb{Z}, \mathcal{H}),\ \forall n \in \mathbb{Z}).$$

We note that the analytic crossed product $N \rtimes_\alpha \mathbb{Z}_+$ determined by N and α is defined to be the σ-weakly closed subalgebra of $N \rtimes_\alpha \mathbb{Z}$ generated by $\pi_\alpha(N)$ and the non-negative powers of S (cf. [8]–[10]). Let $\{\widehat{\alpha_t}\}_{t\in\mathbb{T}}$ be the $*$-automorphism group of $N \rtimes_\alpha \mathbb{Z}$ which is dual to $\{\alpha^n\}_{n\in\mathbb{Z}}$ in the sense of Takesaki [16]. Then we have

Theorem 2.2. *Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ generated by a von Neumann algebra N and a unitary operator v satisfying $vNv^* = N$ and let $\mathfrak{A}$ be the σ-weakly closed subalgebra of M generated by N and non-negative powers of v. Put $\alpha(x) = vxv^*$ ($\forall x \in N$). If there exists a pure, full, $\mathfrak{A}$-invariant* *subspace $\mathfrak{M}$ of $\mathcal{H}$, then there exist a $*$-automorphism group $\{\gamma_t\}_{t\in\mathbb{T}}$ of M and a $*$-isomorphism Φ from M onto $N \rtimes_\alpha \mathbb{Z}$ such that*

$$\Phi(x) = \pi_\alpha(x) \ (\forall x \in N), \quad \Phi(v) = S \quad \text{and} \quad \Phi \circ \gamma_t = \widehat{\alpha}_t \circ \Phi \ (\forall t \in \mathbb{T}).$$

Proof. Let $\mathfrak{M}$ be a pure, full, $\mathfrak{A}$-invariant subspace of $\mathcal{H}$. As in [8, Proposition 3.1], the subspace $\mathfrak{M}$ has the following properties:

$$\text{(i)} \ \ \mathfrak{A}\mathfrak{M} \subset \mathfrak{M}, \qquad \text{(ii)} \ \bigcap_{k>0} v^k\mathfrak{M} = \{0\}, \qquad \text{(iii)} \ \overline{\bigcup_{k<0} v^k\mathfrak{M}} = \mathcal{H}.$$

Putting $\mathfrak{F} = \mathfrak{M} \ominus v\mathfrak{M}$, we have the decomposition of the Hilbert space $\mathcal{H}$:

$$\mathcal{H} = \sum_{n=-\infty}^{\infty} \oplus v^n\mathfrak{F}.$$

Let P_n be the projection from $\mathcal{H}$ onto $v^n\mathfrak{F}$. Since $\mathfrak{F}$ is N-invariant and $vNv^* = N$, it follows that P_n belongs to the commutant of N and $\sum_{n=-\infty}^{\infty} P_n = I$. We define the one-parameter unitary group $\{V_t\}_{t\in\mathbb{T}}$ in the commutant of N defined by

$$V_t = \sum_{n=-\infty}^{\infty} e^{int}P_n \qquad (\forall t \in \mathbb{T}).$$

For each $t \in \mathbb{T}$, we see that

$$\begin{aligned} vV_tv^* &= \sum_{n=-\infty}^{\infty} e^{int}vP_nv^* = \sum_{n=-\infty}^{\infty} e^{int}P_{n+1} \\ &= \sum_{n=-\infty}^{\infty} e^{i(n-1)t}P_n = e^{-it}\sum_{n=-\infty}^{\infty} e^{int}P_n \\ &= e^{-it}V_t. \end{aligned}$$

Setting $\gamma_t(x) = V_t^*xV_t$ for each $t \in \mathbb{T}$ and $x \in M$, we see that $\{\gamma_t\}_{t\in\mathbb{T}}$ is a $*$-automorphism group of M such that $\gamma_t(v) = e^{-it}v$ $(\forall t \in \mathbb{T})$. By [15, 19.9 Theorem], we have this proposition. □

We now fix a pure, full, $\mathfrak{A}$-invariant subspace $\mathfrak{M}$ of $\mathcal{H}$. Then, by Theorem 2.2, there exist a $*$-automorphism group $\{\gamma_t\}_{t\in\mathbb{T}}$ of M and a $*$-isomorphism Φ from M onto $N \rtimes_\alpha \mathbb{Z}$ such that

$$\Phi(x) = \pi_\alpha(x) \ (\forall x \in N), \quad \Phi(v) = S \quad \text{and} \quad \Phi \circ \gamma_t = \widehat{\alpha}_t \circ \Phi \ (\forall t \in \mathbb{T}).$$

