PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 1, Pages 33–38
S 0002-9939(01)06059-2
Article electronically published on April 26, 2001

PROJECTIONLESS C*-ALGEBRAS

LLOLSTEN KAONGA

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We give a sufficient condition for a unital C*-algebra to have no nontrivial projections, and we apply this result to known examples and to free products. We also show how questions of existence of projections relate to the norm-connectedness of certain sets of operators.

We call a unital C*-algebra *projectionless* if it has no projections other than 0 or 1. A nonunital C*-algebra is *projectionless* if it has no nonzero projections. It is easy to check that a nonunital C*-algebra is projectionless if and only if its unitization is projectionless. Hence in this paper we restrict ourselves to unital C*-algebras.

Our main result (Theorem 1) provides a technique for showing that a C*-algebra is projectionless. The key idea appears in the paper of M.-D. Choi [1] who proved that the group C*-algebra of the free group on n generators is projectionless. Choi's technique was used by K. McClanahan [7] who proved that L. Brown's noncommutative unitary C*-algebra U_n^{nc} is projectionless. An intriguing sidelight of our result is that questions about existence of projections in C*-algebras translate to purely operator-theoretic questions concerning connectedness. This paper is an outgrowth of the authors's Ph.D. dissertation [6].

Suppose $\mathcal{A}$ is a C*-algebra. Let $\kappa(\mathcal{A}) = \aleph_0 m$, where m is the minimal cardinality of a subset of $\mathcal{A}$ that generates $\mathcal{A}$ as a C*-algebra. If H is a Hilbert space and $\mathcal{A}$ is unital, let $Rep(\mathcal{A}, H)$ denote the set of all unital representations from $\mathcal{A}$ into $B(H)$. We topologize $Rep(\mathcal{A}, H)$ with the topology of pointwise norm convergence.

Theorem 1. *Suppose $\mathcal{A}$ is a unital C*-algebra and H is a Hilbert space with dimension at least $\kappa(\mathcal{A})$. The following are equivalent:*

(1) *$\mathcal{A}$ is projectionless.*
(2) *There is a connected subset of $Rep(\mathcal{A}, H)$ containing an injective representation and a representation π such that $\pi(\mathcal{A})$ contains no nontrivial projection.*
(3) *There is a connected subset of $Rep(\mathcal{A}, H)$ containing an injective representation ρ and a representation π such that $\pi(\mathcal{A})$ contains no nontrivial projection and such that $\rho(\mathcal{A})$ contains no nonzero elements with rank less than $\kappa(\mathcal{A})$.*

Proof. (1) $\Rightarrow$ (2). Since $\dim H \geq \kappa(\mathcal{A})$, there is an injective element of $Rep(\mathcal{A}, H)$. Hence any connected component of $Rep(\mathcal{A}, H)$ containing an injective representation will satisfy (2).

Received by the editors February 4, 1997 and, in revised form, May 31, 2000.
1991 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.

(2) $\Rightarrow$ (3). Let M be a direct sum of $\kappa(\mathcal{A})$ copies of H, and let $U : H \to M$ be unitary. For each $\pi \in Rep(\mathcal{A}, H)$, let π' denote a direct sum of $\kappa(\mathcal{A})$ copies of π acting on M. The mapping $\alpha : Rep(\mathcal{A}, H) \to Rep(\mathcal{A}, H)$ defined by $\alpha(\pi(\cdot)) = U^*\pi'(\cdot)U$ is continuous and therefore maps connected sets to connected sets. From this fact the implication (2) $\Rightarrow$ (3) follows immediately.

(3) $\Rightarrow$ (1). Suppose K is a connected subset of $Rep(\mathcal{A}, H)$ containing elements π and ρ as in (3). Let $\mathcal{B}$ be the C*-algebra of all bounded continuous functions ϕ from K into $B(H)$ such that $\phi(\pi) \in \pi(\mathcal{A})$. If ϕ is a projection in $\mathcal{B}$, then $\phi(K)$ is a norm-connected collection of projections in $B(H)$ containing $\phi(\pi)$. But $\phi(\pi)$ is trivial since $\phi(p) \in \pi(\mathcal{A})$. Since $\{0\}$ and $\{1\}$ are maximal connected sets of projections in $B(H)$, it is clear that ϕ must be a trivial projection. Hence $\mathcal{B}$ contains no nontrivial projections. However, if we define $\tau : \mathcal{A} \to \mathcal{B}$ by $\tau(a)(\sigma) = \sigma(a)$, then τ is a *-homomorphism, and since $\|\tau(a)\| \geq \|\tau(a)(\rho)\| = \|a\|$, we see that τ is injective. Hence $\mathcal{A}$ contains no nontrivial projections. □

