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EQUIVARIANT COHOMOLOGY WITH LOCAL COEFFICIENTS

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Abstract. We show that for a discrete group G, the equivariant cohomology of a G-space X with G-local coefficients M is isomorphic to the Bredon-Illman cohomology of X with equivariant local coefficients M.

1. Introduction

Equivariant cohomology for spaces equipped with an action of a fixed group G was developed by Bredon ([1]) and Illman ([3]). This theory generalizes cohomology theory for spaces as laid down by Eilenberg-Steenrod. The kind of coefficients needed in the theory are not just fixed abelian groups but contravariant functors from the category of canonical orbits $\mathcal{O}(G)$ into the category of abelian groups. Bredon used this theory to develop obstruction theory for extending maps equivariantly.

Steenrod ([6]) extended the cohomology theory of Eilenberg-Steenrod by replacing a fixed coefficient group by a family of abelian groups parametrized by points of the space in question in order to develop obstruction theory for extending sections of a fibration. In fact, the coefficient in Steenrod's theory, known as a local coefficient system, is an abelian group-valued functor on the fundamental groupoid of the space.

In order to deal with the corresponding problem in the equivariant context, A. Mukherjee and G. Mukherjee ([5]) considered a category $\Pi_G(X)$ associated with a topological group G and a G-space X which generalized the fundamental groupoid of the space to the equivariant setup and defined an equivariant local system of coefficients on X as a contravariant functor M from $\Pi_G(X)$ to the category of abelian groups. They then defined the Bredon-Illman cohomology of X with local coefficients M which we will denote by $H^*_{BI}(X, M)$. They used this cohomology to set up obstruction theory for equivariant sections of a G-fibration where G is a compact Lie group.

Parallelly in time Moerdijk and Svensson ([4]) defined Bredon cohomology with local coefficients for spaces with an action of a discrete group G. With a G-space X they associated a category $\Delta_G(X)$. There exists a canonical functor v_X from $\Delta_G(X)$ to $\Pi_G(X)$. A G-local coefficient is then defined to be an abelian group-valued functor on $\Delta_G(X)$ which factors through $\Pi_G(X)$. For a discrete group G, a G-space X and a G-local coefficient system M, Bredon cohomology $H^*_G(X, M)$

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is then defined as the cohomology $H^*(\Delta_G(X), M)$. The main purpose of defining $H^*_G(X, M)$ is to obtain an equivariant Serre spectral sequence for a G-fibration.

The common feature of these two theories is that they reduce to equivariant singular cohomology of Bredon ([1], [3]) when M is simple, and to the Steenrod cohomology with classical local coefficient system ([6]) when G is trivial. It is then natural to expect that these two theories should agree when G is a discrete group. The purpose of this note is to show that for a discrete group G,

$$H^*_{BI}(X, M) \cong H^*_G(X, M).$$

Throughout the paper G denotes a discrete group. In sections 2 and 3 we recall the definitions of $H^*_{BI}(X, M)$ and $H^*_G(X, M)$. In the final section we prove our result.

2. Definition of $H^*_{BI}(X, M)$

For a discrete group G and a G-space X, $H^*_{BI}(X, M)$ can be defined as follows. Define a category $\Pi_G(X)$ whose objects are G-maps $x_H : G/H \to X$. A morphism in $\Pi_G(X)$ from $x_H : G/H \to X$ to $y_K : G/K \to X$ is a pair $(\hat{a}, [\phi])$, where $\hat{a} : G/H \to G/K$ is the G-map corresponding to $a^{-1}Ha \subseteq K$ and $[\phi]$ is the G-homotopy class of the G-homotopy $\phi : G/H \times I \longrightarrow X$ from x_H to $y_K \circ \hat{a}$. Here two G-homotopies ϕ_1 and ϕ_2 from x_H to $y_K \circ \hat{a}$ are G-homotopic if there exists a G-homotopy $\Phi : G/H \times I \times I \longrightarrow X$ from ϕ_1 to ϕ_2 such that $\Phi(gH, 0, t) = x_H(gH)$ and $\Phi(gH, 1, t) = y_K \circ \hat{a}(gH)$.

An *equivariant local coefficient system* on X is a functor $M : \Pi_G(X)^{op} \longrightarrow \underline{Ab}$ where $\underline{Ab}$ is the category of abelian groups.

Let us fix an ordering on the vertices of $\Delta^n, n \geq 0$, which is induced from the natural ordering of $\{0, \cdots, n\}$. An *equivariant n-simplex* is a G-map

$$\sigma : G/H \times \Delta^n \longrightarrow X.$$

To every equivariant n-simplex $\sigma : G/H \times \Delta^n \longrightarrow X$ we associate a G-map

$$\tilde{\sigma} : G/H \longrightarrow X$$

which is σ restricted to the first vertex of Δ^n.

We can think of $G/H \times \Delta^n$ and $G/K \times \Delta^n$ as trivial bundles over Δ^n. Two simplices $\sigma : G/H \times \Delta^n \longrightarrow X$ and $\tau : G/K \times \Delta^n \longrightarrow X$ are said to be *equivalent* if there exists a fibre-preserving G-map $h : G/H \times \Delta^n \longrightarrow G/K \times \Delta^n$ such that $\tau \circ h = \sigma$.

