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NUMBER OF SINGULARITIES OF A FOLIATION ON $\mathbb{P}^n$

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ABSTRACT. Let $\mathcal{D}$ be a one dimensional foliation on a projective space, that is, an invertible subsheaf of the sheaf of sections of the tangent bundle. If the singularities of $\mathcal{D}$ are isolated, Baum-Bott formula states how many singularities, counted with multiplicity, appear. The isolated condition is removed here. Let m be the dimension of the singular locus of $\mathcal{D}$. We give an upper bound of the number of singularities of dimension m, counted with multiplicity and degree, that $\mathcal{D}$ may have, in terms of the degree of the foliation. We give some examples where this bound is reached. We then generalize this result for a higher dimensional foliation on an arbitrary smooth and projective variety.

INTRODUCTION

Throughout this paper $\mathbb{P}^n$ denotes the projective space of dimension n over an algebraically closed field K of any characteristic and $T_{\mathbb{P}^n}$ the sheaf of sections of the tangent bundle.

Let $\mathcal{D}$ be a 1-dimensional foliation on $\mathbb{P}^n$, that is, $\mathcal{D} \hookrightarrow T_{\mathbb{P}^n}$ is an invertible subsheaf. Put $\mathcal{D} \simeq \mathcal{O}_{\mathbb{P}^n}(-k)$; the number k is called *degree* of the foliation. The singular locus of the foliation is the singular locus of the morphism $\mathcal{D} \to T_{\mathbb{P}^n}$, that is, the set of points $p \in \mathbb{P}^n$ such that $\mathcal{D} \otimes K(p) \to T_{\mathbb{P}^n} \otimes K(p)$ is null. It is a closed subset defined by the Fitting ideal $F_{n-1}(T_{\mathbb{P}^n}/\mathcal{D})$, which is the image of the natural morphism $\Omega^1_{\mathbb{P}^n} \otimes \mathcal{D} \to \mathcal{O}_{\mathbb{P}^n}$. We shall give it a scheme structure by putting $\operatorname{Sing}(\mathcal{D}) = \operatorname{Spec} \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}$, with $\mathcal{I} = F_{n-1}(T_{\mathbb{P}^n}/\mathcal{D})$.

In each affine chart, with affine cooordinates $x_1 \dots, x_n$, $\mathcal{D}$ is given by a vector field $f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$ and the singularities of $\mathcal{D}$ are given by the equations $f_1 = \cdots = f_n = 0$. In other words, $\mathcal{I} = (f_1, \dots, f_n)$.

In general, the singularities of a 1-dimensional foliation on $\mathbb{P}^n$ are isolated, that is, $\operatorname{Sing}(\mathcal{D})$ is a zero dimensional scheme. In this case the number of singularities of $\mathcal{D}$ is given by the classical Baum-Bott formula ([1]):

$$\sum_{p \in \operatorname{Sing}(\mathcal{D})} n_p = (k+1)^n + (k+1)^{n-1} + \cdots + (k+1) + 1$$

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where n_p denotes the Milnor number of $\mathcal{D}$ at p, that is, $n_p = \dim_K(\mathcal{O}_p/(f_1,\dots,f_n))$. Now we want to study the non isolated case. For example, assume that $\mathcal{D}$ is a 1-dimensional foliation on $\mathbb{P}^3$, and the singular locus is made of some curves and some points out of them. Our question is: how many curves can appear? We shall give a complete answer in terms of the degree of the foliation. We shall prove:

Theorem. *Let $\mathcal{D}$ be a one dimensional foliation on $\mathbb{P}^n$ of degree k and let m be the dimension of* $\operatorname{Sing}(\mathcal{D})$. *The number of singularities of $\mathcal{D}$ of dimension m, counted with multiplicity and degree, is bounded by*

$$(k+1)^{n-m} + \cdots + (k+1) + 1$$

and this bound is reached for some foliation.

We shall extend this result for a higher dimensional foliation on an arbitrary smooth and projective variety (Theorem 2.1).

1. One dimensional foliations on $\mathbb{P}^n$

For the sake of clarity we are going to detail the case of a one dimensional foliation on $\mathbb{P}^n$, and we shall later generalize to higher dimensional foliations and to arbitrary projective varieties.

