EXPLICIT EVALUATIONS OF A RAMANUJAN-SELBERG CONTINUED FRACTION

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To the memory of my father, Professor Guang-Da Zhang

Abstract. This paper gives explicit evaluations for a Ramanujan-Selberg continued fraction in terms of class invariants and singular moduli.

§1. Introduction

Let, for $|q| < 1$,

$$N(q) = 1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^4}{1 + \cdots}}}.$$  

Set

$$(a;q)_\infty := \prod_{k=1}^{\infty} (1 - aq^{k-1}).$$  

In his notebooks [14, p. 290], Ramanujan asserted that

$$N(q) = \frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}. $$

This formula was first proved in print by A. Selberg [18]. Other proofs have been given by K. G. Ramanathan [12], G. Andrews et al. [1] and the author [21].

In his “Lost” Notebooks [16, p. 44], Ramanujan also stated that if $|q| < 1$, and

$$L(q) = 1 + \frac{q^2}{1 + \frac{q^4}{1 + \cdots}},$$

then

$$L(q) = \frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}. $$

Here, we just point out that (1.5) can be proved by using the well-known Heine [10] continued fraction formula in the same fashion as the proof of (1.3) in the author’s paper [21]. Set, for $|q| < 1$,

$$S_1(q) = \frac{q^{1/8}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^4}{1 + \cdots}}}}.$$
From (1.1), (1.3) and (1.5), we have
\[ S_1(q) = \frac{q^{1/8}}{N(q)} = \frac{q^{1/8}}{L(q)} = \frac{q^{1/8}(-q^2;q^2)_\infty}{(-q;q^2)_\infty}. \]

We call \( S_1(q) \) the Ramanujan-Selberg continued fraction.

Also, set
\[ S_2(q) = \frac{q^{1/8}}{1 + q + q^2 + q^3 + q^4 + \cdots}. \]

Replacing \( q \) by \(-q\) in (1.1) and (1.3), one can see that
\[ S_2(q) = \frac{q^{1/8}}{N(-q)} = \frac{q^{1/8}}{L(-q)} = \frac{q^{1/8}(-q^2;q^2)_\infty}{(q;q^2)_\infty}. \]

The famous Rogers-Ramanujan continued fraction is defined by
\[ F(q) = \frac{q^{1/5}}{1 + q + q^2 + q^3 + q^4 + \cdots}, \]
and let \( S(q) = -F(-q) \). In his first letter to G. H. Hardy, Ramanujan asserted that
\[ F(e^{-2\pi}) = \sqrt{\frac{5 + \sqrt{5}}{2}} - \frac{\sqrt{5} + 1}{2}, \]
\[ S(e^{-\pi}) = \sqrt{\frac{5 - \sqrt{5}}{2}} - \frac{\sqrt{5} - 1}{2}, \]
and
\[ F(e^{-\pi \sqrt{n}}) \] can be exactly found if \( n \) is any positive rational quantity.

Identities (1.11) and (1.12) were first proved by G. N. Watson [19]. Watson vaguely discussed (1.13) and merely claimed that \( F(e^{-\pi \sqrt{n}}) \) is an algebraic number.

Ramanathan [13] computed \( F(e^{-2\pi \sqrt{n}}) \) and \( S(e^{-\pi \sqrt{n}}) \) for several positive rational numbers \( n \) for which the ideal class groups of \( K = \mathbb{Q}(\sqrt{-n}) \) have the property that each genus contains a single class. By using Weber-Ramanujan’s class invariants and a modular equation of degree 5, Berndt, Chan and the author [4] were able to establish general formulas for \( F(e^{-2\pi \sqrt{n}}) \) and \( S(e^{-\pi \sqrt{n}}) \).

The aim of this note is to establish general formulas for the Ramanujan-Selberg continued fraction and its companion in terms of class invariants, or equivalently in terms of singular moduli.