On the other hand, we take another pure, full, $\mathfrak{A}$-invariant subspace $\mathfrak{N}$ of $\mathcal{H}$. As in the proof of Theorem 2.2, there exists a unitary group $\{W_t\}_{t\in\mathbb{T}}$ in the commutant of N associated with $\mathfrak{N}$. Put $\rho_t(x) = W_txW_t^*$ $(\forall x \in M)$. By Theorem 2.2, there exists a $*$-isomorphism Ψ from M onto $N \rtimes_\alpha \mathbb{Z}$ such that

$$\Psi(x) = \pi_\alpha(x) \ (\forall x \in N), \quad \Psi(v) = S \quad \text{and} \quad \Psi \circ \rho_t = \widehat{\alpha}_t \circ \Psi \ (\forall t \in \mathbb{T}).$$

Therefore, we have

$$\Phi \circ \gamma_t \circ \Phi^{-1} = \widehat{\alpha}_t = \Psi \circ \rho_t \circ \Psi^{-1} \qquad (\forall t \in \mathbb{T}).$$

Since $\Phi^{-1} \circ \Psi$ is the identity map on M, we see that $\gamma_t = \rho_t$ $(\forall t \in \mathbb{T})$ and so $V_txV_t^* = W_txW_t^*$ $(\forall x \in M, \forall t \in \mathbb{T})$. Putting $A_t = W_t^*V_t$ $(\forall t \in \mathbb{T})$, then A_t is the unitary operator in the commutant of M and, for all $s, t \in \mathbb{T}$, we have

$$A_tV_t^*A_sV_t = W_t^*V_tV_t^*W_s^*V_sV_t = W_{t+s}^*V_{t+s} = A_{t+s}.$$

Thus, we have

$$A_{t+s} = A_t\gamma_{-t}(A_s) \qquad (\forall s, t \in \mathbb{T}). \tag{2.1}$$

The unitary family $\{A_t\}_{t\in\mathbb{T}}$ in the commutant of M satisfying (2.1) is called a cocycle with respect to $\mathfrak{M}$. Therefore we have

Theorem 2.3. *Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ generated by a von Neumann algebra N and a unitary operator v satisfying $vNv^* = N$ and let $\mathfrak{A}$ be the σ-weakly closed subalgebra of M generated by N and non-negative powers of v. Let $\mathfrak{M}$ be a pure, full, $\mathfrak{A}$-invariant subspace of $\mathcal{H}$. If $\mathfrak{N}$ is another pure, full, $\mathfrak{A}$-invariant subspace of $\mathcal{H}$, then there exists a cocycle $\{A_t\}_{t\in\mathbb{T}}$ with respect to $\mathfrak{M}$. Conversely, if $\{A_t\}_{t\in\mathbb{T}}$ is a cocycle with respect to $\mathfrak{M}$, then there exists a pure full $\mathfrak{A}$-invariant subspace of $\mathcal{H}$ with the cocycle $\{A_t\}_{t\in\mathbb{T}}$.*

Proof. We only prove the converse. Assume that $\{A_t\}_{t\in\mathbb{T}}$ is a cocycle with respect to $\mathfrak{M}$. Put $W_t = V_t^* A_t$ ($\forall t \in \mathbb{T}$). Then we can easily check that $\{W_t\}_{t\in\mathbb{T}}$ is a unitary group in the commutant of N. Let $W_t = \sum_{n=-\infty}^{\infty} e^{-int} Q_n$ ($\forall t \in \mathbb{T}$) be the spectral decomposition of W_t. Putting $\mathfrak{N} = \sum_{n=0}^{\infty} \oplus Q_n\mathcal{H}$, then $\mathfrak{N}$ is a pure, full, $\mathfrak{A}$-invariant subspace of $\mathcal{H}$. In fact, for each $x \in N$ and $\xi \in \mathcal{H}$, we have $xQ_n\xi \in Q_n\mathcal{H}$ because W_t and A_t belong to N'. Moreover, we see that

$$W_t vQ_n\xi = V_t^* A_t vQ_n\xi = V_t^* vA_tQ_n\xi = e^{-it}vV_t^*A_tQ_n\xi$$

$$= e^{-it}vW_tQ_n\xi = e^{-i(n+1)t}vQ_n\xi.$$

It follows that $vQ_n\xi \in Q_{n+1}\mathcal{H}$ ($\forall n \in \mathbb{N}$). Therefore $\mathfrak{N}$ is $\mathfrak{A}$-invariant. This completes the proof. □

