NUMBER OF SINGULARITIES OF A FOLIATION ON $\mathbb{P}^n$

FERNANDO SANCHO DE SALAS

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Abstract. Let $\mathcal{D}$ be a one dimensional foliation on a projective space, that is, an invertible subsheaf of the sheaf of sections of the tangent bundle. If the singularities of $\mathcal{D}$ are isolated, Baum-Bott formula states how many singularities, counted with multiplicity, appear. The isolated condition is removed here.

Let $m$ be the dimension of the singular locus of $\mathcal{D}$. We give an upper bound of the number of singularities of dimension $m$, counted with multiplicity and degree, that $\mathcal{D}$ may have, in terms of the degree of the foliation. We give some examples where this bound is reached. We then generalize this result for a higher dimensional foliation on an arbitrary smooth and projective variety.

Introduction

Throughout this paper $\mathbb{P}^n$ denotes the projective space of dimension $n$ over an algebraically closed field $K$ of any characteristic and $T_{\mathbb{P}^n}$ the sheaf of sections of the tangent bundle.

Let $\mathcal{D}$ be a 1-dimensional foliation on $\mathbb{P}^n$, that is, $\mathcal{D} \hookrightarrow T_{\mathbb{P}^n}$ is an invertible subsheaf. Put $\mathcal{D} \cong \mathcal{O}_{\mathbb{P}^n}(-k)$; the number $k$ is called degree of the foliation. The singular locus of the foliation is the singular locus of the morphism $\mathcal{D} \rightarrow T_{\mathbb{P}^n}$, that is, the set of points $p \in \mathbb{P}^n$ such that $\mathcal{D} \otimes K(p) \rightarrow T_{\mathbb{P}^n} \otimes K(p)$ is null. It is a closed subset defined by the Fitting ideal $F_{\mathbb{P}^n}^1(T_{\mathbb{P}^n}/\mathcal{D})$, which is the image of the natural morphism $\Omega_{\mathbb{P}^n} \otimes \mathcal{D} \rightarrow \mathcal{O}_{\mathbb{P}^n}$. We shall give it a scheme structure by putting $\text{Sing}(\mathcal{D}) = \text{Spec} \mathcal{O}_{\mathbb{P}^n}/I$, with $I = F_{\mathbb{P}^n}^1(T_{\mathbb{P}^n}/\mathcal{D})$.

In each affine chart, with affine coordinates $x_1, \ldots, x_n$, $\mathcal{D}$ is given by a vector field $f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$ and the singularities of $\mathcal{D}$ are given by the equations $f_1 = \cdots = f_n = 0$. In other words, $I = (f_1, \ldots, f_n)$.

In general, the singularities of a 1-dimensional foliation on $\mathbb{P}^n$ are isolated, that is, $\text{Sing}(\mathcal{D})$ is a zero dimensional scheme. In this case the number of singularities of $\mathcal{D}$ is given by the classical Baum-Bott formula (1):

$$\sum_{p \in \text{Sing}(\mathcal{D})} n_p = (k + 1)^n + (k + 1)^{n-1} + \cdots + (k + 1) + 1$$

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where \( n_p \) denotes the Milnor number of \( D \) at \( p \), that is, \( n_p = \dim_K \langle f_1, \ldots, f_n \rangle \).

Now we want to study the non isolated case. For example, assume that \( D \) is a 1-dimensional foliation on \( \mathbb{P}^3 \), and the singular locus is made of some curves and some points out of them. Our question is: how many curves can appear? We shall give a complete answer in terms of the degree of the foliation. We shall prove:

**Theorem.** Let \( D \) be a one dimensional foliation on \( \mathbb{P}^n \) of degree \( k \) and let \( m \) be the dimension of \( \text{Sing}(D) \). The number of singularities of \( D \) of dimension \( m \), counted with multiplicity and degree, is bounded by

\[(k + 1)^{n-m} + \cdots + (k + 1) + 1\]

and this bound is reached for some foliation.

We shall extend this result for a higher dimensional foliation on an arbitrary smooth and projective variety (Theorem 2.1).