Remark 1. The set E of injective elements of $Rep(\mathcal{A}, H)$ whose range contains no nonzero elements with rank less than $\kappa(\mathcal{A})$ is connected. To see this suppose $\rho \in E$. Then $\rho_1 \in E$ if and only if $rank\rho(a) = rank\rho_1(a)$ for each $a \in \mathcal{A}$. It follows from [4] that $\rho_1 \in E$ if and only if ρ and ρ_1 are approximately unitarily equivalent, i.e., ρ_1 is in the closure of the set of elements of $Rep(\mathcal{A}, H)$ that are unitarily equivalent to ρ. Since the set of unitary operators in $B(H)$ is norm-connected, it follows that E is connected.

Suppose $\mathcal{G}$ is a generating set for the C*-algebra $\mathcal{A}$ with minimal cardinality. The mapping $\pi \mapsto \pi\,|\mathcal{G}$ from $Rep(\mathcal{A}, H)$ to $\prod_{x\in\mathcal{G}} B(H)$ is an embedding with the norm topology on $B(H)$ and the product topology on the product. Let $\mathcal{E}(\mathcal{A}, \mathcal{G}, H)$ denote the image of $Rep(\mathcal{A}, H)$ in $\prod_{x\in\mathcal{G}} B(H)$ under this embedding, and call an element $\pi\,|\mathcal{G}$ *faithful* if the representation π is injective. With this identification, Theorem 1 has the following translation.

Corollary 1. *Suppose $\mathcal{G}$ is a generating set for the C*-algebra $\mathcal{A}$ and H is a Hilbert space with dimension $\kappa(\mathcal{A})$. The following are equivalent:*

(1) *$\mathcal{A}$ has no nontrivial projections.*
(2) *There is a connected subset of $\mathcal{E}(\mathcal{A}, \mathcal{G}, H)$ containing a faithful element and an element whose generated C*-algebra contains no nontrivial projections.*

Remark 2. The preceding corollary is particularly useful when the C*-algebra $\mathcal{A}$ is defined in terms of a generating set and relations on that set. In this case $\mathcal{E}(\mathcal{A}, \mathcal{G}, H)$ is precisely the element of $\prod_{x\in\mathcal{G}} B(H)$ that satisfies the defining relations. When $\mathcal{A}$ is finitely generated, $\mathcal{E}(\mathcal{A}, \mathcal{G}, H)$ can naturally be identified with a set of operators on the Hilbert space $\prod_{x\in\mathcal{G}} H$.

The following examples show how we can apply our results to obtain known results. Later we shall obtain new applications.

Example 1. We first obtain the result of Choi [1]. Let F be the free group with free generating set $\mathcal{G}$, and let $\mathcal{A}$ be the group C*-algebra of F. Then $\phi \in \mathcal{E}(\mathcal{A}, \mathcal{G}, H)$ if and only if $\phi(x)$ is unitary for each $x \in \mathcal{G}$. Given such a ϕ, we can choose a function $\psi : \mathcal{G} \to B(H)$ such that $\psi(x) = \psi(x)^*$ and $e^{i\psi(x)} = \phi(x)$ for every $x \in \mathcal{G}$. For $t \in [0, 1]$ we define $\phi_t(x) = \exp((1 - t)i\psi(x))$. Hence $\{\phi_t\}$ defines a path in $\mathcal{E}(\mathcal{A}, \mathcal{G}, H)$ joining ϕ to the constant 1 function whose generated C*-algebra is just the scalars and contains no nontrivial projections. Hence $\mathcal{A}$ is projectionless.

