ON A SEMILINEAR SCHRÖDINGER EQUATION WITH CRITICAL SOBOLEV EXPONENT

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Abstract. We consider the semilinear Schrödinger equation $-\Delta u + V(x)u = K(x)|u|^{2^* - 2}u + g(x, u)$, $u \in W^{1,2}(\mathbb{R}^N)$, where $N \geq 4$, $V, K, g$ are periodic in $x_j$ for $1 \leq j \leq N$, $K > 0$, $g$ is of subcritical growth and $0$ is in a gap of the spectrum of $-\Delta + V$. We show that under suitable hypotheses this equation has a solution $u \neq 0$. In particular, such a solution exists if $K \equiv 1$ and $g \equiv 0$.

1. Introduction and statement of the main result

In this paper we shall be concerned with the semilinear Schrödinger equation

\begin{equation}
-\Delta u + V(x)u = K(x)|u|^{2^* - 2}u + g(x, u), \quad u \in W^{1,2}(\mathbb{R}^N),
\end{equation}

where $N \geq 4$, $2^* := 2N/(N-2)$ is the critical Sobolev exponent and $g$ is of subcritical growth. More precisely, we make the following assumptions:

(A1): $V, K \in C(\mathbb{R}^N)$, $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $K(x) > 0$ in $\mathbb{R}^N$ and $V, K, g$ are 1-periodic in $x_j$ for $j = 1, \ldots, N$.

(A2): $|g(x, u)| \leq c_0(1 + |u|^{p-1})$ on $\mathbb{R}^N \times \mathbb{R}$ for some $c_0 > 0$ and $p \in (2, 2^*)$.

(A3): $g(x, u)/u \to 0$ uniformly in $x$ as $u \to 0$.

(A4): $0 \leq 2G(x, u) - ag(x, u)$ on $\mathbb{R}^N \times \mathbb{R}$, where $G(x, u) := \int_0^u g(x, s)ds$.

(A5): $0 \notin \sigma(-\Delta + V)$ and $\sigma(-\Delta + V) \cap (-\infty, 0) \neq \emptyset$, where $\sigma$ denotes the spectrum in $L^2(\mathbb{R}^N)$.

Note that we do not exclude the case of $g \equiv 0$. It is well-known that under our hypotheses on $V$ the spectrum of $-\Delta + V$ in $L^2(\mathbb{R}^N)$ is bounded below and is the union of disjoint closed intervals; see e.g. p. 161 and Theorem 4.5.9 in [12]. So (A5) is equivalent to 0 being in a spectral gap of $-\Delta + V$. According to (A3), $g(x, 0) \equiv 0$. Hence $u = 0$ is necessarily a solution of (1.1).

Our main result is the following

Theorem 1.1. Suppose that conditions (A1)-(A5) are satisfied, $N \geq 4$ and $K(x_0) = \max_{\mathbb{R}^N} K(x)$. If $K(x) - K(x_0) = o(|x - x_0|^2)$ as $x \to x_0$ and $V(x_0) < 0$, then equation (1.1) has a solution $u \neq 0$. 

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Remark 1.2. (i) If \( N = 4 \), then it suffices that \( K(x) - K(x_0) = O(|x - x_0|^2) \) as \( x \to x_0 \) (see the comment at the end of Section 4). This condition is obviously satisfied if \( K \) is of class \( C^2 \).

(ii) The flatness condition \( K(x) - K(x_0) = o(|x - x_0|^2) \) has been imposed by several authors; see e.g. [7].

As an immediate consequence of Theorem 1.1 we obtain the following:

Corollary 1.3. If conditions (A1)--(A5) are satisfied, \( N \geq 4 \) and \( K(x) \equiv K \) is a positive constant, then equation (1.1) has a solution \( u \neq 0 \).

