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ON THE CONNECTIVITY OF THE JULIA SET OF A FINITELY GENERATED RATIONAL SEMIGROUP

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Abstract. In this paper we show that the Julia set $J(G)$ of a finitely generated rational semigroup G is connected if the union of the Julia sets of generators is contained in a subcontinuum of $J(G)$. Under a nonseparating condition, we prove that the Julia set of a finitely generated polynomial semigroup is connected if its postcritical set is bounded.

1. Introduction

Let S denote a Riemann surface and $\mathcal{H}(S)$ the set of nonconstant holomorphic mappings on S. Under composition, $\mathcal{H}(S)$ is a semigroup. We study the subsemigroups of $\mathcal{H}(S)$ from the point of view of dynamics. Given a subsemigroup G of $\mathcal{H}(S)$, the domain of normality for G is called the *Fatou set* $F(G)$, and its complement $S\backslash F(G)$ is called the *Julia set* $J(G)$. One can think of this as a dichotomy by the dynamical behavior of the semigroup at random. Here, we will deal with the case when S is the Riemann sphere $\hat{\mathbf{C}}$.

By a *rational semigroup* we mean a subsemigroup G of $\mathcal{H}(\hat{\mathbf{C}})$ containing at least one element with degree ≥ 2. If a rational semigroup consists only of polynomials, we call it *a polynomial semigroup*. If a rational semigroup G can be freely generated by its finite elements $\{f_1, \cdots, f_n\}$, we write $G = \langle f_1, \cdots, f_n\rangle$, where n is a positive integer, and call it *a finitely generated rational semigroup*.

This setting is a generalization of the study of the iteration of one rational function. As a matter of fact, a principal aim for the study of rational semigroups is to see how far the classical theory of the iteration of a rational function applies in this more general setting. W. Zhou and F. Ren [5] initiated the study of the dynamics of rational semigroups by introducing the concept of random iteration. There have been some basic results along this line. For instance, the closure of the set of repelling periodic points is the Julia set [1].

In this paper, we deal with the connectivity of the Julia set. It is well known that the Julia set of a polynomial is connected if and only if the orbits of its critical points are bounded [2]. In the case of polynomial semigroup, however, we no longer have the corresponding conclusion. As a counterexample, we take $p_1(z) = z^2 + 4z$ and $p_2(z) = z^2 - 4z + 4$. Then $J(p_1) = [-4, 0]$ and $J(p_2) = [0, 4]$.

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Since $J(p_1) \cup J(p_2) = [-4, 4]$ is connected, it follows from Theorem 1 that $J(\langle p_1, p_2 \rangle)$ is connected (see Figure 1). On the other hand, for the critical point -2 of p_1, $\{p_2^{\circ k} \circ p_1(-2)\}_{k=1}^{\infty}$ is not bounded. Each $p_2^{\circ k} \circ p_1$ has -2 as its critical point. Therefore, the boundedness of the postcritical orbits set is not necessary for the Julia set to be connected. Nevertheless, it is a sufficient condition as is proved in Theorem 2 below.

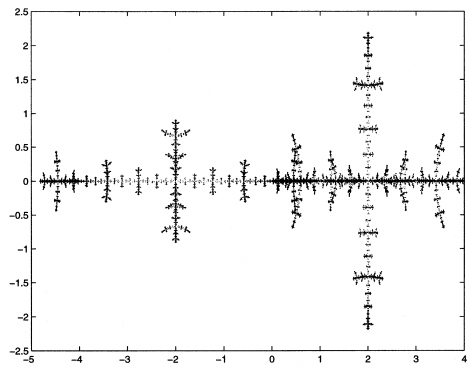

FIGURE 1. The Julia set $J(\langle p_1, p_2 \rangle)$ is connected, where $p_1(z) = z^2 + 4z$ and $p_2(z) = z^2 - 4z + 4$.

2. Preliminaries

By a *continuum*, we mean a nonempty compact connected metric space. A *subcontinuum* is a continuum as a subspace. The following lemma comes from continuum theory (see for example Nadler [3]). This general property will be used in the proof of our main result.

Lemma 1. *Let X and Y be two continua and let $f : X \longrightarrow Y$ be an open map. If B is a subcontinuum of Y, then each component A of $f^{-1}(B)$ is mapped onto B by f.*

For simplicity, we follow the definitions in [4].