Example 2. Next we recover results of McClanahan [7]. Suppose n is a positive integer and $\mathcal{A}$ is the C*-algebra generated by $\mathcal{G} = \{a_{ij} : 1 \le i, j \le n\}$ subject to the relation that the $n \times n$ matrix (a_{ij}) is unitary. In [7] $\mathcal{A}$ is denoted by U_n^{nc}. Then $\phi \in \mathcal{E}(\mathcal{A}, \mathcal{G}, H)$ if and only if the matrix $(\phi(a_{ij})) = U$ is unitary in $\mathcal{M}_n(B(H))$. Choose a selfadjoint element $A \in \mathcal{M}_n(B(H))$ so that $e^{iA} = U$. For each $t \in [0,1]$ define $\phi_t : \mathcal{G} \to B(H)$ so that $(\phi_t(a_{ij})) = e^{i(1-t)A}$. Then $\{\phi_t\}$ is a path in $\mathcal{E}(\mathcal{A}, \mathcal{G}, H)$ joining ϕ to ϕ_1 whose generated C*-algebra is just the scalars and contains no nontrivial projections. Hence $\mathcal{A}$ is projectionless.

Example 3. We now consider a result of [3]. Suppose $\mathcal{G} = \{a_i : i \in I\}$ is a nonempty collection, $\{\lambda_i : i \in I\}$ is a collection of positive real numbers and $\mathcal{A}$ is the C*-algebra generated by $\mathcal{G}$ subject to the relations $\|a_i\| \le \lambda_i$ for every $i \in I$. If $\pi \in Rep(\mathcal{A}, H)$, define π_t for $0 \le t \le 1$ by $\pi_t(a_i) = (1-t)\pi(a_i)$ for each $i \in I$. Then $\{\pi_t\}$ is a path in $Rep(\mathcal{A}, H)$ joining π to π_1, whose range consists of scalars and thus has no nontrivial projections. Hence $\mathcal{A}$ is projectionless.

We next show how our results apply to free products. Suppose $\{\mathcal{A}_i : i \in I\}$ is a nonempty family of unital C*-algebras. The free product $\mathcal{A} = *_{i\in I}\mathcal{A}_i$ is a unital C*-algebra containing $\bigcup_{i\in I}\mathcal{A}_i$ (with the same unit) as a generating set satisfying the defining property: if H is a Hilbert space and, for each $i \in I$, $\pi_i \in Rep(\mathcal{A}_i, H)$, there is a representation $\pi = *_{i\in I}\pi_i \in Rep(\mathcal{A}, H)$ such that, for each $i \in I$, $\pi\,|\mathcal{A}_i = \pi_i$. The mapping $\alpha : \prod_{i\in I} Rep(\mathcal{A}_i, H) \to Rep(\mathcal{A}, H)$ defined by $\alpha(\gamma) = *_{i\in I}\gamma(i)$ is a surjective homeomorphism, since $\mathcal{A}$ is generated by the union of the $\mathcal{A}_i$'s.

Theorem 2. *Suppose* $\{\mathcal{A}_i : i \in I\}$ *is a nonempty family of unital C*-algebras,* H *is a Hilbert space and, for each* $i \in I$, π_i *is an injective element of* $Rep(\mathcal{A}_i, H)$. *If, for each* $i \in I$, *there is a* ρ_i *in the connected component of* π_i *in* $Rep(\mathcal{A}_i, H)$ *such that* $*_{i\in I}\rho_i\left(*_{i\in I}\mathcal{A}_i\right) = C^*\left(\bigcup_{i\in I}\rho_i(\mathcal{A}_i)\right)$ *is projectionless, then* $*_{i\in I}\mathcal{A}_i$ *is projectionless.*

Proof. First note that none of the hypotheses are changed if, for some cardinal m, we replace all of the π_i's and ρ_i's with direct sums of m copies of themselves. Hence we can assume that $\dim H \ge \kappa(*_{i\in I}\mathcal{A}_i)$ and that, for each $i \in I$ and each nonzero $x \in \mathcal{A}_i$, we have $rank(\pi_i(x)) = \dim H$. We next choose a faithful representation $\tau \in Rep(*_{i\in I}\mathcal{A}_i, H)$ such that, for every nonzero $y \in *_{i\in I}\mathcal{A}_i$, we have $rank(\tau(y)) = \dim H$. For each $i \in I$ let K_i be the connected component of π_i in $Rep(\mathcal{A}_i, H)$. Then $\alpha\left(\prod_{i\in I} K_i\right)$ is connected in $Rep(*_{i\in I}\mathcal{A}_i, H)$ and contains both $*_{i\in I}\pi_i$ and $*_{i\in I}\rho_i$. However, it follows from the remark following Theorem 1 that $\tau\,|\mathcal{A}_i \in K_i$ for each $i \in I$. Hence $\tau \in \alpha\left(\prod_{i\in I} K_i\right)$. Now Theorem 1 applies and we conclude $*_{i\in I}\mathcal{A}_i$ is projectionless. □