Equation (1.1) with \( K \equiv 0 \) and \( V, g \) satisfying (A1)--(A3), (A5) and a stronger version of (A4) (the subcritical case) has been considered by several authors; see e.g. [1] [3] [5] [11] [13] [16] [17] [18] and the references there. Equation (1.1) under conditions similar to (A1)--(A5) was discussed in [10], and our Theorem 1.1 is an extension of the main result there. We also note that when \( g \equiv 0 \), (A5) cannot be replaced by the hypothesis that \( 0 \notin \sigma(-\Delta + V) \). Indeed, as was observed in [4], equation \(-\Delta u + \lambda u = |u|^{2^* - 2}u \), where \( \lambda \neq 0 \), has only the trivial solution \( u = 0 \) in \( W^{1,2}(\mathbb{R}^N) \).

Recall [19] that there is a one-to-one correspondence between solutions of (1.1) and critical points of the functional

\[
J(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V u^2) \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} K |u|^{2^*} \, dx - \int_{\mathbb{R}^N} G(x, u) \, dx.
\]

Moreover, \( J \in C^1(E, \mathbb{R}) \), where \( E := W^{1,2}(\mathbb{R}^N) \). Later we shall see that the functional \( J \) has the so-called linking geometry.

In what follows we shall usually abbreviate \( L^p(\mathbb{R}^N) \) by \( L^p \) and the Sobolev space \( W^{m,p}(\mathbb{R}^N) \) by \( W^{m,p} \). The norms will be respectively denoted by \( \| \cdot \|_p \) and \( \| \cdot \|_{m,p} \). The open ball centered at \( a \) and having radius \( r \) will be denoted by \( B(a, r) \). The spaces \( L^p \) and \( W^{m,p} \) are real except in Section 2 where they are complex.

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2. The linear operator

Let \( \mathcal{L}_q : \mathcal{D}(\mathcal{L}_q) \subset L^q(\mathbb{R}^N) \to L^q(\mathbb{R}^N), 2 \leq q < \infty \), be the operator given by \( \mathcal{L}_q u := (-\Delta + V(x))u \). If \( q = 2 \), we shall write \( \mathcal{L} \) instead of \( \mathcal{L}_2 \). In this section we assume that \( V \in L^{\infty}(\mathbb{R}^N), N \geq 1 \), and we do not require \( V \) to be periodic.

Lemma 2.1. \( \mathcal{L}_q \) is a closed operator with domain \( \mathcal{D}(\mathcal{L}_q) = W^{2,q}(\mathbb{R}^N) \).

Proof. The operator \( u \mapsto (V(x) - 1)u \) is bounded in \( L^q \). Therefore it suffices to prove the above statement for \( -\Delta + 1 \). However, this is an immediate consequence of the fact that \((-\Delta + 1)^{-1}\) is an isomorphism of \( L^q \) onto \( W^{2,q} \) (a property of the Bessel potentials; see formula (41) and Theorem 3 of Chap. V in [14]).

Recall that in this section the spaces \( L^p \) and \( W^{m,p} \) are complex. By a result of Hempel and Voigt [8] (see also Arendt [2] Example 5.3) \( \sigma(\mathcal{L}_q) = \sigma(\mathcal{L}) \) and \( (\mathcal{L}_q - \lambda)^{-1}|_{L^{\infty} \cap L^q} = (\mathcal{L} - \lambda)^{-1}|_{L^{\infty} \cap L^q} \) for all complex \( \lambda \notin \sigma(\mathcal{L}) \).

Let \( (E(\lambda))_{\lambda \in \mathbb{R}} \) be the spectral family of \( \mathcal{L} \). Then for a fixed \( \mu \), \( E(\mu)L^2 \) is the subspace of \( L^2 \) corresponding to \( \lambda \leq \mu \).

Proposition 2.2. If \( V \in L^{\infty}(\mathbb{R}^N) \) satisfies (A5), then \( \|u\|_{1,\infty} \leq c_0 \|u\|_2 \) for some constant \( c_0 > 0 \) and all \( u \in E(0)L^2 \).
Proof. Let $\Gamma$ be a positively oriented smooth Jordan curve (in $\mathbb{C}$) containing $\sigma(L) \cap (-\infty, 0)$ in its interior and the remaining part of $\sigma(L)$ in its exterior. Since $L$ is a closed operator,

$$E(0) = -\frac{1}{2\pi i} \int_{\Gamma} (\Delta + V - \lambda)^{-1} \, d\lambda$$