Let $\mathcal{D}$ be a 1-dimensional foliation on $\mathbb{P}^n$, and $\operatorname{Sing}(\mathcal{D})$ its singular locus, of ideal $\mathcal{I}$. Let C be an irreducible component of $\operatorname{Sing}(\mathcal{D})$, with the reduced structure. We shall denote by $n_C(\mathcal{D})$ the multiplicity of $\mathbb{P}^n$ along $\operatorname{Sing}(\mathcal{D})$ at C (see Fulton [3], page 79). When C is an isolated point, this number coincides with the Milnor number.

Theorem 1.1. *Let $\mathcal{D}$ be a one-dimensional foliation on $\mathbb{P}^n$ of degree k. Let m be the (non-necessarily pure) dimension of* $\operatorname{Sing}(\mathcal{D})$ *and $d = n - m$. Let $C_1,\dots,C_l$ be the irreducible components of dimension m of* $\operatorname{Sing}(\mathcal{D})$*, with its reduced structure. Then*

$$\sum_{i=1}^{l} n_{C_i}(\mathcal{D}) \cdot \mathit{deg}\ (C_i) \le (k+1)^d + (k+1)^{d-1} + \cdots + (k+1) + 1.$$

Proof. The main ingredient of the proof is the following:

Lemma 1.2. *Let us denote $i_{\mathcal{D}}\colon \mathcal{D} \hookrightarrow T_{\mathbb{P}^n}$. There exists a morphism $f\colon \mathcal{O}_{\mathbb{P}^n}(1)^m \to T_{\mathbb{P}^n}$ such that the singular locus of the morphism $i_{\mathcal{D}} \oplus f\colon \mathcal{D} \oplus \mathcal{O}_{\mathbb{P}^n}(1)^m \to T_{\mathbb{P}^n}$ has dimension m.*

Proof. Let $\mathcal{E} = \underline{\operatorname{Hom}}(\mathcal{O}_{\mathbb{P}^n}(1)^m, T_{\mathbb{P}^n})$ be the bundle of homomorphisms of $\mathcal{O}_{\mathbb{P}^n}(1)^m$ to $T_{\mathbb{P}^n}$. In other words, it is the bundle associated to the sheaf of modules $T_{\mathbb{P}^n}(-1) \oplus \overset{m}{\cdots} \oplus T_{\mathbb{P}^n}(-1)$. Denote $\pi\colon \mathcal{E} \to \mathbb{P}^n$ the structural morphism and E the global sections of $\mathcal{E}$, which is a finite dimensional vector space. From the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0$$

it follows easily that $\mathcal{E}$ is generated by its global sections. That is, one has an epimorphism of bundles over $\mathbb{P}^n$

$$\phi\colon \mathbb{P}^n \times E \to \mathcal{E}.$$

Let $\tau\colon \pi^*\mathcal{O}_{\mathbb{P}^n}(1)^m \to \pi^* T_{\mathbb{P}^n}$ be the universal homomorphism. It is not difficult to see that the singular locus of the morphism $i_{\mathcal{D}} \oplus \tau\colon \pi^*(\mathcal{D} \oplus \mathcal{O}_{\mathbb{P}^n}(1)^m) \to \pi^* T_{\mathbb{P}^n}$ has codimension d (see, for example, [5]). Then $\phi^{-1}(\operatorname{Sing}(i_{\mathcal{D}} \oplus \tau))$ has codimension d

in $\mathbb{P}^n \times E$. Let us consider the projection $p\colon \mathbb{P}^n \times E \to E$. By dimensions, there exists $e \in E$, such that the intersection of $\phi^{-1}(\mathrm{Sing}(i_{\mathcal{D}} \oplus \tau))$ with the fibre of e has codimension d in such fibre. That is, there exists a morphism $e\colon \mathcal{O}_{\mathbb{P}^n}(1)^m \to T_{\mathbb{P}^n}$ such that the singular locus of the morphism $i_{\mathcal{D}} \oplus e\colon \mathcal{D} \oplus \mathcal{O}_{\mathbb{P}^n}(1)^m \to T_{\mathbb{P}^n}$ has codimension d. □