Remark 3. The fact that each π_i is injective in no way implies that $*_{i\in I}\pi_i$ is injective; for example, each π_i could be the inclusion map into some tensor product of the $\mathcal{A}_i$'s. The condition on $*_{i\in I}\rho_i$ holds if the range of each ρ_i is the scalars. Hence if any tensor product of the $\mathcal{A}_i$'s is projectionless, then the free product is projectionless. The group C*-algebra of the free group on n generators is isomorphic to the free product of n copies of $C(\mathbb{T})$ ($\mathbb{T}$ is the unit circle in $\mathbb{C}$) and since the tensor product of these algebras is isomorphic to $C(\mathbb{T}^n)$, which is projectionless, we again obtain Choi's result [1].

If $\mathcal{A}$ is a unital C*-algebra and n is a positive integer, Phillips [8, Section 1] defines a C*-algebra $W_n(\mathcal{A})$ so that if $\mathcal{A}$ is defined in terms of a generating set $\mathcal{G} = \{a_\lambda : \lambda \in \Lambda\}$ and a set of defining relations $\mathcal{R}$, the $W_n(\mathcal{A})$ is the C*-algebra with generators $\{b_{ij}(\lambda) : 1 \le i, j \le n, \lambda \in \Lambda\}$ so that the $n \times n$ matrices $\{B_\lambda : \lambda \in \Lambda\}$ satisfy the relations in $\mathcal{R}$. Thus McClanahan's U_n^{nc} is $W_n(C(\mathbb{T}))$. Note that if H^n denotes a direct sum of n copies of H, then we can identify $B(H^n)$ with $\mathcal{M}_n(B(H))$. This induces a natural mapping $\beta : Rep(\mathcal{A}, H^n) \to Rep(W_n(\mathcal{A}), H)$ so that, for every $\lambda \in \Lambda, \pi(a_\lambda)$ is the $n \times n$ matrix $(\beta(\pi)(b_{ij}(\lambda)))$. It is clear that β is actually a homeomorphism. There are cases in which $\mathcal{A}$ is not projectionless, but for which $W_n(\mathcal{A})$ is projectionless for some values of n.

Proposition 1. *Suppose $\mathcal{A}$ is a unital C*-algebra and $n \in \mathbb{N}$. If, for some Hilbert space H with $\dim H \ge \kappa(\mathcal{A})$, there is a connected subset of $Rep(\mathcal{A}, H)$ containing an injective element π and an element ρ so that $\rho(\mathcal{A})$ is isomorphic to a C*-subalgebra of $\mathcal{M}_n$, then $W_n(\mathcal{A})$ is projectionless. In fact, there is a faithful representation of $W_n(\mathcal{A})$ whose connected component K in $Rep(W_n(\mathcal{A}), H)$ contains a representation whose range is the scalars.*

Proof. Since H is isomorphic to the direct sum H^n of n copies of H, we can assume that $\pi, \rho \in Rep(\mathcal{A}, H^n)$. We can also assume, as in preceding proofs, that, for every $x \in \mathcal{A}$, $rank\pi(x)$ and $rank(\rho(x))$ only take on values 0 and $\dim H$. It follows from the assumption on $\rho(\mathcal{A})$ that ρ is unitarily equivalent to a representation whose range is contained in $\mathcal{M}_n(\mathbb{C}I)$ (where I is the identity operator on H). Since the set of unitaries in $B(H^n)$ is connected, we can assume that $\rho(\mathcal{A}) \subset \mathcal{M}_n(\mathbb{C}I)$. Next suppose $\tau \in Rep(W_n(\mathcal{A}), H)$ is injective and τ is unitarily equivalent to a direct sum of $\kappa(\mathcal{A})$ copies of τ. Thus, if $\tau = \beta(\sigma)$, then σ is also unitarily equivalent to a direct sum of $\kappa(\mathcal{A})$ copies of itself. It is clear that the mapping $a_\lambda \to (b_{ij}(\lambda))$ extends to a *-isomorphism from $\mathcal{A}$ into $\mathcal{M}_n(W_n(\mathcal{A}))$; it follows that σ must be injective. Hence, by the remark following Theorem 1, π, and hence ρ, is in the connected component of $Rep(\mathcal{A}, H^n)$ containing σ. It follows that $\beta(\rho)$ is in the connected component in $Rep(W_n(\mathcal{A}), H)$ of $\beta(\sigma) = \tau$. However, the range of $\beta(\rho)$ is the scalars. Thus, by Theorem 1, $W_n(\mathcal{A})$ is projectionless. $\square$