We next consider the case that $G = \mathbb{R}$. Let N be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $\{u_t\}_{t\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group on $\mathcal{H}$ satisfying the condition $u_tN{u_t}^* = N$ ($\forall t \in \mathbb{R}$). Let M_0 be the von Neumann algebra generated by N and $\{u_t\}_{t\in\mathbb{R}}$, and let $\mathfrak{B}$ be the σ-weakly closed subalgebra of M_0 generated by N and $\{u_t\}_{t\geq 0}$.

Definition 2.4. *Let $\mathfrak{M}$ be a closed subspace of $\mathcal{H}$. We shall say that $\mathfrak{M}$ is: $\mathfrak{B}$-invariant, if $\mathfrak{B}\mathfrak{M} \subset \mathfrak{M}$; reducing, if $M_0\mathfrak{M} \subset \mathfrak{M}$; pure, if $\mathfrak{M}$ contains no non-trivial reducing subspace; and full, if the smallest reducing subspace containing $\mathfrak{M}$ is all of $\mathcal{H}$.*

Since u_t and N satisfy the condition $u_tN{u_t}^* = N$ ($\forall t \in \mathbb{R}$), we can define the σ-weakly continuous $*$-automorphism β_t of N implemented by the unitary operator u_t ($\forall t \in \mathbb{R}$). Recall that the continuous crossed product $N \rtimes_\beta \mathbb{R}$ of N by $\{\beta_t\}_{t\in\mathbb{R}}$ is the von Neumann algebra acting on a Hilbert space $L^2(\mathbb{R}, \mathcal{H})$ generated by the operators $\pi_\beta(x)$ and $\lambda(t)$ defined by the equations, for $\forall x \in N$,

$$\{\pi_\beta(x)\xi\}(t) = \beta_{-t}(x)\xi(t) \quad (\forall \xi \in L^2(\mathbb{R}, \mathcal{H}),\ \forall t \in \mathbb{R})$$

and

$$\{\lambda(t)\xi\}(s) = \xi(s-t) \quad (\forall \xi \in L^2(\mathbb{R}, \mathcal{H}),\ \forall s, t \in \mathbb{R}).$$

The analytic crossed product $N \rtimes_\beta \mathbb{R}_+$ determined by N and $\{\beta_t\}_{t\in\mathbb{R}}$ is defined to be the σ-weakly closed subalgebra of $N \rtimes_\beta \mathbb{R}$ generated by $\pi_\beta(N)$ and $\{\lambda(t)\}_{t\geq 0}$.

Theorem 2.5. *Let M_0 be the von Neumann algebra generated by N and a unitary group $\{u_t\}_{t\in\mathbb{R}}$ satisfying $u_tNu_t^* = N(\forall t \in \mathbb{R})$ and let $\mathfrak{B}$ be the σ-weakly closed subalgebra of M_0 generated by N and $\{u_t\}_{t\geq 0}$. If there exists a pure, full, $\mathfrak{B}$-invariant subspace $\mathfrak{M}$ of $\mathcal{H}$, then there exist a one-parameter group $\{\theta_t\}_{t\in\mathbb{R}}$ of $*$-automorphisms on M_0 and a $*$-isomorphism Θ from M_0 onto $N \rtimes_\beta \mathbb{R}$ such that*

$$\Theta(x) = \pi_\beta(x)\ (\forall x \in N), \quad \Theta(u_t) = \lambda(t) \quad \textit{and} \quad \Theta\circ\theta_t = \widehat{\beta}_t \circ \Theta\ (\forall t \in \mathbb{R}),$$

where $\{\widehat{\beta}_t\}_{t\in\mathbb{R}}$ is the $*$-automorphism of $N \rtimes_\beta \mathbb{R}$ which is dual to $\{\beta_t\}_{t\in\mathbb{R}}$. Further Θ maps $\mathfrak{A}$ onto $N \rtimes_\beta \mathbb{R}_+$.