Now the proof of the theorem: by the lemma, let us consider a morphism $f\colon \mathcal{O}_{\mathbb{P}^n}(1)^m \to T_{\mathbb{P}^n}$ such that the singular locus of $i_{\mathcal{D}} \oplus f\colon \mathcal{D} \oplus \mathcal{O}_{\mathbb{P}^n}(1)^m \to T_{\mathbb{P}^n}$ has dimension m. It is clear that $\mathrm{Sing}(\mathcal{D}) \subset \mathrm{Sing}(i_{\mathcal{D}} \oplus f)$. Let $C_1, \dots, C_l, W_1, \dots, W_s$ be the irreducible components (with the reduced structure) of $\mathrm{Sing}(i_{\mathcal{D}} \oplus f)$, where $C_1, \dots, C_l$ are the irreducible components of $\mathrm{Sing}(\mathcal{D})$. By Baum-Bott-Kempf-Laksov theorem (see [2] for the integrable case or [4] for the general case),

$$\sum n_{C_i}(i_{\mathcal{D}} \oplus f) \cdot C_i + \sum n_{W_i}(i_{\mathcal{D}} \oplus f) \cdot W_i = c_d(T_{\mathbb{P}^n} - \mathcal{D} \oplus \mathcal{O}_{\mathbb{P}^n}(1)^m) \tag{1}$$

where $n_{C_i}(i_{\mathcal{D}} \oplus f)$ and $n_{W_i}(i_{\mathcal{D}} \oplus f)$ are the multiplicities of $\mathbb{P}^n$ along $\mathrm{Sing}(i_{\mathcal{D}} \oplus f)$ at C_i and W_i, respectively. By the Euler sequence, $T_{P^n} = -\mathcal{O}_{\mathbb{P}^n} + \mathcal{O}_{\mathbb{P}^n}(1)^{n+1}$ in K-theory, and $\mathcal{D} = \mathcal{O}_{\mathbb{P}^n}(-k)$. So, if H denotes a hyperplane of $\mathbb{P}^n$, then

$$c_d(T_{\mathbb{P}^n} - \mathcal{D} \oplus \mathcal{O}_{\mathbb{P}^n}(1)^m) = \left[\frac{(1+H)^{d+1}}{1-kH}\right]_d = ((k+1)^d + \cdots + (k+1) + 1) \cdot H^d.$$

On the other hand, $n_{C_i}(i_{\mathcal{D}} \oplus f) \geq n_{C_i}(\mathcal{D})$ (since the ideal defining $\mathrm{Sing}(\mathcal{D})$ contains the one defining $\mathrm{Sing}(i_{\mathcal{D}} \oplus f)$). Then, taking degree in (1), one concludes.

Now let us see that this bound is reached. Let $\mathcal{D}_d$ be a 1-dimensional foliation on $\mathbb{P}^d$ of degree $k \geq -1$, with isolated singularities $q_1, \dots, q_l$. Then

$$\sum_{q \in \mathrm{Sing}(\mathcal{D}_d)} n_q(\mathcal{D}_d) = (k+1)^d + \cdots + (k+1) + 1.$$

Let $\mathbb{A}^d$ be an affine chart containing all the singularities of $\mathcal{D}_d$. Let $x_1, \dots, x_d$ be affine coordinates of $\mathbb{A}^d$ and $\mathcal{D}'_d = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$ a generator of $\mathcal{D}_d$ in $\mathbb{A}^d$. Let us consider the natural projection $\pi\colon \mathbb{A}^d \times \mathbb{A}^{n-d} \to \mathbb{A}^d$. Then $\mathcal{D}' = \pi^* \mathcal{D}'_d$ is a vector field on $\mathbb{A}^n$ with singularities $C'_i = \pi^{-1}(q_i)$, which have codimension d. Moreover, it is clear that $n_{C'_i}(\mathcal{D}') = n_{q_i}(\mathcal{D}_d)$. Now, $\mathcal{D}'$ extends to a 1-dimensional foliation $\mathcal{D}$ on $\mathbb{P}^n$. If C_i is the closure of C'_i in $\mathbb{P}^n$, then $n_{C_i}(\mathcal{D}) = n_{q_i}(\mathcal{D}_d)$. Then $\sum n_{C_i}(\mathcal{D}) = (k+1)^d + \cdots + (k+1) + 1$. A computation shows that the codimension of $\mathrm{Sing}(\mathcal{D})$ is d (in fact, there are no singularities of codimension $\leq d$ at the infinity). Besides, the degree of $\mathcal{D}$ is still k. In conclusion, the number of singularities of $\mathcal{D}$ of codimension d is $(k+1)^d + \cdots + (k+1) + 1$, with $k =$ degree of $\mathcal{D}$. □