Corollary 2. *If X is a compact Hausdorff space with m connected components and $1 \le m \le n < \infty$, then $W_n(C(X))$ is projectionless.*

Proof. We first show that if X is connected and H is any Hilbert space, then $Rep(C(X), H)$ is connected. Suppose X is connected, $x_0 \in X$, and H is any Hilbert space. Define $\pi_0 : C(X) \to B(H)$ by $\pi_0(f) = f(x_0)I$, where I is the identity operator on H. Suppose next that $k \in \mathbb{N}$ and $\mathcal{Q} = \{Q_1, \ldots, Q_k\}$ is an orthogonal family of projections in $B(H)$ whose sum is I. We define a map $\gamma_{k,\mathcal{Q}} : X^k \to Rep(C(X), H)$ by

$$\gamma_{k,\mathcal{Q}}(x_1, \ldots, x_k)(f) = \sum_{j=1}^{k} f(x_j)Q_j.$$

It is clear that $\gamma_{k,\mathcal{Q}}$ is continuous and, since X^k is connected, $\gamma_{k,\mathcal{Q}}(X^k)$ is connected in $Rep(C(X), H)$ and contains $\pi_0 = \gamma_{k,\mathcal{Q}}(x_0, \ldots, x_0)$. Hence $\bigcup_{k,\mathcal{Q}} \gamma_{k,\mathcal{Q}}(X^k)$ is connected in $Rep(C(X), H)$. However, it follows from the spectral theorem that $\bigcup_{k,\mathcal{Q}} \gamma_{k,\mathcal{Q}}(X^k)$ is dense in $Rep(C(X), H)$. Thus $Rep(C(X), H)$ is connected.

Now suppose the connected components of X are $X_1, \ldots, X_m$ and $y_j \in X_j$ for $1 \le j \le m$. Let $P_j = \chi_{X_j} \in C(X)$ be the characteristic function of X_j for $1 \le j \le m$. Suppose H is an infinite-dimensional Hilbert space and $\pi : C(X) \to$

$B(H^m) = \mathcal{M}_n(B(H))$ is defined by

$$\pi(f) = \begin{pmatrix} f(y_1)I & 0 & \cdots & 0 \\ 0 & f(y_2)I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(y_m)I \end{pmatrix} = diag(f(y_1)I, \ldots, f(y_m)I),$$

where I is the identity operator on H. Suppose that $\rho : C(X) \to B(H^m)$ is any unital faithful representation such that ρ is unitarily equivalent to a direct sum of $\dim H$ copies of itself. Then there is a unitary operator U such that

$$U^*\rho\left(\sum_{1\le j\le m} \zeta_j P_j\right)U = diag(\zeta_1 I, \ldots, \zeta_m I)$$

for all scalars $\zeta_1, \ldots, \zeta_m$. Since $\rho(C(X))$ is commutative, there are representations $\rho_1, \ldots, \rho_m : C(X) \to B(H)$ such that $\rho = diag(\rho_1, \ldots, \rho_m)$, i.e.,

$$\rho(f) = diag(\rho_1(P_1 f), \ldots, \rho_m(P_m f))$$

for every $f \in C(X)$. It is clear that we can view each ρ_j as a representation on $C(X_j)$, and by the preceding paragraph, the connected component of ρ_j in $Rep(C(X), H)$ contains the representation π_j defined by $\pi_j(f) = f(y_j)I$. It follows that the connected component of ρ in $Rep(C(X), H^m)$ contains $diag(\pi_1, \ldots, \pi_m) = \pi$. Also $\pi(C(X))$ is isomorphic to a C*-subalgebra of $\mathcal{M}_n$ for every integer $n \ge m$. Hence by Proposition 1, $W_n(C(X))$ is projectionless. □

Questions and Comments. 1. If X is a compact subset of $\mathbb{C}^k$, then $C(X)$ is isomorphic to the C*-algebra generated by $a_1, \ldots, a_k$ subject to the relations that the a_j's are commuting normal elements and that the joint spectrum of $(a_1, \ldots, a_k)$ is contained in X. In this case, the first paragraph of the proof of the preceding corollary showed that if X is connected and H is a Hilbert space, then the set of all tuples $(A_1, \ldots, A_k)$ of commuting normal operators with joint spectrum in X is norm-connected. This shows how questions about projectionless C*-algebras relate to topological questions about Hilbert space operators.