Proof. Let $\mathfrak{M}$ be a pure, full, $\mathfrak{B}$-invariant subspace of $\mathcal{H}$. Then it is clear that $\mathfrak{M}$ has the following properties:

$$\text{(i)}\ \ \mathfrak{B}\mathfrak{M} \subset \mathfrak{M}, \qquad \text{(ii)}\ \ \bigcap_{t>0} u_t\mathfrak{M} = \{0\}, \qquad \text{(iii)}\ \ \overline{\bigcup_{t<0} u_t\mathfrak{M}} = \mathcal{H}.$$

Let P_t be the projection from $\mathcal{H}$ onto $u_t\mathfrak{M}\,(\forall t \in \mathbb{R})$. Since $N\mathfrak{M} \subset \mathfrak{M}$ and $N = u_tNu_t^*$, for each $t \in \mathbb{R}$, $u_t\mathfrak{M}$ is N-invariant, and so P_t belongs to N'. Since $\mathfrak{M}$ is pure and full, we can easily check that the projections $\{P_t\}_{t\in\mathbb{R}}$ are a spectral family. Thus, we obtain the strongly continuous unitary group $\{U_t\}_{t\in\mathbb{R}}$ of N' defined by

$$U_t = \int_{\mathbb{R}} e^{-i\lambda t} dP_\lambda \quad (\forall t \in \mathbb{R}).$$

Since $u_s^*P_\lambda u_s = u_{\lambda-s}$, we have for every $s, t \in \mathbb{R}$,

$$\begin{aligned} u_s^*U_tu_s &= \int_{\mathbb{R}} e^{-i\lambda t} du_s^*P_\lambda u_s = \int_{\mathbb{R}} e^{-i\lambda t} dP_{\lambda-s} \\ &= e^{-ist}\int_{\mathbb{R}} e^{-i\lambda t} dP_\lambda = e^{-ist}U_t. \end{aligned}$$

Therefore $U_txU_t^* = x\ (\forall x \in N\ \forall t \in \mathbb{R})$ and $U_tu_sU_t^* = e^{-ist}u_s\ (\forall s, t \in \mathbb{R})$. Thus the $*$-automorphism group $\{\theta_t\}_{t\in\mathbb{R}}$ of M_0 defined by $\theta_t(x) = U_txU_t^*\ (\forall x \in M_0,\ \forall t \in \mathbb{R})$ satisfies $\theta_t(u_s) = e^{-ist}u_s\ (\forall s, t \in \mathbb{R})$. Therefore we have the proposition from [15, 19.9 Theorem]. □

We now fix a pure, full, $\mathfrak{B}$-invariant subspace $\mathfrak{M}$ of $\mathcal{H}$. As in the case of $G = \mathbb{Z}$, we can consider the notion of cocycle with respect to $\mathfrak{M}$. By Theorem 2.5, there exist a one-parameter group $\{\theta_t\}_{t\in\mathbb{R}}$ of $*$-automorphisms on M_0 which is implemented by the unitary group $\{U_t\}_{t\in\mathbb{T}}$ and a $*$-isomorphism Θ from M_0 onto $N \rtimes_\beta \mathbb{R}$ such that

$$\Theta(x) = \pi_\beta(x)\ (\forall x \in N), \quad \Theta(u_t) = \lambda(t) \quad \text{and} \quad \Theta\circ\theta_t = \widehat{\beta}_t \circ \Theta\ (\forall t \in \mathbb{R}).$$

We take another pure, full, $\mathfrak{B}$-invariant subspace $\mathfrak{N}$ of $\mathcal{H}$. As in the proof of Theorem 2.5, there exists a one-parameter unitary group $\{W_t\}_{t\in\mathbb{R}}$ associated with $\mathfrak{N}$. Put $\sigma_t(x) = W_txW_t^*$ for any $x \in M_0$. Then, by Theorem 2.5, there exists a $*$-isomorphism Π from M_0 onto $N \rtimes_\beta \mathbb{R}$ such that

$$\Pi(x) = \pi_\beta(x)\ (\forall x \in N), \quad \Pi(u_t) = \lambda(t) \quad \text{and} \quad \Pi\circ\sigma_t = \widehat{\beta}_t \circ \Pi\ (\forall t \in \mathbb{R}).$$