according to formula (III.6.19) in [10]. So

$$u = -\frac{1}{2\pi i} \int_{\Gamma} (\Delta + V - \lambda)^{-1} u \, d\lambda$$

whenever $u \in E(0)L^2$. Since $\Gamma$ is compact and $-\Delta + V - \lambda$ is invertible for each $\lambda \in \Gamma$ (as an operator from $D(L)$ into $L^2$), it is easy to see from (2.2) and the Sobolev embedding theorem that $\|u\|_{q_1} \leq c_1 \|u\|_{2^*} \leq c_2 \|u\|_2$, where $q_1 = 2N/(N-4)$ if $N > 4$ and $q_1$ may be chosen arbitrarily large if $N \leq 4$ (here and in what follows $c_1$, $c_2$, etc. denote positive constants whose numerical values are immaterial). Keeping in mind that $L_q$ is closed and $L_q - \lambda$ is invertible on $\Gamma$ for all $q$, we may employ the usual bootstrap argument: we get $\|u\|_{q_2} \leq c_3 \|u\|_{2,q_1} \leq c_4 \|u\|_{q_1} \leq c_5 \|u\|_2$, where $q_2 = 2N/(N-8)$; after a finite number of iterations $q_k > N$ and by (2.2) again, $\|u\|_{2,q_k} \leq c \|u\|_2$. Now the conclusion follows by the Sobolev embedding $W^{2,q_k} \hookrightarrow W^{1,\infty}$.

Proposition 2.3 (Troestler [17]). If $V \in L^\infty(\mathbb{R}^N)$ satisfies (A5) and $2 \leq q < \infty$, then $E(0)|_{L^2(L^q)}$ is $L^q$-continuous. In particular, $E(0)$ and $I - E(0)$ extend to continuous projections of $L^q$ onto the complementary subspaces $cL^q(E(0)L^2)$ and $cL^q_1((I - E(0))L^2)$ ($cL^q$ denotes the closure).

Proof. By (2.1), $\|E(0)u\|_{q} \leq \|E(0)u\|_{2,q} \leq c_0 \|u\|_{q}$ for all $u \in L^2 \cap L^q$ and some $c_0 > 0$. Hence $E(0)$ and $I - E(0)$ may be extended to continuous projections of $L^q$ onto the complementary subspaces.

Proposition 2.4. If $V \in L^\infty(\mathbb{R}^N)$, then for each $\mu \in \mathbb{R}$ there exist constants $c_1$ and $c_2 = c_2(\mu)$ such that $\|u\|_{q} \leq c_1 \|u\|_{2} \leq c_2 \|u\|_{2}$ whenever $u \in E(\mu)L^2$. Here $q = 2N/(N-4)$ if $N > 4$, $q$ may be taken arbitrarily large if $N = 4$ and $q = \infty$ if $N < 4$.

Proof. The operator $L^\mu := L|_{E(\mu)L^2} : E(\mu)L^2 \to E(\mu)L^2$ is bounded. Let $\Gamma$ be a positively oriented smooth Jordan curve enclosing the spectrum of $L^\mu$. Then (2.2) still holds for all $u \in E(\mu)L^2$ (with $(-\Delta + V - \lambda)^{-1}$ replaced by $(L^\mu - \lambda)^{-1}$). Therefore $\|u\|_q \leq c_1 \|u\|_{2} \leq c_2 \|u\|_{2}$.

3. Existence of a Palais-Smale sequence

In this section we assume that the hypotheses (A1)–(A5) are satisfied. Recall $E = W^{1,2}(\mathbb{R}^N)$ and let $E^- := E(0)L^2 \cap E$ and $E^+ := (I - E(0))L^2 \cap E (E(\lambda))$ is as in the preceding section). Then the quadratic form $\int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2) \, dx$ is positive definite on $E^+$ and negative definite on $E^-$ [15, Sections 8 and 9]. Hence we may introduce a new inner product $(\cdot, \cdot)$ in $E$ such that the corresponding norm $\|\cdot\|$ is equivalent to $\|\cdot\|_{1,2}$ and $\int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2) \, dx = \|u^+\|^2 - \|u^-\|^2$, where $u^\pm \in E^\pm$. 

