ON SOME INEQUALITIES INVOLVING THE ZEROS AND WEIGHTED $L^p$ NORMS OF POLYNOMIALS

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Abstract. Using Parseval’s identity and the Hardy-Littlewood-Pólya inequality on the maximal decreasing rearrangement, we establish some sharp inequalities involving the weighted $L^p$ norm and the zeros of polynomials.

1. Introduction

Let $\{z_j\}$ be the zeros (counting the multiplicity) of the polynomial

$$p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$$

and let

$$\|p\|_2 = \left( \frac{1}{2\pi} \int_0^{2\pi} |p(e^{it})|^2 dt \right)^{\frac{1}{2}}.$$

Recently, Kroo and Pritsker [1, Theorem 2.4] proved

Theorem A.

$$\prod_{j=1}^{n} (1 + |z_j|^2) \leq 2^{n-1} \|p\|_2^2.$$  

The inequality is best possible and the equality is achieved if and only if

$$p(z) = z^n + a, \text{ where } |a| = 1.$$ 

Moreover

$$\|p\|_2^2 \leq 2^{-n} \left( \frac{2n}{n} \right) \prod_{j=1}^{n} (1 + |z_j|^2).$$

The inequality is best possible and the equality is achieved if and only if

$$p(z) = (z + a)^n, \text{ where } |a| = 1.$$ 

The main goal of this note is to establish the following generalization of (1.2).

Theorem 1. Let $w(t)$ be a non-negative weight function defined on $[-\pi, \pi]$ with the following properties:

1) $w(-t) = w(t)$,

2) $w(t)$ is monotonically decreasing on $[0, \pi]$, and

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(3) \( w(t) \) is integrable on \([-\pi, \pi]\).

Then, for \( c \geq 2 \),

\[
\|p\|_c \leq 2^{-\frac{c}{n}} \prod_{j=1}^{n} (1 + |z_j|^2)^{\frac{1}{2}} \|P^*\|_c,
\]

where \( P^*(z) = (1 + z)^n \) and \( \|p\|_c = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |p(e^{it})|^c w(t) dt \right)^{\frac{1}{c}}. \)

Since the letter \( p \) is reserved for polynomials, to avoid unnecessary confusion, the letter \( c \) takes the place of the standard letter \( p \) for the notation of the \( L^p \) norm.

The method used in [1] does not seem applicable in dealing with the weighted \( L^p \) norm, whence a completely different approach is employed in this work. In Section 3, we will give a very different proof of (1.1) based on a simple property of the Blaschke product and the Parseval identity.

2. Proof of Theorem 1

One key ingredient of the proof is based on the following extremal property of the decreasing symmetric rearrangements of non-negative functions (see [2, p. 278] for the definition of the symmetric rearrangement of a function).

**Lemma 2.1.** Let \( g_j \) be continuous and non-negative on the interval \([-a, a]\) and let \( g_j^* \) be the symmetrical rearrangement of \( g_j \). Then, for any measurable set \( E \subset [-a, a] \) of measure \( m(E) = 2c \) with \( c \leq a \),

\[
\int_E \prod g_j(x) dx \leq \int_{-c}^{c} \prod g_j^*(x) dx.
\]

The result is a slight generalization of Theorem 378 in [2, p. 278]. For the reader’s convenience we briefly outline the proof.

**Proof.** Let \( E_j \subset [-a, a], j = 1, 2, \ldots, n, \) be measurable and let \( K_{E_j}(x) \) be the characteristic functions of \( E_j \) and let \( E^* = [-c, c] \) and \( E_j^* = [-c_j, c_j] \), where \( 2c_j = m(E_j) \). Then, the decreasing symmetric rearrangement of \( K_{E_j}(x) \) is \( K_{E_j}(x) \). Clearly, we have

\[
\int_E \prod K_{E_j}(x) dx = m \left( \bigcap_{j=1}^{n} E_j \cap E \right) \leq \min_j m(E \cap E_j) \leq \min_j m(E^* \cap E_j^*) = \int_{E^*} \prod K_{E_j^*}(x) dx.
\]

Let \( s(x) > 0 \) be a simple function. Then (see Section 10.13 of [2]) we can represent \( s(x) \) in the form

\[
s(x) = a_1 K_{E_1}(x) + a_2 K_{E_2}(x) + \cdots + a_m K_{E_m}(x)
\]

with \( E_1 + 1 \subset E_1 \), so that one has

\[
s^*(x) = a_1 K_{E_1^*}(x) + a_2 K_{E_2^*}(x) + \cdots + a_m K_{E_m^*}(x),
\]
where $a_j > 0$ and $j = 1, 2, \ldots, m$. For the simple functions $s_j(x)$, the inequality

$$
\int_E \prod s_j(x) dx \leq \int_E \prod s_j^*(x) dx
$$

follows from a linear combination of (2.1).

Finally, we establish the lemma in the general case by approximating $g$ in terms of simple functions.

**Corollary 1.** Let $p(z) = \prod_{j=1}^n (z - z_j)$ and $p^*(z) = \prod_{j=1}^n (z + x_j)$, where $x_j = |z_j|$. Then

$$
\int_{-\pi}^{\pi} |p(e^{it})|^c w(t) dt \leq \int_{-\pi}^{\pi} |p^*(e^{it})|^c w(t) dt
$$

for all $c \geq 0$.

