A PROBLEM OF PRESCRIBING GAUSSIAN CURVATURE ON $S^2$

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Abstract. A class of functions $K(x) = K(x_1, x_2, x_3)$ and the corresponding solutions of
\[ \Delta u + K(x)e^{2u} = 1 \]
are obtained as a special case of the solutions of
\[ \Delta^m u + K(x)e^{au} = f(x), \quad x = (x_1, x_2, \ldots, x_n), \]
where $\Delta^m$ is defined as $\Delta(\Delta^{m-1})$.

On the two-sphere $S^2 = \{ x \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1 \}$ with metric $ds_0^2 = dx_1^2 + dx_2^2 + dx_3^2$, if the metric is conformally changed to $ds^2 = e^{2u}ds_0^2$, then the Gaussian curvature $K(x)$ of the new metric is determined by the equation
\[
\Delta u + K(x)e^{2u} = 1, \quad x \in S^2, \tag{1}
\]
where $\Delta$ denotes the Laplacian relative to the metric $ds_0^2$.

The question raised by L. Nirenberg is: Which functions $K(x)$ can be prescribed so that (1) has a solution $u$?

Integrating (1) over the whole sphere, we get
\[
\int_{S^2} Ke^{2u} \, d\mu = 4\pi,
\]
where $d\mu$ denotes the surface measure on $S^2$. Thus, an obvious necessary condition is that $K$ must be positive somewhere. Another necessary condition was found by Kazdan and Warner [3] via integration by parts. Moser [4] proved that if $K$ is an even function on $S^2$, then (1) has a solution. In [1], Cheng and Smoller considered the case of rotationally symmetric functions $K$.

The purpose of this note is to show that a class of functions $K$ and the corresponding solutions can be produced by means of elementary arguments. We first prove the following result:

Theorem 1. If $a > 0$ is an arbitrary constant and the functions $K(x)$ and $f(x)$ are positive and such that
\[
\Delta^m \ln \left( \frac{K(x)}{f(x)} \right) = 0, \quad x = (x_1, x_2, \ldots, x_n), \tag{2}
\]
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then the solution of
\begin{equation}
\Delta^m u + K(x)e^{au} = f(x) \tag{3}
\end{equation}
is given by
\begin{equation}
u = \frac{1}{a} \ln \left( \frac{f(x)}{K(x)} \right). \tag{4}
\end{equation}

Proof. We divide (3) by $f(x)$ and write it as
\begin{equation}
\frac{1}{f(x)} \Delta^m u + e^{au} \ln(K(x)) = 1. \tag{5}
\end{equation}
Now by the change of variables
\begin{equation}
au + \ln \left( \frac{K(x)}{f(x)} \right) = \nu \tag{6}
\end{equation}
and with the help of (2), we obtain
\begin{equation}
\Delta^m u = \frac{1}{a} \Delta^m \nu. \tag{7}
\end{equation}
Thus the equation (5) reduces to
\begin{equation}
\frac{1}{af(x)} \Delta^m \nu + e^\nu = 1. \tag{8}
\end{equation}
Clearly, $\nu = 0$ is a solution of (8). We then get from (6)
\begin{equation}
u = \frac{1}{a} \ln \left( \frac{f(x)}{K(x)} \right). \tag{9}
\end{equation}
This completes the proof.

If in Theorem 1 we choose $m = 1$, $a = 2$, $n = 3$ and $f(x) = 1$, then we have

**Theorem 2.** If the function $K(x) = K(x_1, x_2, x_3)$ is positive and $\ln(K(x))$ is harmonic, then $u = \frac{1}{2} \ln\left( \frac{1}{K(x)} \right)$ is a solution of (1).

**Remark.** Evidently $u = \frac{1}{2} \ln\left( \frac{f(x)}{K(x)} \right)$ is also a solution of the nonlinear Dirichlet problem $\Delta u + K(x)e^{au} = f(x)$ in $D$ and $u = 0$ on $\partial D$ as long as $f(x) = K(x)$ on $\partial D$ and $\ln\left( \frac{K(x)}{f(x)} \right)$ is harmonic in $D$.

**References**


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