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Set $\psi(u) := (2^*)^{-1} \int_{\mathbb{R}^N} K|u|^2^* \, dx + \int_{\mathbb{R}^N} G(x,u) \, dx$; then
\begin{equation}
(3.1)
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 + Vu^2 \right) \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} K|u|^{2^*} \, dx - \int_{\mathbb{R}^N} G(x,u) \, dx
= \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - \psi(u).
\end{equation}

Let $z_0 \in E^+ \setminus \{0\}$,
$$M := \{ u = u^- + sz_0 : u^- \in E^-, \ s \geq 0 \text{ and } \|u\| \leq R \}$$
and denote the boundary of $M$ in $E^- \oplus \mathbb{R}_{z_0}$ by $\partial M$. We summarize the properties of $J$ in the following:

**Proposition 3.1.** (i) There exist $\alpha, \rho > 0$ and $R > \rho$ ($R$ depending on $z_0$) such that $J(u) \geq \alpha$ for all $u \in E^+ \cap \partial B(0, \rho)$ and $J(u) \leq 0$ for all $u \in \partial M$.

(ii) $\psi \geq 0$, $\psi$ is weakly sequentially lower semicontinuous and $\psi'$ is weakly sequentially continuous.

Functionals satisfying (i) above are said to have the linking geometry.

**Proof.** (i) See e.g. \cite{11,18,19}. The proofs given there are for nonlinearities of subcritical growth but the argument remains unchanged in our case (the part showing $J|_{\partial M} \leq 0$ is in fact somewhat simpler here; observe only that $(2^*)^{-1} K(x)|u|^{2^*} + G(x,u) \geq c_0 |u|^2$ for some $c_0 > 0$).

(ii) It is obvious that $\psi \geq 0$. Let $u_n \rightharpoonup u$. Then $u_n \rightharpoonup u$ a.e. in $\mathbb{R}^N$, possibly after passing to a subsequence. Hence it follows from the Fatou lemma that $\psi$ is weakly sequentially lower semicontinuous. Moreover, since $u_n \rightharpoonup u$ in $L^p_\text{loc}$, it is easy to see from (A2) and (A3) that
$$\int_{\mathbb{R}^N} g(x,u_n)v \, dx \to \int_{\mathbb{R}^N} g(x,u)v \, dx \text{ for each } v \in E.$$
Finally, $u_n \rightharpoonup u$ in $L^{(N+2)/(N-2)}_{\text{loc}}$; therefore $K|u_n|^{2^*-2}u_n \to K|u|^{2^*-2}u$ in $L^{p}_{\text{loc}}$ and
$$\int_{\mathbb{R}^N} K|u_n|^{2^*-2}u_n \varphi \, dx \to \int_{\mathbb{R}^N} K|u|^{2^*-2}u \varphi \, dx \text{ whenever } \varphi \in C_0^{\infty}.$$

Taking into account that the sequence $(K|u_n|^{2^*-1})$ is bounded in $L^{2N/(N+2)}$, we may replace $\varphi$ by $v \in E$. This completes the proof.

**Proposition 3.2.** If $J$ is a functional of the form appearing in the second line of (8.1) and if (i), (ii) of Proposition 3.1 are satisfied, then there exists a Palais-Smale sequence $(u_n)$ for $J$ such that $J(u_n) \to c \in [\alpha, \sup_M J]$.

This is a special case of Theorem 3.4 in \cite{11}; see also Theorem 6.10 in \cite{19}.

We have thus shown that the functional $J$ associated with (1.1) possesses a Palais-Smale sequence $(u_n)$ with $J(u_n) \to c$.