\section{Explicit formulas for \( S_1(q) \) and \( S_2(q) \)}

For \( q = \exp(-\pi \sqrt{n}) \), where \( n \) is positive rational, let
\[ G_n := 2^{-1/4}q^{1/24}(-q;q^2)_\infty \]
and
\[ g_n := 2^{-1/4}q^{1/24}(q;q^2)_\infty. \]

We shall refer to \( G_n \) and \( g_n \) as the Ramanujan-Weber class invariants, which can be roughly viewed as generators of the Hilbert class field of the complex quadratic field of \( K = \mathbb{Q}(\sqrt{-n}) \). The reader is referred to the important paper of B. Birch [7] and the excellent books of Cox [9] and Lang [11]. We also use modular equations in
the sequel, and refer to [2] pp. 213, 214] for this terminology. The singular modulus \( n \) is that unique positive number between 0 and 1 satisfying

\[
\sqrt{n} = \frac{\phantom{|}{2F_1(\frac{1}{2}; \frac{1}{2}; 1; 1 - \alpha_n)}{2F_1(\frac{1}{2}; \frac{1}{2}; 1; \alpha_n)},}
\]

where \( 2F_1 \) is the hypergeometric function. Moreover (cf. [2, p. 102]),

\[
2F_1(\frac{1}{2}; \frac{1}{2}; 1; \alpha) = \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \alpha \sin^2 \phi}}.
\]

Then we have [3, p. 185]

\[
G_n = (4\alpha_n(1 - \alpha_n))^{-1/24}
\]

and

\[
g_n = (4\alpha_n(1 - \alpha_n)^2)^{-1/24}.
\]

Let \( \alpha \) and \( \beta \) be moduli. We say that \( \beta \) is of degree \( d \) over \( \alpha \) if

\[
\frac{2F_1(\frac{1}{2}; \frac{1}{2}; 1; 1 - \beta)}{2F_1(\frac{1}{2}; \frac{1}{2}; 1; \beta)} = d \frac{2F_1(\frac{1}{2}; \frac{1}{2}; 1; 1 - \alpha)}{2F_1(\frac{1}{2}; \frac{1}{2}; 1; \alpha)}.
\]

Therefore, if \( \alpha = \alpha_n \) and \( \beta \) is of degree \( d \) over \( \alpha \), then, by (2.3), \( \beta = \alpha_n^d \). A modular equation of second degree is an equation connecting \( \alpha = \alpha_n \) and \( \beta = \alpha_n^4 \) which will be used in our proofs.

**Theorem** (modular equations of second degree [2 p. 214]). Let \( \beta \) be of second degree over \( \alpha \) and

\[
m = \frac{2F_1(\frac{1}{2}; \frac{1}{2}; 1; \alpha)}{2F_1(\frac{1}{2}; \frac{1}{2}; 1; \beta)}.
\]

Then

\[
m\sqrt{1 - \alpha} + \sqrt{\beta} = 1
\]

and

\[
m^2\sqrt{1 - \alpha} + \beta = 1.
\]

Now, we state and prove the main theorems.

**Theorem 2.1.** Let \( q = e^{-\pi \sqrt{n}} \) and \( \alpha = \alpha_n \). Then

\[
S_1(q) = \frac{q^{1/8}}{\sqrt{2}}.
\]

**Proof.** First, it is easy to show that (cf. [2, p. 37, (22.3)])

\[
(-q^2; q^2)_\infty = \frac{1}{(q^2; q^4)_\infty},
\]

which is a very famous theorem of Euler. By (1.7), (2.11), (2.1) and (2.2) we have

\[
S_1(q) = \frac{q^{1/8}}{(-q; q^2)_\infty (q^4; q^4)_\infty} = \frac{1}{\sqrt{2G_n g_{4n}}}.
\]

Set \( \alpha = \alpha_n \) and \( \beta = \alpha_{4n} \). Then \( \beta \) is of second degree over \( \alpha \). From (2.8) and (2.9), we find that

\[
\sqrt{\beta} = \frac{1 - \sqrt{1 - \alpha}}{1 + \sqrt{1 - \alpha}}.
\]
and
\begin{equation}
1 - \beta = \frac{4\sqrt{1 - \alpha}}{(1 + \sqrt{1 - \alpha})^2}. \tag{2.14}
\end{equation}