1. **One Dimensional Foliations on \( \mathbb{P}^n \)**

For the sake of clarity we are going to detail the case of a one dimensional foliation on \( \mathbb{P}^n \), and we shall later generalize to higher dimensional foliations and to arbitrary projective varieties.

Let \( D \) be a 1-dimensional foliation on \( \mathbb{P}^n \), and \( \text{Sing}(D) \) its singular locus, of ideal \( I \). Let \( C \) be an irreducible component of \( \text{Sing}(D) \), with the reduced structure. We shall denote by \( n_C(D) \) the multiplicity of \( \mathbb{P}^n \) along \( \text{Sing}(D) \) at \( C \). For the sake of clarity we are going to detail the case of a one dimensional foliation on \( \mathbb{P}^n \).

**Theorem 1.1.** Let \( D \) be a one-dimensional foliation on \( \mathbb{P}^n \) of degree \( k \). Let \( m \) be the (non-necessarily pure) dimension of \( \text{Sing}(D) \) and \( d = n - m \). Let \( C_1, \ldots, C_l \) be the irreducible components of dimension \( m \) of \( \text{Sing}(D) \), with its reduced structure. Then

\[
\sum_{i=1}^{l} n_{C_i}(D) \cdot \deg(C_i) \leq (k + 1)^d + (k + 1)^{d-1} + \cdots + (k + 1) + 1.
\]

**Proof.** The main ingredient of the proof is the following:

**Lemma 1.2.** Let us denote \( i_D : D \hookrightarrow T_{\mathbb{P}^n} \). There exists a morphism \( f : \mathcal{O}_{\mathbb{P}^n}(1)^m \rightarrow T_{\mathbb{P}^n} \) such that the singular locus of the morphism \( i_D \oplus f : D \oplus \mathcal{O}_{\mathbb{P}^n}(1)^m \rightarrow T_{\mathbb{P}^n} \) has dimension \( m \).

**Proof.** Let \( \mathcal{E} = \text{Hom}(\mathcal{O}_{\mathbb{P}^n}(1)^m, T_{\mathbb{P}^n}) \) be the bundle of homomorphisms of \( \mathcal{O}_{\mathbb{P}^n}(1)^m \) to \( T_{\mathbb{P}^n} \). In other words, it is the bundle associated to the sheaf of modules \( T_{\mathbb{P}^n}(-1)^\oplus m \cdot \oplus T_{\mathbb{P}^n}(-1) \). Denote \( \pi : \mathcal{E} \rightarrow \mathbb{P}^n \) the structural morphism and \( E \) the global sections of \( \mathcal{E} \), which is a finite dimensional vector space. From the Euler exact sequence

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{m+1} \rightarrow T_{\mathbb{P}^n} \rightarrow 0
\]

it follows easily that \( \mathcal{E} \) is generated by its global sections. That is, one has an epimorphism of bundles over \( \mathbb{P}^n \)

\[
\phi : \mathbb{P}^n \times E \rightarrow \mathcal{E}.
\]

Let \( \tau : \pi^*\mathcal{O}_{\mathbb{P}^n}(1)^m \rightarrow \pi^*T_{\mathbb{P}^n} \) be the universal homomorphism. It is not difficult to see that the singular locus of the morphism \( i_D \oplus \pi : \pi^*(D \oplus \mathcal{O}_{\mathbb{P}^n}(1)^m) \rightarrow \pi^*T_{\mathbb{P}^n} \) has codimension \( d \) (see, for example, [5]). Then \( \phi^{-1}(\text{Sing}(i_D \oplus \tau)) \) has codimension \( d \)
in \(\mathbb{P}^n \times E\). Let us consider the projection \(p: \mathbb{P}^n \times E \to E\). By dimensions, there exists \(e \in E\), such that the intersection of \(\phi^{-1}(\text{Sing}(i_D \oplus \tau))\) with the fibre of \(e\) has codimension \(d\) in such fibre. That is, there exists a morphism \(e: \mathcal{O}_{\mathbb{P}^n}(1)^m \to T_{\mathbb{P}^n}\) such that the singular locus of the morphism \(i_D \oplus e: D \oplus \mathcal{O}_{\mathbb{P}^n}(1)^m \to T_{\mathbb{P}^n}\) has codimension \(d\).