Put $B_t = W_t^*U_t\ (\forall t \in \mathbb{R})$. Then B_t is a unitary operator in the commutant of M_0 satisfying $B_{t+s} = B_t\theta_{-t}(B_s)\ (\forall s, t \in \mathbb{R})$. We shall say that the unitary family $\{B_t\}_{t\in\mathbb{R}}$ is a cocycle with respect to $\mathfrak{M}$. As in Theorem 2.3, we have the following:

Theorem 2.6. *Let M_0 be the von Neumann algebra generated by N and a unitary group $\{u_t\}_{t\in\mathbb{R}}$ satisfying $u_t N u_t^* = N (\forall t \in \mathbb{R})$ and let $\mathfrak{B}$ be the σ-weakly closed subalgebra of M_0 generated by N and $\{u_t\}_{t\geq 0}$. Let $\mathfrak{M}$ be a pure, full, $\mathfrak{B}$-invariant subspace of $\mathcal{H}$. If we take another pure, full, $\mathfrak{B}$-invariant subspace $\mathfrak{N}$ of $\mathcal{H}$, then there exists a cocycle $\{B_t\}_{t\in\mathbb{R}}$ with respect to $\mathfrak{M}$. Conversely, if $\{B_t\}_{t\in\mathbb{R}}$ is a cocycle with respect to $\mathfrak{M}$, then there exists a pure, full, $\mathfrak{B}$-invariant subspace of $\mathcal{H}$ with the cocycle $\{B_t\}_{t\in\mathbb{R}}$.*

3. Representation of the continuous case $G = \mathbb{R}$

Let N be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $\{u_t\}_{t\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group on $\mathcal{H}$. We consider the von Neumann algebra M_0 generated by N and $\{u_t\}_{t\in\mathbb{R}}$, and the σ-weakly closed subalgebra $\mathfrak{B}$ of M_0 generated by N and $\{u_t\}_{t\geq 0}$. Let A be the infinitesimal generator of $\{u_t\}_{t\in\mathbb{R}}$ defined by

$$A\xi = \lim_{t\to 0+} \frac{u_t\xi - \xi}{t} \qquad (\forall \xi \in D(A)),$$

where $D(A)$ is the set of all elements for which the limit exists. It is well-known that the Cayley transform v of A, that is, $v = (I + A)(I - A)^{-1}$, is a unitary operator on $\mathcal{H}$. For the unitary operator v, let M be the von Neumann algebra generated by N and v, and let $\mathfrak{A}$ be the σ-weakly closed subalgebra generated by N and the non-negative powers of v as in §2.

The next proposition embodies an important idea of [6] and is the key result of our approach. For completeness, we give the proof.

Proposition 3.1. *Keep the notations as above. Let $\mathfrak{M}$ be a closed subspace of $\mathcal{H}$. Then $\mathfrak{M}$ is $\mathfrak{A}$-invariant if and only if $\mathfrak{M}$ is $\mathfrak{B}$-invariant.*

Proof. We only need to prove that a closed subspace $\mathfrak{M}$ of $\mathcal{H}$ is v-invariant if and only if $\mathfrak{M}$ is u_t-invariant for all $t > 0$. Let A be a infinitesimal generator of $\{u_t\}_{t\in\mathbb{R}}$. Setting $R(\lambda, A) = (\lambda I - A)^{-1}$, we see that

$$v = R(1, A) + AR(1, A) = 2R(1, A) - I, \tag{3.1}$$

and making use of the Laplace transform representation of $R(1, A)$, we have

$$v\xi = 2\int_0^\infty e^{-t} u_t \xi \, dt - \xi \qquad \forall \xi \in \mathcal{H}. \tag{3.2}$$

Let $\xi \in \mathfrak{M}$. Since $u_t\xi \in \mathfrak{M}$ for all $t > 0$, we have $v\xi\ \mathfrak{M}$.