**Proposition 3.3.** The Palais-Smale sequence above is bounded.

**Proof.** It follows from (A2)–(A3) that for each $\varepsilon > 0$ there exists $c_1(\varepsilon)$ such that $|g(x,u)| \leq \varepsilon |u| + c_1(\varepsilon)|u|^{2^*-1}$. By (A4),
$$c + 1 + \|u_n\| \geq J(u_n) - \frac{1}{2} \langle J'(u_n), u_n \rangle \geq \frac{1}{N} \int_{\mathbb{R}^N} K|u_n|^{2^*} \, dx$$
for almost all \( n \), and since \( K(x) \) is bounded below by a positive constant,

\[
\|u_n\|^2 \leq c_2 + c_3 \|u_n\|.
\]

Using the Hölder and Sobolev inequalities we obtain, for large \( n \),

\[
\|u_n^+\|^2 = \langle J'(u_n), u_n^+ \rangle + \int_{\mathbb{R}^N} K|u_n|^{2^*-2} u_n u_n^+ \, dx + \int_{\mathbb{R}^N} g(x, u_n) u_n^+ \, dx.
\]

Hence by (3.2),

\[
\|u_n^+\|^2 \leq \|u_n\|^2 \|u_n^+\| + c_4 \|u_n\|^{2^*-1} \|u_n^+\| + c_5 (\varepsilon \|u_n\| + c_1 (\varepsilon \|u_n\|^{2^*-1}) \|u_n^+\|.
\]

and a similar inequality holds for \( \|u_n^-\| \). Choosing \( \varepsilon \) sufficiently small, we see that \( (u_n) \) must be bounded.

4. Proof of Theorem 1.1

In the preceding section we have shown that there exists a bounded Palais-Smale sequence \( (u_n) \) such that \( J(u_n) \to c \in [\alpha, \sup_M J] \). Clearly, \( (u_n) \) is either

(i) Vanishing: For each \( r > 0 \), \( \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, r)} u_n^2 \, dx = 0 \), or

(ii) Non-vanishing: There exist \( r, \eta > 0 \) and a sequence \( (y_n) \subset \mathbb{R}^N \) such that

\[
\lim_{n \to \infty} \int_{B(y_n, r)} u_n^2 \, dx \geq \eta.
\]

In (ii) we may assume \( y_n \in \mathbb{Z}^N \) by taking a larger \( r \) if necessary. Suppose (ii) holds and let \( \tilde{u}_n(x) := u_n(x + y_n) \). Since \( J \) is invariant with respect to the translation of \( x \) by elements of \( \mathbb{Z}^N \) (i.e. \( J(u(x)) = J(u(x + y)) \) whenever \( y \in \mathbb{Z}^N \)), \( \|\tilde{u}_n\| = \|u_n\| \) and \( \|J'(\tilde{u}_n)\| = \|J'(u_n)\| \). Hence \( \tilde{u}_n \to \tilde{u} \) after passing to a subsequence, \( J'(\tilde{u}) = 0 \) and since \( \lim_{n \to \infty} \int_{B(0, r)} \tilde{u}_n^2 \, dx \geq \eta, \tilde{u} \neq 0 \). So \( \tilde{u} \) is a nontrivial solution of (1.1).

To complete the proof of Theorem 1.1 it remains therefore to show that vanishing cannot occur. This will be done in the following two propositions. Let

\[
S := \inf_{u \in \mathcal{E}\setminus \{0\}} \frac{\|\nabla u\|^2}{\|u\|^2},
\]

Proposition 4.1. If \( 0 < c < c^* := \frac{S^{N/2}}{N \|K\|^{N-2/2}} \), then \( (u_n) \) cannot be vanishing.

Proof. If \( (u_n) \) is vanishing, then it follows from P.L. Lions’ lemma [19, Lemma 1.21] that \( u_n \to 0 \) in \( L^r \) whenever \( 2 < r < 2^* \). Let \( (z_n) \) be a bounded sequence in \( E \). Since for each \( \varepsilon > 0 \) there is \( c_1(\varepsilon) \) such that \( |g(x, u)| \leq \varepsilon |u| + c_1(\varepsilon)|u|^{p-1} \),

\[
\int_{\mathbb{R}^N} |g(x, u_n)| \, |z_n| \, dx \leq c_2 \varepsilon \|u_n\| \|z_n\| + c_3(\varepsilon) \|u_n\|^{p-1} \|z_n\|.
\]

Using this and a similar argument for \( G \) we see that

\[
\int_{\mathbb{R}^N} g(x, u_n) z_n \, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} G(x, u_n) \, dx \to 0.
\]