2. It is natural to look at properties of operators and define the unital C*-algebra generated by elements with these properties as the defining relations, and ask if the C*-algebra is projectionless. For example, the C*-algebra generated by a with the relation that a is an isometry (i.e., $a^*a = 1$) is not projectionless, since aa^* is a nontrivial projection. An operator T is *subnormal* if it is the restriction of a normal operator to an invariant subspace. An excellent reference for subnormal operators is the book of J. B. Conway [2]. It was shown by Halmos and Bram [2, Theorem 1.9] that an operator T is subnormal if and only if, for each positive integer n, the $n \times n$ matrix

$$\left((T^*)^j T^i\right) \ge 0.$$

Suppose X is a connected compact subset of $\mathbb{C}$. Is the set $\mathcal{S}(X)$ of all subnormal operators on ℓ^2 with spectrum in X norm connected? If X is a disc, is the answer yes? What about the case where X is an annulus centered at the origin? If T is subnormal and λ is in the boundary of the spectrum of T, then T is approximately unitarily equivalent to $T \oplus \lambda I$ on $\ell^2 \oplus \ell^2$, and since the normal operators with spectrum in X are connected, it follows that the connected component in $\mathcal{S}(X)$ containing T contains an operator unitarily equivalent to $T \oplus N$ where N is a normal operator whose spectrum is X. Hence there is no Fredholm index obstruction to $\mathcal{S}(X)$ being connected.

3. One must be careful when defining C*-algebras with relations. For example, the C*-algebra generated by a with the defining relation that a is nilpotent or that the spectrum of a is $\{0\}$ does not qualify since neither of these relations is closed even in the presence of a norm boundedness condition. For a precise discussion of defining relations, the reader can consult [5].

4. In Corollary 2 what happens when $n < m$? If $m = 2$, and $n = 1$, then $W_n(C(X))$ is just $C(X)$, which contains a nontrivial projection. What if $m = 3$ and $n = 2$? A general question related to this one is: If $\mathcal{A}$ is a unital C*-algebra, $n \in \mathbb{N}$ and $\mathcal{M}_n(\mathcal{A})$ contains $n + 1$ nonzero mutually orthogonal projections, must $\mathcal{A}$ have a nontrivial projection?

5. If $\mathcal{A}$ and $\mathcal{B}$ are projectionless unital C*-algebras, is their minimal tensor product projectionless? An affirmative answer to this question would imply that the free product of projectionless C*-algebras is projectionless.

References

[1] M.-D. Choi, *The full C*-algebra of the free group on two generators*, Pacific J. Math. **87** (1980) 41-48. MR **82b:**46069

[2] J. B. Conway, *The theory of subnormal operators*, Amer. Math. Soc. Surveys and Monographs **36** (1991). MR **92h:**47026

[3] J. Froelich and H. Salas, *The C*-algebra generated by a C*-universal operator*, preprint.

[4] D. W. Hadwin, *Nonseparable approximate equivalence*, Trans. Amer. Math. Soc. **266** (1981) 203-231. MR **82e:**46078

[5] D. W. Hadwin, L. Kaonga, and B. Mathes, *Noncommutative continuous functions*, preprint.

[6] L. Kaonga, *Decomposable functions and universal C*-algebras*, Ph.D. Thesis, University of New Hampshire (1994).

[7] K. McClanahan, *C*-algebras generated by elements of a unitary matrix*, J. Funct. Anal. 107 (1992). MR **93j:**46062

[8] N. C. Phillips, *Classifying algebras for K-theory of σ-C*-algebras*, Canadian J. Math. **XLI** 6 (1989) 1021-1089. MR **91h:**46119

Aprisma Technologies, Durham, New Hampshire 03824

E-mail address: `lkaonga@aprisma.com`