To prove the converse, we first show that $R(\lambda, A)\mathfrak{M} \subset \mathfrak{M}$ for all $\lambda > 0$. Now the resolvent is analytic on the resolvent set and can be expanded in a power series as follows:

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n \left\{R(\lambda_0, A)\right\}^{n+1}, \tag{3.3}$$

valid for $|\lambda_0 - \lambda||R(\lambda_0, A)| < 1$. For $\lambda_0 > 0$, we have $|R(\lambda_0, A)| \leqq \frac{1}{\lambda_0}$ so that the above series holds for $|\lambda - \lambda_0| < \lambda_0$. It follows from this expansion that $R(\lambda_0, A)\mathfrak{M} \subset \mathfrak{M}$ implies $R(\lambda, A)\mathfrak{M} \subset \mathfrak{M}$ for all $|\lambda - \lambda_0| < \lambda_0$. Assuming $v\mathfrak{M} \subset \mathfrak{M}$, one infers from (3.1) that $R(1, A)\mathfrak{M} \subset \mathfrak{M}$ and hence by a stepwise process using (3.3) that

$R(\lambda, A)\mathfrak{M} \subset \mathfrak{M}$ for all $\lambda > 0$. Hence for ξ in $\mathfrak{M}$ and η in the orthogonal complement of $\mathfrak{M}$

$$0 = \langle R(\lambda, A)\xi, \eta\rangle = \int_0^\infty e^{-\lambda t}\langle u_t\xi, \eta\rangle\, dt \qquad (\forall\lambda > 0).$$

By the Laplace transform uniqueness theorem, we have $\langle u_t\xi, \eta\rangle = 0$ and hence $u_t\mathfrak{M} \subset \mathfrak{M}$ for all $t > 0$. This completes the proof. □

From Proposition 3.1, we have the following:

Proposition 3.2. *Keep the notation as above. Then*
(i) $M = M_0$. *Moreover, if* M_0 *has a separating vector, then* $\mathfrak{A} = \mathfrak{B}$.
(ii) *A closed subspace* $\mathfrak{M}$ *of* $\mathcal{H}$ *is pure, full,* $\mathfrak{A}$*-invariant if and only if* $\mathfrak{M}$ *is pure, full,* $\mathfrak{B}$*-invariant.*

Proof. We only prove (i). By Proposition 3.1, a closed subspace $\mathfrak{M}$ is reducing for M if and only if $\mathfrak{M}$ is reducing for M_0. Hence the commutant of M is equal to the commutant of M_0, and so $M = M_0$.

We next prove that $\mathfrak{A} = \mathfrak{B}$. To do this, we need the following notations. If $\mathfrak{C}$ is an algebra of M_0 and $\mathfrak{L}$ is a lattice of projections in $B(\mathcal{H})$, then we write

$$\mathrm{Lat}\mathfrak{C} = \{P \in B(\mathcal{H})_p \mid (I - P)TP = 0,\ \forall T \in \mathfrak{C}\}$$

and

$$\mathrm{Alg}\mathfrak{L} = \{T \in B(\mathcal{H}) \mid (I - P)TP = 0,\ \forall P \in \mathfrak{L}\},$$

where $B(\mathcal{H})_p$ is the set of all projections in $B(\mathcal{H})$.

By Proposition 3.1, it is clear that $\mathrm{Lat}\mathfrak{A} = \mathrm{Lat}\mathfrak{B}$. Since $\mathrm{AlgLat}\mathfrak{B}$ contains $\mathfrak{B}$, we have the following inclusions:

$$\mathfrak{A} \subset \mathfrak{B} \subset \mathrm{AlgLat}\mathfrak{A}.$$

If $\mathfrak{A} \subsetneqq \mathrm{AlgLat}\mathfrak{A}$, then there exists a non-zero element $x \in \mathrm{AlgLat}\mathfrak{A}$ such that $x \notin \mathfrak{A}$. Hence there is a normal linear functional ϕ in the predual $(M_0)_*$ of M_0 such that $\phi(x) = 1$ and $\phi|_{\mathfrak{A}} = 0$. Since M_0 has a separating vector, by [7, Corollary 1.13.7], there are non-zero vectors ξ and η in $\mathcal{H}$ such that $\phi(y) = \langle y\xi, \eta\rangle$ $(\forall y \in M)$. Hence, for each $y \in \mathfrak{A}$, we have

$$\langle y\xi, \eta\rangle = \phi(y) = 0.$$

This implies that $[\mathfrak{A}\xi] \perp \eta$, where $[\mathfrak{A}\xi]$ denotes the closed subspace of $\mathcal{H}$ spanned by $\mathfrak{A}\xi$. Since $[\mathfrak{A}\xi] \in \mathrm{Lat}\mathfrak{A}$ and $x \in \mathrm{AlgLat}\mathfrak{A}$, $x\xi$ belongs to $[\mathfrak{A}\xi]$. This implies that

$$1 = \phi(x) = \langle x\xi, \eta\rangle = 0.$$

This is a contradiction and so $\mathfrak{A} = \mathfrak{B}$. This completes the proof. □

We remark that if the commutant of a von Neumann algebra M_0 is properly infinite, then M_0 always has a separating vector (cf. [1, Corollary 11]).

