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THE HOPF CONJECTURE FOR MANIFOLDS WITH LOW COHOMOGENEITY OR HIGH SYMMETRY RANK

THOMAS PÜTTMANN AND CATHERINE SEARLE

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ABSTRACT. We prove that the Euler characteristic of an even-dimensional compact manifold with positive (nonnegative) sectional curvature is positive (nonnegative) provided that the manifold admits an isometric action of a compact Lie group G with principal isotropy group H and cohomogeneity k such that $k - (\operatorname{rank} G - \operatorname{rank} H) \leq 5$. Moreover, we prove that the Euler characteristic of a compact Riemannian manifold M^{4l+4} or M^{4l+2} with positive sectional curvature is positive if M admits an effective isometric action of a torus T^l , i.e., if the symmetry rank of M is $\geq l$.

The Gauss-Bonnet theorem states that the Euler characteristic of a closed surface M is determined by its total curvature:

$$\chi(M) = 2\pi \int_M K.$$

In particular, if the curvature is positive (nonnegative), the Euler characteristic of the surface is positive (nonnegative). H. Hopf [H] generalized in 1925 the Gauss-Bonnet theorem to even-dimensional hypersurfaces of Euclidean space and posed in the early 1930's (according to Berger [Be]) the question of whether a compact even-dimensional manifold which admits a metric of positive (nonnegative) sectional curvature must have positive (nonnegative) Euler characteristic.

Indications that the Hopf conjecture should be true came from the generalizations of the Gauss-Bonnet theorem: Fenchel [F] and Allendoerfer [A] proved in 1940 independently a Gauss-Bonnet formula for submanifolds of Euclidean space with arbitrary codimension. Three years later Allendoerfer and Weil [AW] (using E. Cartan's result that any Riemannian manifold can locally be embedded into Euclidean space) established the theorem in its final intrinsic version: For any even-dimensional manifold the Euler characteristic can be obtained by integrating a function derived from the curvature tensor, the so-called Gauss-Bonnet integrand. Chern [C1] gave the first intrinsic proof of this theorem in 1944.

After this, many attempts were made to settle the stronger algebraic Hopf conjecture: A curvature tensor with positive (nonnegative) sectional curvature yields

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a positive (nonnegative) Gauss-Bonnet integrand. Milnor (unpublished, see [C2]) actually proved the algebraic Hopf conjecture in dimension 4, but finally in 1976 Geroch [G] found curvature tensors with positive sectional curvature in all even dimensions ≥ 6 that do not provide a positive Gauss-Bonnet integrand.

A different approach to the Hopf conjecture is to consider first Riemannian manifolds that have a certain amount of symmetry. Hopf himself and Samelson [HS] proved in 1941 that the Euler characteristic of every compact homogeneous space G/H is nonnegative and positive if and only if rank $G = \operatorname{rank} H$ holds. The key observations in their proof are that a regular element in the compact Lie group G has at most finitely many fixed points in the homogeneous space G/H and that each of these fixed points has fixed point index 1. In 1972 Wallach [W] then showed that for any even-dimensional homogeneous space of positive sectional curvature one actually has rank $G = \operatorname{rank} H$. Therefore, the Hopf conjecture is true for homogeneous spaces. Recently, Podestà and Verdiani [PV] proved among other things that the Hopf conjecture also holds for cohomogeneity one manifolds. We show that much weaker symmetry assumptions are sufficient.

Theorem 1. Let M be a compact even-dimensional Riemannian manifold with positive (nonnegative) sectional curvature. Let $G \times M \to M$ be an isometric action of a compact Lie group G with principal isotropy group H and cohomogeneity k. If

$$k - (\operatorname{rank} G - \operatorname{rank} H) \le 5,$$

then M has positive (nonnegative) Euler characteristic.

Proof. If M is nonorientable, then the action of G can be lifted to an action by orientation preserving isometries (see [Br, Corollary I.9.4]) on the orientable double covering space of M. We can therefore assume that M is orientable.