It follows that, by (2.6) and (2.14),
\begin{equation}
g_{4n} = \left( \frac{4\beta}{(1 - \beta)^2} \right)^{-1/24} = \left( \frac{2\sqrt{\beta}}{1 - \beta} \right)^{-1/12} = \left( \frac{1}{2\sqrt{1 - \alpha}} \right)^{-1/12}, \tag{2.15}
\end{equation}

Therefore, from (2.12), (2.5) and (2.15),
\begin{equation}
S_1(q) = \frac{1}{\sqrt{2}} (4\alpha(1 - \alpha))^{1/24} \left( \frac{\alpha^2}{4(1 - \alpha)} \right)^{1/24},
\end{equation}

This completes the proof.

**Corollary 2.2.** Let \( q = e^{-\pi\sqrt{\tau}} \), \( G = G_n \) and \( g = g_n \). Then
\begin{equation}
S_1(q) = 2^{-5/8} \left( 1 - \sqrt{1 - G^{-24}} \right)^{1/8}, \tag{2.16}
\end{equation}

and
\begin{equation}
S_1(q) = 2^{-1/2} \left( (1 + 2g^{24}) - \sqrt{(1 + 2g^{24})^2 - 1} \right)^{1/8}. \tag{2.17}
\end{equation}

**Proof.** From (2.5) and (2.6), we have
\begin{equation}
\alpha = \frac{1}{2} \left( 1 - \sqrt{1 - G^{-24}} \right), \tag{2.18}
\end{equation}

and
\begin{equation}
\alpha = (1 + 2g^{24}) - \sqrt{(1 + 2g^{24})^2 - 1}. \tag{2.19}
\end{equation}

Then, by (2.10), Corollary (2.2) follows immediately.

**Theorem 2.3.** Let \( q = e^{-\pi\sqrt{\tau}} \) and \( \alpha = \alpha_n \). Then
\begin{equation}
S_2(q) = \frac{1}{\sqrt{2}} \left( \frac{\alpha}{1 - \alpha} \right)^{1/8}. \tag{2.20}
\end{equation}

**Proof.** By (1.9), (2.11) and (2.2), we have
\begin{equation}
S_2(q) = \frac{\sqrt[8]{q^{1/8}}}{(q; q^2)_\infty(q^4; q^4)_\infty} = \frac{1}{\sqrt{2g_ng_{4n}}}. \tag{2.21}
\end{equation}

Then the theorem follows from (2.2), (2.6) and (2.15) immediately.

By (2.18) and (2.19), \( S_2(q) \) can be also expressed either in terms of \( G \) or \( g \).

The Theorems and Corollaries above provide explicit evaluations of the Ramanujan-Selberg continued fraction in terms of the Ramanujan-Weber class invariants or singular moduli. For values of \( G_n \) and \( g_n \), see the paper of Berndt, Chan and the author [5], and the author’s papers [22], [23], for values of \( \alpha_n \), see the paper of Berndt, Chan and the author [5]. Ramanujan calculated numerous class invariants.
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Example 1. We have (cf. [3, p. 282])

\[ \alpha_{58} = (13\sqrt{58} - 99)^2(99 - 70\sqrt{2})^2. \]

Then by (2.10), we find that

\[ S_1 \left( e^{-\pi\sqrt{58}} \right) = 2^{-1/2}(13\sqrt{58} - 99)^{1/4}(99 - 70\sqrt{2})^{1/4}. \]

Example 2. In his first notebook, Ramanujan [14, p. 310] claimed that

\[ \alpha_{10} = (\sqrt{10} - 3)^2(3 - 2\sqrt{2})^2 = \frac{3\sqrt{2} - \sqrt{5} - 2}{3\sqrt{2} + \sqrt{5} + 2}. \]

For a proof, see [3, p. 282]. Then

\[ \frac{\alpha_{10}}{1 - \alpha_{10}} = \frac{3\sqrt{10} - 1}{2} - 3\sqrt{2}, \]

and, by (2.20),

\[ S_2 \left( e^{-\pi\sqrt{10}} \right) = \frac{1}{\sqrt{2}} \left( \frac{3\sqrt{10} - 1}{2} - 3\sqrt{2} \right)^{1/8}. \]

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