Definition 1. Let G be a rational semigroup and z a point of $\hat{\mathbf{C}}$. The *backward orbit* $O^-(z)$ of z and the *set of exceptional points* $E(G)$ are defined by $O^-(z) = \{w \in \hat{\mathbf{C}} | \text{there is some } g \in G \text{ such that } g(w) = z\}$ and $E(G) = \{z \in \hat{\mathbf{C}} | \mathrm{Card} O^-(z) \leq 2\}$ respectively.

Definition 2. Let G be a rational semigroup. The *postcritical set* $P(G)$ is defined by $P(G) = \overline{\bigcup_{g \in G} \{\text{critical values of } g\}}$. Let G be a polynomial semigroup. The *finite postcritical set* $P^*(G)$ is defined by $P^*(G) = P(G) \backslash \{\infty\}$.

We abuse notation and write $F(g)$ for $F(\langle g \rangle)$ and $J(g)$ similarly. The following three lemmas are recalled from Hinkkanen and Martin [1].

Lemma 2. *Let G be a rational semigroup. For any $g \in G$, $g(F(G)) \subset F(G)$, $g^{-1}(J(G)) \subset J(G)$, $F(G) \subset F(g)$, and $J(g) \subset J(G)$.*

Lemma 3. *Let G be a rational semigroup. Then $E(G) = \{z \in \hat{\mathbf{C}} | Card O^-(z) < \infty\}$ and $Card E(G) \leq 2$.*

Lemma 4. *Let G be a rational semigroup. If a point z is not in $E(G)$, then $J(G) \subset \overline{O^-(z)}$. In particular, taking a point $z \in J(G) \backslash E(G)$, we have $\overline{O^-(z)} = J(G)$.*

3. The connectivity of the Julia set

First, we give a general result for the connectivity of the Julia set of a finitely generated rational semigroup.

Theorem 1. *Let $G = \langle f_1, \cdots, f_n \rangle$ be a finitely generated rational semigroup with $\{f_1, \cdots, f_n\}$ as the generator system. We assume that the Julia set of each generator is nonempty. Then the Julia set $J(G)$ is connected if and only if the union of the Julia sets of generators $\bigcup_{i=1}^n J(f_i)$ is contained in some subcontinuum $K \subset J(G)$.*

Proof. Let $g = f_{i_1} \circ \cdots \circ f_{i_k}$ be an element in G, where k is a positive integer, $i_j \in \{1, \cdots, n\}, j = 1, \cdots, k$. Define subsets K_j inductively as follows. Let $K_0 = K$. Let $K_j = f_{i_j}^{-1}(K_{j-1}) \cup K_{j-1}$ for $j = 1, \cdots, k$. Obviously, by Lemma 2, $K \subset K_{j-1} \subset K_j \subset J(G)$ for each $j = 1, \cdots, k$. Each K_j is connected. For if $K_0 = K$ is connected, we assume K_{j-1} is connected. Since f_{i_j} is an open map on $\hat{\mathbf{C}}$, by Lemma 1, each of the finitely many components of $f_{i_j}^{-1}(K_{j-1})$ is mapped onto K_{j-1}. Notice that $J(f_{i_j}) \neq \emptyset$ is in K_{j-1} and completely invariant under f_{i_j}. Therefore, each component of $f_{i_j}^{-1}(K_{j-1})$ meets K_{j-1}, and K_j is hence connected. Specifically, K_k is connected.

We take K_g to be K_k and set K^* to be $\bigcup_{g \in G} K_g$. Because K is contained in each K_g, K^* is a connected subset in $J(G)$. There is at least one f_i with degree ≥ 2. Then, K contains at least three points. It follows from Lemma 4 that we can take a point z in $K \backslash E(G)$. From the construction of K_g, it follows that $g^{-1}(z) \subset K_g$. Thus, $O^-(z) \subset K^*$. Consequently, $J(G) = \overline{O^-(z)} \subset \overline{K^*} \subset \overline{J(G)} = J(G)$. This implies that $J(G)$ is connected. The proof is complete.

Remark 1. The connectivity of the Julia set of each generator is not required in Theorem 1.

As mentioned in Section 1, the Julia set of a finitely generated polynomial semigroup may be connected even though the postcritical set is not bounded. Using Theorem 1, we present the following theorem. Recall that a plane continuum is called *nonseparating* if its complement in the plane is connected.