We now consider the case that N and u_t satisfy the condition $u_tNu_t^* = N$ $(\forall t \in \mathbb{R})$. If there exists a pure, full, $\mathfrak{B}$-invariant subspace $\mathfrak{M}$ of $\mathcal{H}$, then, by Theorem 2.5, M_0 is $*$-isomorphic to a continuous crossed product. Moreover, from

Proposition 3.2(ii), $\mathfrak{M}$ is also a pure, full, $\mathfrak{A}$-invariant subspace. Hence, by Theorem 2.2, if N and v satisfy the condition $vNv^* = N$, then M is $*$-isomorphic to a discrete crossed product. So it is natural to ask when N and v satisfy the condition $vNv^* = N$. Recall that the unitary operator v has the following form:

$$v\xi = 2\int_0^\infty e^{-t}u_t\xi dt - \xi \qquad (\forall \xi \in \mathcal{H}).$$

It is clear that if, for every $t \in \mathbb{R}$, u_t belongs to the commutant of N, then v is also in N'. In this case, v and N satisfy the condition $vNv^* = N$. But, in general, v and N do not satisfy the condition $vNv^* = N$. In fact, we can give the following:

Example 3.3. Let N be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $\{\beta_t\}_{t\in\mathbb{R}}$ be a $*$-automorphism group of N such that there exists a t_0 in $\mathbb{R}$ such that β_{t_0} is outer. Recall that a continuous crossed product $N \rtimes_\beta \mathbb{R}$ is the von Neumann algebra generated by $\pi_\beta(N)$ and $\{\lambda(t)\}_{t\in\mathbb{R}}$. For each $x \in N$ and $t \in \mathbb{R}$, it is clear that

$$\pi_\beta(\beta_t(x)) = \lambda(t)\pi_\beta(x)\lambda(t)^*$$

and

$$\pi_\beta(N) = \lambda(t)\pi_\beta(N)\lambda(t)^* \qquad (\forall t \in \mathbb{R}).$$

For the unitary group $\{\lambda(t)\}_{t\in\mathbb{R}}$, we obtain the unitary operator v defined by the form

$$v\xi = 2\int_0^\infty e^{-t}\lambda(t)\xi dt - \xi \qquad (\forall \xi \in L^2(\mathbb{R}, \mathcal{H})).$$

By choosing an appropriate representation for $N \rtimes_\beta \mathbb{R}$, we shall assume that $N \rtimes_\beta \mathbb{R}$ has a separating vector. In this case, by Proposition 3.2, the von Neumann algebra generated by $\pi_\beta(N)$ and v coincides with $N \rtimes_\beta \mathbb{R}$, and the σ-weakly closed subalgebra generated by $\pi_\beta(N)$ and the non-negative powers of v also coincides with $N \rtimes_\beta \mathbb{R}_+$. Hence, by Theorem 2.2, if $\pi_\beta(N)$ and v satisfy the condition $v\pi_\beta(N)v^* = \pi_\beta(N)$, then there is a $*$-isomorphism Φ from $N \rtimes_\beta \mathbb{R}$ onto $N \rtimes_\alpha \mathbb{Z}$ such that $\Phi(N \rtimes_\beta \mathbb{R}_+) = N \rtimes_\alpha \mathbb{Z}_+$ for some $*$-automorphism α of N. Since there exists a faithful normal canonical conditional expectation of $N \rtimes_\alpha \mathbb{Z}$ onto $\pi_\alpha(N)$ (cf. [8]–[10]), there is a faithful normal conditional expectation of $N \rtimes_\beta \mathbb{R}$ onto $\pi_\beta(N)$. However, Katayama showed in [4, Theorem 3.5] that if there exists a t_0 in $\mathbb{R}$ such that β_{t_0} is outer, then there does not exist any normal conditional expectation of $N \rtimes_\beta \mathbb{R}$ onto $\pi_\beta(N)$, which is a contradiction. Hence, $\pi_\beta(N)$ and v do not satisfy the condition $v\pi_\beta(N)v^* = \pi_\beta(N)$.