Hence

\[
J(u_n) - \frac{1}{2} \langle J'(u_n), u_n \rangle = \frac{1}{N} \int_{\mathbb{R}^N} K|u_n|^{2^*} \, dx + o(1) \to c.
\]
Recall \((E(\lambda))_{\lambda \in \mathbb{R}}\) is the spectral family of \(-\Delta + V\) in \(L^2\). Let \(u = u^+ + u^- \in E^+ \oplus E^\perp\) and write \(u^+ = w + z\), where \(w \in E(\mu)L^2\), \(z \in (I - E(\mu))L^2\), \(\mu > 0\) large (to be determined). By Proposition \ref{prop:5.2}, \(w \in E\), hence also \(z \in E\); moreover, \(\|u^+_n\|_q \leq c_4\|u^-_n\|_2 \leq c_5\|u_n\|\) and \(\|w_n\|_q \leq c_4\|w_n\|_2 \leq c_5\|u_n\|\), where \(q = 2N/(N-4)\) if \(N > 4\) and \(q\) may be taken arbitrarily large if \(N = 4\). Let \(r\) be such that \((2^* - 1)/r + 1/q = 1\). Then \(2 < r < 2^*\) (for \(N = 4\), \(q\) needs to be larger than 4). Since \(\|u^-\|_q\) is bounded and \(u_n \to 0\) in \(L^r\), we obtain using \((4.2)\) and the Hölder inequality that
\[
\|u^-\|_q^2 = -\langle J'(u_n), u^-\rangle - \int_{\mathbb{R}^N} K|u_n|^{2^*-2}u_n u^- dx - \int_{\mathbb{R}^N} g(x, u_n)u^- dx \\
\leq \|K\|_\infty \|u_n\|^{2^*-1}_r \|u^-\|_q + o(1) \to 0.
\]
Similarly,
\[
\|w_n\|_q^2 = \int_{\mathbb{R}^N} K|u_n|^{2^*-2}u_n w_n dx + o(1) \to 0.
\]
Hence
\[(4.4) \qquad u_n - z_n = w_n + u^-_n \to 0,
\]
and therefore
\[
\|z_n\|_q^2 = \int_{\mathbb{R}^N} (|\nabla z_n|^2 + Vz_n^2) dx = \int_{\mathbb{R}^N} K|u_n|^{2^*-2}u_n z_n dx + o(1) \\
= \int_{\mathbb{R}^N} K|u_n|^{2^*} dx + o(1).
\]
Furthermore, for each \(\delta > 0\) we may find \(\mu > 0\) such that
\[(4.6) \qquad (1-\delta) \int_{\mathbb{R}^N} |\nabla z_n|^2 dx \leq \int_{\mathbb{R}^N} (|\nabla z_n|^2 + Vz_n^2) dx.
\]
Indeed, since \(z_n \in (I - E(\mu))L^2 \cap E\), we have \(\int_{\mathbb{R}^N} (|\nabla z_n|^2 + Vz_n^2) dx \geq \mu \|z_n\|_2^2\) and
\[
\delta \int_{\mathbb{R}^N} |\nabla z_n|^2 dx \geq \delta(\mu - \|V\|_\infty)\|z_n\|_2^2 \geq -\int_{\mathbb{R}^N} Vz_n^2 dx
\]
whenever \(\mu\) is large enough. Combining \((4.3)\), \((4.1)\), \((4.6)\) and \((4.5)\) gives
\[
(1-\delta)S\|K\|_\infty^{-2/2^*} \left( \int_{\mathbb{R}^N} K|u_n|^{2^*} dx \right)^{2/2^*} \leq (1-\delta)S\|u_n\|_2^2.
\]
\[
= (1-\delta)S\|z_n\|_2^2 + o(1) \leq (1-\delta) \int_{\mathbb{R}^N} |\nabla z_n|^2 dx + o(1)
\]
\[
\leq \int_{\mathbb{R}^N} K|u_n|^{2^*} dx + o(1).
\]
Passing to the limit and using \((4.3)\) we obtain
\[
(1-\delta)S\|K\|_\infty^{-2/2^*}(cN)^{2/2^*} \leq cN;
\]
hence either \(c = 0\) which is impossible or \((1-\delta)^{N/2}c^* \leq c < c^*\) which is also impossible because \(\delta\) may be chosen arbitrarily small.