Theorem 2. *Let $G = \langle p_1, \cdots, p_n \rangle$ be a finitely generated polynomial semigroup with the generator system $\{p_1, \cdots, p_n\}$ and each generator of degree ≥ 2. If the finite postcritical set $P^*(G)$ is bounded and if we further assume that at least one Julia set is nonseparating, then the Julia set $J(G)$ is connected.*

Proof. If $\bigcup_{i=1}^n J(p_i)$ is connected, the conclusion follows from Theorem 1 immediately.

We assume that $L = \bigcup_{i=1}^n J(p_i)$ is not connected. Since each finite critical orbit of p_i is contained in $P^*(G)$, each $J(p_i)$ is a continuum. We may assume that $J(p_1)$ is a nonseparating continuum so that $J(p_1)$ is identical with the filled Julia set $K(p_1)$. We suppose that $J(p_1) \cap J(p_i) = \emptyset$. If $J(p_1)$ belongs to the unbounded Fatou component of p_i, then $K(p_1)$ and $K(p_i)$ are disjoint. Taking a critical point c of p_i in $K(p_i)$, we note that $p_1^k(c) \in P^*(G)$ and $\{p_1^k(c)\}$ is unbounded since $c \notin K(p_1)$. The contradiction shows that if $J(p_1) \cap J(p_i) = \emptyset$, then $J(p_1)$ lies in some bounded Fatou component U of p_i. The same argument shows that $P^*(G) \subset J(p_1)$ since

$P^*(G)$ is forward invariant under p_1. In particular, $P^*(p_i) \subset J(p_1)$, from which it follows that p_i has no Siegel disks and, indeed, that there is just one attracting cycle. In fact it is an attracting fixed point which belongs to $J(p_1) \subset U$. Further, U is the domain of attraction of this fixed point: indeed, U is completely invariant under p_i.

Further, if $J(p_j)$ meets neither $J(p_1)$ nor $J(p_i)$, then $J(p_1)$ belongs to some Fatou component V of $J(p_j)$. Thus, $J(p_1) \subset U \cap V$ while $\partial U \cap \partial V = \emptyset$. Hence, either $U \subset V$ or $V \subset U$.

Thus, we may find a sequence of integers $i_1, i_2, \cdots, i_m$ and domains $U_{i_1}, U_{i_2}, \cdots, U_{i_m}$ such that $\partial U_{i_j} \cap \partial U_{i_{j+1}} = \emptyset$, $J(p_1) \subset U_{i_1} \subset U_{i_2} \subset \cdots \subset U_{i_m}$, where U_{i_j} is an invariant Fatou component of p_{i_j} and each $J(p_\nu), 1 \leq \nu \leq n$, is in the component of L, which contains some $J(p_{i_k})$.

By the above construction the attracting fixed point α of p_{i_k} in U_{i_k} lies in $J(p_1)$ as do all the singularities of $p_{i_k}^{-n}$. Thus, for any $n \in \mathbf{N}, K_n = p_{i_k}^{-n}(J(p_1))$ is connected and contains α, so that the closed set $K(k) = \overline{\bigcup_{n=0}^{\infty} K_n}$ is connected. It also meets $J(p_{i_k})$. Hence, $(\bigcup_{\nu=1}^{n} J(p_\nu)) \cup (\bigcup_{k=1}^{m} K(k))$ is a connected subset of $J(G)$, using Lemma 2. It follows from Theorem 1 that $J(G)$ is connected. $\square$

Remark 2. One cannot omit the nonseparating condition. For instance, we let $f_1(z) = z^2$ and $f_2(z) = z^3/4$. The Julia set $J(\langle f_1, f_2 \rangle)$ is not connected.

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References

[1] A. Hinkkanen and G. J. Martin, *The dynamics of semigroups of rational functions I*, Proc. London Math. Soc. **73**(1996), 358-384. MR **97e:**58198

[2] J. Milnor, *Dynamics in one complex variable: Introductory lectures*, Vieweg Verlag, 1999. CMP 2000:03

[3] S. B. Nadler, *Continuum Theory: An introduction*, Marcel Dekker, 1992. MR **93m:**54002

[4] H. Sumi, *On dynamics of hyperbolic rational semigroups*, J. Math. Kyoto Univ. **37**(1997), 717-733. MR **99h:**30028

[5] W. Zhou and F. Ren, *The Julia sets of the random iteration of rational functions*, Chinese Sci. Bull. **37**(1992), 969-971.

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