Finally, we discuss the relation between a continuous crossed product and a discrete crossed product.

Theorem 3.4. *If a crossed product $N \rtimes_\beta \mathbb{R}$ admits a separating vector (for example, $N \rtimes_\beta \mathbb{R}$ is properly infinite), then the following two conditions are equivalent:*

(i) *There exists a $*$-isomorphism Φ form $N \rtimes_\beta \mathbb{R}$ onto $N \rtimes_\alpha \mathbb{Z}$ such that $\Phi(N \rtimes_\beta \mathbb{R}_+) = N \rtimes_\alpha \mathbb{Z}_+$ for some $*$-automorphism α of N.*

(ii) *β_t is inner for all $t \in \mathbb{R}$.*

Proof. (i) $\Rightarrow$ (ii) Since Φ is the $*$-isomorphism satisfying $\Phi(N \rtimes_\beta \mathbb{R}_+) = N \rtimes_\alpha \mathbb{Z}_+$, we have $\Phi(\pi_\beta(N)) = \pi_\alpha(N)$. Since there exists a normal conditional expectation of $N \rtimes_\alpha \mathbb{Z}$ onto $\pi_\alpha(N)$, there also exists a normal conditional expectation of $N \rtimes_\beta \mathbb{R}$ onto $\pi_\beta(N)$. Hence, by [4, Theorem 3.6], we have that β_t is inner for each $t \in \mathbb{R}$.

(ii) $\Rightarrow$ (i) Since β_t is inner for all $t \in \mathbb{R}$, there exists a unitary operator v_t in N such that β_t is implemented by the unitary operator v_t. Putting $u_t = \lambda(t)\pi_\beta(v_t)^*$ for all t in $\mathbb{R}$, we can show that $\{u_t\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group in $\pi_\beta(N)'$. Moreover, we see that the von Neumann algebra generated by $\pi_\beta(N)$ and $\{u_t\}_{t\in\mathbb{R}}$ is equal to $N \rtimes_\beta \mathbb{R}$. Similarly, the σ-weakly closed subalgebra of $N \rtimes_\beta \mathbb{R}$ generated by $\pi_\beta(N)$ and $\{u_t\}_{t>0}$ is $N \rtimes_\beta \mathbb{R}_+$. For the unitary group $\{u_t\}_{t\in\mathbb{R}}$, we can construct the unitary operator v on $L^2(\mathbb{R}, \mathcal{H})$ as follows:

$$v\xi = 2\int_0^\infty e^{-t} u_t \xi \, dt - \xi \qquad (\forall \xi \in L^2(\mathbb{R}, \mathcal{H})).$$

Since $N \rtimes_\beta \mathbb{R}$ admits a separating vector, by Proposition 3.2, $N \rtimes_\beta \mathbb{R}$ is also generated by $\pi_\beta(N)$ and v, and $N \rtimes_\beta \mathbb{R}_+$ is generated by $\pi_\beta(N)$ and the non-negative powers of v. Since, for every $t \in \mathbb{R}$, u_t belongs to the commutant of $\pi_\beta(N)$, v is also in $\pi_\beta(N)'$. Thus v and $\pi_\beta(N)$ satisfy the condition $v\pi_\beta(N)v^* = \pi_\beta(N)$. Putting $\mathfrak{M} = L^2(\mathbb{R}_+, \mathcal{H})$, it is clear that $\mathfrak{M}$ is a pure, full, $N \rtimes_\beta \mathbb{R}_+$-invariant subspace of $L^2(\mathbb{R}, \mathcal{H})$. For each $t \in \mathbb{R}$, we have

$$u_t\mathfrak{M} = \lambda(t)\pi_\beta(v_t)^*\mathfrak{M} \subset \lambda(t)\mathfrak{M} = \pi_\beta(v_t)^*\lambda(t)\pi_\beta(v_t)\mathfrak{M} \subset u_t\mathfrak{M}.$$

It follows that $u_t\mathfrak{M} = \lambda(t)\mathfrak{M}$ ($\forall t \in \mathbb{R}$), and so $\mathfrak{M}$ is pure and full for $\{u_t\}_{t\in\mathbb{R}}$. Therefore, by Proposition 3.2, $\mathfrak{M}$ is also pure and full for v. Thus, by Theorem 2.2, we have (i). This completes the proof. □

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