Now the proof of the theorem: by the lemma, let us consider a morphism \(f: \mathcal{O}_{\mathbb{P}^n}(1)^m \to T_{\mathbb{P}^n}\) such that the singular locus of \(i_D \oplus f: D \oplus \mathcal{O}_{\mathbb{P}^n}(1)^m \to T_{\mathbb{P}^n}\) has dimension \(m\). It is clear that \(\text{Sing}(D) \subset \text{Sing}(i_D \oplus f)\). Let \(C_1, \ldots, C_l, W_1, \ldots, W_s\) be the irreducible components (with the reduced structure) of \(\text{Sing}(i_D \oplus f)\), where \(C_1, \ldots, C_l\) are the irreducible components of \(\text{Sing}(D)\). By Baum-Bott-Kempf-Laksov theorem (see [2] for the integrable case or [3] for the general case),

\[
(1) \quad \sum n_{C_i}(i_D \oplus f) \cdot C_i + \sum n_{W_i}(i_D \oplus f) \cdot W_i = c_d(T_{\mathbb{P}^n} - D \oplus \mathcal{O}_{\mathbb{P}^n}(1)^m)
\]

where \(n_{C_i}(i_D \oplus f)\) and \(n_{W_i}(i_D \oplus f)\) are the multiplicities of \(\mathbb{P}^n\) along \(\text{Sing}(i_D \oplus f)\) at \(C_i\) and \(W_i\), respectively. By the Euler sequence, \(T_{\mathbb{P}^n} = -\mathcal{O}_{\mathbb{P}^n} + \mathcal{O}_{\mathbb{P}^n}(1)^{n+1}\) in \(K\)-theory, and \(D = \mathcal{O}_{\mathbb{P}^n}(-k)\). So, if \(H\) denotes a hyperplane of \(\mathbb{P}^n\), then

\[
c_d(T_{\mathbb{P}^n} - D \oplus \mathcal{O}_{\mathbb{P}^n}(1)^m) = \left[\frac{(1 + H)^{d+1}}{1 - k H}\right]_d = ((k + 1)^d + \cdots + (k + 1) + 1) \cdot H^d.
\]

On the other hand, \(n_{C_i}(i_D \oplus f) \geq n_{C_i}(D)\) (since the ideal defining \(\text{Sing}(D)\) contains the one defining \(\text{Sing}(i_D \oplus f)\)), then, taking degree in (1), one concludes.

Now let us see that this bound is reached. Let \(D_d\) be a 1-dimensional foliation on \(\mathbb{P}^d\) of degree \(k \geq -1\), with isolated singularities \(q_1, \ldots, q_l\). Then

\[
\sum_{q \in \text{Sing}(D_d)} n_q(D_d) = (k + 1)^d + \cdots + (k + 1) + 1.
\]

Let \(A^d\) be an affine chart containing all the singularities of \(D_d\). Let \(x_1, \ldots, x_d\) be affine coordinates of \(A^d\) and \(A'_d = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}\) a generator of \(D_d\) in \(A^d\). Let us consider the natural projection \(\pi: A^d \times A^{n-d} \to A^d\). Then \(D' = \pi^* D'_d\) is a vector field on \(A^n\) with singularities \(C'_i = \pi^{-1}(q_i)\), which have codimension \(d\). Moreover, it is clear that \(n_{C'_i}(D') = n_q(D_d)\). Now, \(D'\) extends to a 1-dimensional foliation \(D\) on \(\mathbb{P}^n\). If \(C_i\) is the closure of \(C'_i\) in \(\mathbb{P}^n\), then \(n_{C_i}(D) = n_q(D_d)\). Then \(\sum n_{C_i}(D) = (k + 1)^d + \cdots + (k + 1) + 1\). A computation shows that the codimension of \(\text{Sing}(D)\) is \(d\) (in fact, there are no singularities of codimension \(\leq d\) at the infinity). Besides, the degree of \(D\) is still \(k\). In conclusion, the number of singularities of \(D\) of codimension \(d\) is \((k + 1)^d + \cdots + (k + 1) + 1\), with \(k\) degree of \(D\).