We consider the fixed point set

$$M^T = \{ p \in M \mid \psi(p) = p \text{ for all } \psi \in T \}$$

of a maximal torus T of G. Note that M^T is equal to the fixed point set of a generating element $\psi \in T$, i.e., of an element ψ with $\overline{\{\psi^m \mid m \in \mathbb{Z}\}} = T$. If the fixed point set M^T is empty, then there exists a Killing field without zeros. This implies that M cannot have positive sectional curvature (by Berger's theorem, see e.g. [W]) and that the Euler characteristic is zero. We can therefore assume that M^T is nonempty. Now each of the finitely many components of M^T is a totally geodesic submanifold of M with even codimension and the Euler characteristic of M is the sum of the Euler characteristics of the components (see [K, Chapter II]). By Theorem IV.5.3 of [Br] each component N of M^T satisfies

$$\dim N \le k - (\operatorname{rank} G - \operatorname{rank} H) \le 5$$

Since N is even-dimensional and the Hopf conjecture holds in dimensions 2 and 4 we are done. $\hfill \Box$

Note that $k - (\operatorname{rank} G - \operatorname{rank} H) \leq \dim M - 2 \operatorname{rank} G$ (see [Br, Corollary IV.5.4]) if the action of G is effective. Hence we get as a special case of Theorem 1 that any compact even-dimensional Riemannian manifold M^{2l+4} with positive (nonnegative) sectional curvature has positive (nonnegative) Euler characteristic if M^{2l+4} admits an effective isometric torus action $T^l \times M \to M$. Using a result from [GS] we can improve this result in the case of positive sectional curvature.

Theorem 2. Let M^{4l+2} or M^{4l+4} be a Riemannian manifold with positive sectional curvature that admits an almost effective isometric T^l -action. Then for any $T^1 \subset T^l$ the Euler characteristics of all the components of the fixed point set $\operatorname{Fix}(M;T^1)$ are positive. In particular, $\chi(M) > 0$.

Proof. As above we can assume that M is orientable in order to have even-dimensional fixed point sets. The proof is done by induction. For l = 0 note that the Hopf conjecture is true in dimensions 2 and 4. For the induction step consider M^{4l+6} or M^{4l+8} with an almost effective T^{l+1} -action. Consider any circle $T^1 \subset T^{l+1}$ and any component N of its fixed point set in M. We will show that $\chi(N) > 0$. Choose a $\tilde{T}^1 \subset T^{l+1}$ such that $N \subset \operatorname{Fix}(M; \tilde{T}^1)$ and such that the component \tilde{N} of $\operatorname{Fix}(M; \tilde{T}^1)$ that contains N has maximal dimension. It follows from the slice theorem and from the representation theory of tori that $T^{l} = T^{l+1}/\tilde{T}^{1}$ acts almost effectively on \tilde{N} . If $\operatorname{codim} \tilde{N} \geq 4$, then we know from the induction assumption that in particular N as a component of $Fix(\tilde{N}; T^1)$ has positive Euler characteristic and hence we are done. In the case where $\operatorname{codim} N = 2$ we know from [GS] that M is differentiably covered by a sphere or a complex projective space. From results of Bredon [Br, Chapters III and VII] it follows that all the components of the fixed point set of any circle action on M have positive Euler characteristic. Thus in particular N has positive Euler characteristic.

After this paper was accepted for publication we were informed that Xiaochun Rong obtained Theorem 2 independently (see [R]). In his paper he gives many more results on the topology of positively curved manifolds with high symmetry rank.

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FAKULTÄT FÜR MATHEMATIK, RUHR-UNIVERSITÄT BOCHUM, D-44780 BOCHUM, GERMANY *E-mail address*: puttmann@math.ruhr-uni-bochum.de

INSTITUTO DE MATEMATICAS, UNIDAD CUERNAVACA-UNAM, APARTADO POSTAL 273-3, ADMON. 3, CUERNAVACA, MORELOS, 62251, MEXICO

 $E\text{-}mail\ address:\ \texttt{csearleQmatcuer.unam.mx}$

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