**Lemma 2.2.** Let $f \in L^1[0, \pi]$.

(a) Suppose $f$ is non-negative and monotonically decreasing on $[0, \pi]$. Then

$$
\int_{0}^{\pi} f(t) \cos t \, dt > 0.
$$

(b) Let $c \geq 2$ and let $f$ be as in (a). Suppose, for $0 \leq x < \infty$,

$$
F(x) = (1 + x^2)^{-\frac{2}{3}} \int_{0}^{\pi} |x + e^{it}|^c f(t) \, dt.
$$

Then $F(x)$ achieves its absolute maximum at $x = 1$.

**Proof.** (a) We note that, with the replacement of $t$ by $t + \frac{\pi}{2}$, the integral becomes

$$
- \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f \left( t + \frac{\pi}{2} \right) \sin t \, dt = \int_{0}^{\frac{\pi}{2}} \left( f \left( \frac{\pi}{2} - t \right) - f \left( \frac{\pi}{2} + t \right) \right) \sin t \, dt,
$$

which, from the monotonicity assumption on $f$, is clearly non-negative.

(b) A simple calculation shows that

$$
F'(x) = \frac{c(1 - x^2)}{(1 + x^2)^{\frac{5}{2}}} \int_{0}^{\pi} g(t) \cos t \, dt,
$$

where $g(t) = \left( 1 + \frac{2x}{1 + x^2} \cos t \right)^{\frac{5}{2} - 1} f(t)$.

We observe that, for $c \geq 2$, the function $g(t)$ is monotonically decreasing on $[0, \pi]$ for all $0 \leq x < \infty$. Hence, from part (a), we conclude that the integral in (2.2) is positive. That the absolute maximum of $F$ occurs at $x = 1$ now follows from the fact that $F'(x) = 0$ only if $x = 1$, $F'(x) > 0$ for $0 \leq x < 1$, and $F'(x) < 0$ for $x > 1$.

We now come to the proof of Theorem 1.

**Proof.** For a given $p(z)$, we consider the quantity

$$
v(p) = \int_{-\pi}^{\pi} \frac{|p(e^{it})|^c w(t) \, dt}{\prod_{j=1}^n (1 + |z_j|^2)^{\frac{c}{2}}},
$$

Using Corollary 1 and then applying Lemma 2.2(b) inductively on each $|z_j|$, we deduce that

$$
v(p) \leq v(p^*) \leq v(P^*),
$$
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where

\[ v(P^*) = 2^{-\frac{n}{2}} \int_{-\pi}^{\pi} |(e^{it} + 1)^n|^c w(t) \, dt. \]

This establishes Theorem 1.

\[ \square \]

3. AN ALTERNATIVE PROOF OF (1.1)

The proof is essentially based on the following:

**Lemma 3.1.** Let \( A \) be a subset of \( N = \{1, 2, \ldots, n\} \) and \( \tilde{A} \) be the complement \( N \setminus A \). Define

\[ d(A) = \prod_{j \in A} |z_j| \quad \text{and} \quad d(\tilde{A}) = \prod_{j \in \tilde{A}} |z_j|. \]

Then

\[ d^2(A) + d^2(\tilde{A}) \leq \|p\|_2^2. \]

**Proof.** Let \( B(z, A) = \prod_{j \in A} \frac{z - z_j}{1 - z_j} \) (we note that \( B(z, A) \) has a removable singularity at \( z_j \) in case \( |z_j| = 1 \)). Then,

\[ |B(z, A)| = 1 \text{ if } |z| = 1, \]

and we can write

\[ p(z) = B(z, A)P_A(z), \]

where

\[ P_A(z) = \prod_{j \in A} (1 - z z_j) \prod_{j \in A} (z - z_j) = d_0 z^n + \cdots + d_n \]

with \( d_0 = \prod_{j \in A} (-z_j) \) and \( d_n = \prod_{j \in \tilde{A}} (-z_j) \). From (3.1), (3.2), and Parseval’s identity,

\[ \|p\|_2^2 = \|P_A\|_2^2 = \sum_{j=0}^{n} |d_j|^2 \geq d^2(A) + d^2(\tilde{A}). \]

\[ \square \]

We now prove (1.1).

**Proof of (1.1).** We first make a simple and crucial observation about the product \( \prod_{j=1}^{n} (1 + |z_j|^2) \):

(a) Its summand consists of \( 2^n \) terms (some of them may be repeated if the \( z_j \) are not distinct) and

(b) it can be rewritten as

\[ \prod_{j=1}^{n} (1 + |z_j|^2) = \frac{1}{2} \sum_{i=1}^{2^n} d^2(A_i) + d^2(\tilde{A}_i), \]

where \( A_i \) runs through all the subsets of \( N \).

Applying (3.3) to (3.5), the desired conclusion follows.

\[ \square \]
We observe that if equality holds in (1.1), then the equality holds for (3.4) for every subset $A$ of $N$. In particular, it implies that $P_A$ consists of two terms, from which one easily deduces that
\[ p(z) = z^n + a. \]

A direct computation shows that $|a| = 1$.

From the inequality
\[ \left( \frac{a^t + b^t}{2} \right)^{\frac{1}{t}} \leq \left( \frac{a^s + b^s}{2} \right)^{\frac{1}{s}} \text{ if } s \geq t \text{ and } a, b \geq 0, \]
we deduce the following:

**Corollary 2.** Let $0 < c \leq 2$. Then
\[ \prod_{j=1}^{n} (1 + |z_j|^c) \leq 2^{n - \frac{n}{2}} \| p \|_2^c. \]

**References**


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