**Remark:** A birational point of view. A 1-dimensional foliation on \(\mathbb{P}^n\) may be viewed as a derivation of the field of meromorphic functions \(K(X_1, \ldots, X_n)\). In the above example of a 1-dimensional foliation with maximum number of singularities of dimension \(m\), what we have done is: take a derivation \(D\) of \(K(X_1, \ldots, X_{n-m})\) (a 1-dimensional foliation of \(\mathbb{P}^{n-m}\)) with isolated singularities and extend it as a derivation of \(K(X_1, \ldots, X_n)\), in the trivial way. The singular locus of this foliation has dimension \(m\), the same degree as \(D\), and the number of singularities of dimension \(m\) coincides with the number of singularities of \(D\).
2. **Higher dimensional foliations on a projective variety**

Let $X$ be a smooth and projective variety of dimension $n$ over an algebraically closed field $K$. Let $T_X$ be the sheaf of sections of the tangent bundle on $X$. Let $\mathcal{O}_X(1)$ be an ample invertible sheaf and $H$ an associated divisor. It is well known that $T_X \otimes \mathcal{O}_X(l)$ is generated by its global sections, for $l$ large enough. Let us denote $\mu$ the minimum integer such that $T_X \otimes \mathcal{O}_X(\mu)$ is generated by its global sections. Let $\mathcal{P} \hookrightarrow T_X$ be a locally free subsheaf of rank $r$. If $\mathcal{P}$ is integrable, then it is called an $r$-dimensional foliation, but we shall not assume integrability conditions. Let $\text{Sing}(\mathcal{P})$ be the singular locus of $\mathcal{P}$, that is, the set of points where the morphism between the associated bundles is not injective.

One has

**Theorem 2.1.** Let $m$ be the dimension of $\text{Sing}(\mathcal{P})$, $d = n - m$ the codimension. Let $C_1, \ldots, C_l$ be the irreducible components of dimension $m$ of $\text{Sing}(\mathcal{P})$. Then

$$\sum_{i=1}^{l} n_{C_i}(\mathcal{P}) \cdot \deg(C_i) \leq \deg \left[ \frac{c(T_X)}{c(\mathcal{P})(1 - \mu H)^{m-r+1}} \right]_d$$

where $c(T_X)$, $c(\mathcal{P})$ denote the total Chern class of $T_X$, $\mathcal{P}$ and $[\cdot]_d$ denotes to take the component of degree $d$.

**Proof.** It follows the same steps as the one for one-dimensional foliations on $\mathbb{P}^n$. Taking into account that $T_X(\mu)$ is generated by its global sections, one obtains that there exists a morphism $i \circ f : \mathcal{O}(\mathcal{P}) \otimes f : \mathcal{P} \otimes \mathcal{O}_X(\mu)^m \rightarrow T_X$ such that the singular locus of $i \circ f$ has dimension $m$. Applying Kempf-Laksov theorem to this morphism, and taking degree, one concludes.

**Example.** If $X = \mathbb{P}^n$, then the above formula results in

$$\sum_{i=1}^{l} n_{C_i}(\mathcal{P}) \cdot \deg(C_i) \leq \deg \left[ c_d(\mathcal{O}_{\mathbb{P}^n}(1)^{d+r} - \mathcal{P}) \right].$$

When $\text{Sing}(\mathcal{P})$ has “good” dimension (that is, $m = r - 1$), the number of singularities is given by the classical formula of Baum, Bott, Kempf, Laksov ([2], [4]). So Theorem 2.1 extends this classical formula for the case of “bad” dimension of the singularities.

**References**


**Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, Spain**

E-mail address: fsancho@gugu.usal.es