Let
\[
\varphi_\varepsilon(x) := \frac{c_N \psi(x) \varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}},
\]
where \( c_N = (N(N-2))^{(N-2)/4} \), \( \varepsilon > 0 \) and \( \psi \in C^\infty_0(\mathbb{R}^N, [0, 1]) \) is such that \( \psi(x) = 1 \) for \( |x| \leq r/2 \) and \( \psi(x) = 0 \) for \( |x| \geq r \) \((r \text{ to be determined})\). We shall need the following asymptotic estimates as \( \varepsilon \to 0^+ \) (see e.g. pp. 35 and 52 in [19]):

\[
(4.7) \quad \|\nabla \varphi_\varepsilon\|_2^2 = S^{N/2} + O(\varepsilon^{N-2}), \quad \|\nabla \varphi_\varepsilon\|_1 = O(\varepsilon^{(N-2)/2}),
\]

\[
(4.8) \quad \|\varphi_\varepsilon\|_2^2 = S^{N/2} + O(\varepsilon^N), \quad \|\varphi_\varepsilon\|_{2^* - 1} = O(\varepsilon^{(N-2)/2}), \quad \|\varphi_\varepsilon\|_1 = O(\varepsilon^{(N-2)/2})
\]

and

\[
(4.9) \quad \max_{t \geq 0} I(tu) = \frac{1}{N} \left( \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + V u^2) \, dx}{\left( \int_{\mathbb{R}^N} K |u|^{2^*} \, dx \right)^{(N-2)/2}} \right)^{N/2}
\]

whenever the integral in the numerator above is positive, and the maximum is 0 otherwise. Let \( \|u\|_{2^*, K} := \int_{\mathbb{R}^N} K |u|^{2^*} \, dx \). It is easy to see from (4.9) that if

\[
(4.10) \quad m_\varepsilon := \sup_{\|u\|_{2^*, K} = 1} \int_{\mathbb{R}^N} (|\nabla u|^2 + V u^2) \, dx \leq \frac{S}{\|K\|^{(N-2)/N}},
\]

then \( \sup_\varepsilon \int_{\mathbb{R}^N} (|\nabla u|^2 + V u^2) \, dx \) is finite for all \( \varepsilon > 0 \).

Below we shall repeatedly use (4.7) and (4.8). Since \( \int_{\mathbb{R}^N} (|\nabla \varphi_\varepsilon|^2 + V (\varphi_\varepsilon)^2) \, dx \leq 0 \), \( \int_{\mathbb{R}^N} |\nabla \varphi_\varepsilon|^2 \, dx \leq c_1 \|\varphi_\varepsilon\|_2^2 \leq c_1 \|\varphi_\varepsilon\|_2 \to 0 \) as \( \varepsilon \to 0 \); moreover, \( \|\varphi_\varepsilon\|_{2^*} \leq c_2 \|\varphi_\varepsilon\| \to 0 \) and \( \|\varphi_\varepsilon\|_{2^*} \to 0 \) as \( \varepsilon \to 0 \). Suppose \( \|u\|_{2^*, K} = 1 \) and write \( u = u^- + s \varphi_\varepsilon = (u^- + s \varphi^-) + s \varphi^+ \). It follows from Proposition 2.2 and the argument above that \( \|u^-\|_{2^*} \leq c_3 \) and \( |s| \leq c_3 \) for some constant \( c_3 \) independent of \( \varepsilon \). By Proposition 2.2 and convexity of \( \| \cdot \|_{2^*, K} \) we obtain