Remark: A birational point of view. A 1-dimensional foliation on $\mathbb{P}^n$ may be viewed as a derivation of the field of meromorphic functions $K(X_1, \dots, X_n)$. In the above example of a 1-dimensional foliation with maximum number of singularities of dimension m, what we have done is: take a derivation D of $K(X_1, \dots, X_{n-m})$ (a 1-dimensional foliation of $\mathbb{P}^{n-m}$) with isolated singularities and extend it as a derivation of $K(X_1, \dots, X_n)$, in the trivial way. The singular locus of this foliation has dimension m, the same degree as D, and the number of singularities of dimension m coincides with the number of singularities of D.

2. Higher dimensional foliations on a projective variety

Let X be a smooth and projective variety of dimension n over an algebraically closed field K. Let T_X be the sheaf of sections of the tangent bundle on X. Let $\mathcal{O}_X(1)$ be an ample invertible sheaf and H an associated divisor. It is well known that $T_X \otimes \mathcal{O}_X(l)$ is generated by its global sections, for l large enough. Let us denote μ the minimum integer such that $T_X \otimes \mathcal{O}_X(\mu)$ is generated by its global sections. Let $\mathcal{P} \hookrightarrow T_X$ be a locally free subsheaf of rank r. If $\mathcal{P}$ is integrable, then it is called an r-dimensional foliation, but we shall not assume integrability conditions. Let $\operatorname{Sing}(\mathcal{P})$ be the singular locus of $\mathcal{P} \hookrightarrow T_X$, that is, $\operatorname{Sing}(\mathcal{P})$ is the set of points where the morphism between the associated bundles is not injective. One has

Theorem 2.1. *Let m be the dimension of $\operatorname{Sing}(\mathcal{P})$, $d = n - m$ the codimension. Let $C_1, \dots, C_l$ be the irreducible components of dimension m of $\operatorname{Sing}(\mathcal{P})$. Then*

$$\sum_{i=1}^{l} n_{C_i}(\mathcal{P}) \cdot deg\ (C_i) \leq deg\ \left[\frac{c(T_X)}{c(\mathcal{P})(1-\mu H)^{m-r+1}}\right]_d$$

where $c(T_X)$, $c(\mathcal{P})$ denote the total Chern class of T_X, $\mathcal{P}$ and $[\quad]_d$ denotes to take the component of degree d.

Proof. It follows the same steps as the one for one-dimensional foliations on $\mathbb{P}^n$. Taking into account that $T_X(\mu)$ is generated by its global sections, one obtains that there exists a morphism $f\colon \mathcal{O}_X(-\mu)^m \to T_X$ such that the singular locus of $i_{\mathcal{P}} \oplus f\colon \mathcal{P} \oplus \mathcal{O}_X(\mu)^m \to T_X$ has dimension m. Applying Kempf-Laksov theorem to this morphism, and taking degree, one concludes. $\square$

Example. If $X = \mathbb{P}^n$, then the above formula results in

$$\sum_{i=1}^{l} n_{C_i}(\mathcal{P}) \cdot \deg\ (C_i) \leq \deg\ \left[c_d(\mathcal{O}_{\mathbb{P}^n}(1)^{d+r} - \mathcal{P})\right].$$

When $\operatorname{Sing}(\mathcal{P})$ has "good" dimension (that is, $m = r-1$), the number of singularities is given by the classical formula of Baum, Bott, Kempf, Laksov ([2], [4]). So Theorem 2.1 extends this classical formula for the case of "bad" dimension of the singularities.

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