\[
(4.11) \quad 1 = \|u\|_{2^*, K}^2 \geq \|s \varphi_\varepsilon\|_{2^*, K}^2 + 2s^2 \int_{\mathbb{R}^N} (s \varphi_\varepsilon)^{2^* - 1} u^- \, dx \geq \|s \varphi_\varepsilon\|_{2^*, K}^2 - c_4 \|\varphi_\varepsilon\|_{2^* - 1}^2 \|u^-\|_2^2.
\]
Moreover, by Proposition 2.2 again,
\begin{equation}
\int_{\mathbb{R}^N} (\nabla \varphi_\varepsilon \cdot \nabla u^- + V \varphi_\varepsilon u^-) \, dx \leq c_5 (\| \nabla \varphi_\varepsilon \|_1 + \| \varphi_\varepsilon \|_1) \| u^- \|_2
\end{equation}
\begin{equation}
= O(\varepsilon^{(N-2)/2}) \| u^- \|_2.
\end{equation}
Since \( V(x) \leq -\beta < 0 \) for \( x \in \text{supp} \varphi_\varepsilon \) and \( K(x) - K(0) = o(|x|^2) \) as \( x \to 0 \),
\begin{equation}
\int_{\mathbb{R}^N} V \varphi_\varepsilon^2 \, dx \leq \begin{cases} -d \varepsilon^2 & \text{if } N \geq 5, \\ -d \varepsilon^2 \log \varepsilon & \text{if } N = 4, \end{cases}
\end{equation}
for some \( d > 0 \) and
\begin{equation}
\| \varphi_\varepsilon \|_{2^*,K}^2 = \| K \|_\infty \int_{\mathbb{R}^N} \varphi_\varepsilon^2 \, dx + \int_{\mathbb{R}^N} (K(x) - K(0)) \varphi_\varepsilon^2 \, dx
\end{equation}
\begin{equation}
= \| K \|_\infty S^{N/2} + o(\varepsilon^2).
\end{equation}
Let \( N \geq 5 \). Using (4.12), (4.14), (4.11), (4.13) and the fact that
\begin{equation}
-u^- \|_2 + O(\varepsilon^{(N-2)/2}) \| u^- \|_2 \leq O(\varepsilon^{N-2}),
\end{equation}
we obtain
\begin{equation}
m_\varepsilon \leq -\| u^- \|_2^2 + \frac{\int_{\mathbb{R}^N} (|\nabla \varphi_\varepsilon|^2 + V \varphi_\varepsilon^2) \, dx}{\| \varphi_\varepsilon \|_{2^*,K}^2} \| \varphi_\varepsilon \|_{2^*,K}^2 + O(\varepsilon^{(N-2)/2}) \| u^- \|_2
\end{equation}
\begin{equation}
\leq -c_6 \| u^- \|_2^2 + \frac{\int_{\mathbb{R}^N} (|\nabla \varphi_\varepsilon|^2 + V \varphi_\varepsilon^2) \, dx}{\| K \|_\infty^{(N-2)/N} S^{(N-2)/2} + o(\varepsilon^2)} (1 + c_4 \| \varphi_\varepsilon \|_{2^*,K}^2 \| u^- \|_2^{2/2^*})
\end{equation}
\begin{equation}
+ O(\varepsilon^{(N-2)/2}) \| u^- \|_2
\end{equation}
\begin{equation}
= -c_6 \| u^- \|_2^2 + \frac{S^{N/2} - d \varepsilon^2 + O(\varepsilon^{N-2})}{\| K \|_\infty^{(N-2)/N} S^{(N-2)/2} + o(\varepsilon^2)} + O(\varepsilon^{(N-2)/2}) \| u^- \|_2
\end{equation}
\begin{equation}
\leq \frac{S}{\| K \|_\infty^{(N-2)/N}} - d_0 \varepsilon^2 + o(\varepsilon^2),
\end{equation}
where \( d_0 > 0 \). If \( N = 4 \), then in a similar way,
\begin{equation}
m_\varepsilon \leq \frac{S}{\| K \|_\infty^{(N-2)/N}} - d_0 \varepsilon^2 \log \varepsilon + o(\varepsilon^2).
\end{equation}
Hence (4.10) holds provided \( \varepsilon \) is sufficiently small.

Note that if \( K(x) - K(0) = O(|x|^2) \) as \( x \to 0 \), then (4.14) holds with \( O(\varepsilon^2) \) replacing \( o(\varepsilon^2) \). This does not affect the estimate of \( m_\varepsilon \) if \( N = 4 \). Hence for such \( N \) the conclusion of Theorem 1.1 remains valid under the weaker hypothesis on \( K \) as in Remark 1.2.

REFERENCES


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