Let $S^n_G(X, M)$ be the group of all functions c which maps an equivariant n-simplex σ to an element of $M(\tilde{\sigma})$ so that if two simplices are equivalent as above, then $c(\sigma) = M(h_*)(c(\tau))$ where $h_* : \tilde{\sigma} \longrightarrow \tilde{\tau}$ is the morphism in $\Pi_G(X)$ induced by h. Then $S^\bullet_G(X, M)$ is a cochain complex and the Bredon-Illman cohomology of X with local coefficients M ([5]) is defined to be

$$H^*_{BI}(X, M) = H^*(S^\bullet_G(X, M)).$$

3. Definition of $H^*_G(X, M)$

Let X be a G-space. The category $\Delta_G(X)$ associated with X is defined to consist of equivariant n-simplices of X as its objects. A morphism between $\sigma : G/H \times \Delta^n \longrightarrow X$ and $\tau : G/K \times \Delta^m \longrightarrow X$ in $\Delta_G(X)$ is a pair (ϕ, α) where $\phi : G/H \longrightarrow G/K$ is a G-map and $\alpha : \{0, \cdots, n\} \longrightarrow \{0, \cdots, m\}$ is an order-preserving map so that $\tau \circ (\phi \times \alpha) = \sigma$. Here the map $\Delta^n \longrightarrow \Delta^m$ induced by α is also denoted by α.

There is a canonical functor $v_X : \Delta_G(X) \longrightarrow \Pi_G(X)$ which maps an equivariant n-simplex σ to $\tilde{\sigma}$ where $\tilde{\sigma}$ is σ restricted to the first vertex of Δ^n. Unlike [4] we consider restriction to the first vertex instead of the last vertex to make the notation consistent with the construction of $H^*_{BI}(X, M)$.

For any small category $\mathcal{C}$, let $\underline{Ab}(\mathcal{C})$ be the category of all contravariant functors from $\mathcal{C}$ to $\underline{Ab}$ with morphisms natural transformations.

A functor $M \in \underline{Ab}(\Delta_G(X))$ is said to be *G-local* if for some $M' \in \underline{Ab}(\Pi_G(X))$, $M \cong v^*_X(M')$. Note that the notion of a G-local coefficient system is the same as the equivariant local coefficient system as defined in [5]. For a G-local coefficient system M, the equivariant cohomology of X with coefficients M ([4]) is defined to be

$$H^*_G(X, M) = H^*(\Delta_G(X), M).$$

4. The isomorphism

In Theorem 2.2 of [4] it is proved that $H^*_G(X, M)$ is isomorphic to the Bredon cohomology $H^*_{Br}(X, M)$ when M is simple. We generalise this result to

Theorem 4.1. *Let X be a G-space and M a functor from $\Pi_G(X)^{op}$ to $\underline{Ab}$. Then there is an isomorphism*

$$H^*_{BI}(X, M) \cong H^*_G(X, M).$$

(On the right we identify M with $v^*_X(M)$.)

Proof. Let Δ be the category whose objects are $\underline{n} = \{0, \cdots, n\}$ with usual order and whose morphisms are order-preserving maps. Let $\mathcal{O}(G)$ be the category whose objects are G/H and whose morphisms are G-maps $\hat{a} : G/H \longrightarrow G/K$.

As in [4] we let $\tilde{X}$ be the bisimplicial set whose (p, q) simplices are triples (u, α, σ), where

$$\begin{aligned} &u = (\underline{n_0} \xrightarrow{u_1} \underline{n_1} \longrightarrow \cdots \xrightarrow{u_p} \underline{n_p}) \in N_p(\Delta), \\ &\alpha = (G/H_0 \xrightarrow{\alpha_1} G/H_1 \longrightarrow \cdots \xrightarrow{\alpha_q} G/H_q) \in N_q(\mathcal{O}(G)), \\ &\sigma : G/H_q \times \Delta^{n_p} \longrightarrow X \text{ is a } G\text{-map}. \end{aligned}$$

The face and degeneracy maps on $\tilde{X}$ are induced from those on $N(\Delta)$ and $N(\mathcal{O}(G))$. Then

$$\text{diagonal}(\tilde{X}) \cong N(\Delta_G(X)).$$

To every $(u, \alpha, \sigma) \in \tilde{X}^{p,q}$ we associate a G-map:

$$\overline{\sigma} = \sigma \circ (\alpha_q \circ \cdots \circ \alpha_1 \times u_p \circ \cdots \circ u_1) : G/H_0 \times \Delta^{n_0} \longrightarrow X.$$

Define $C^{p,q}(X, M)$ to be all functions on $\tilde{X}^{p,q}$ which send an element (u, α, σ) of $\tilde{X}^{p,q}$ to an element of $M(v_X(\overline{\sigma}))$. It follows quite easily that $C^{p,q}(X, M)$ is a bicomplex with obvious differentials d_h and d_v induced from the face maps of $\tilde{X}$. Denote the total complex of $C^{\bullet\bullet}(X, M)$ by $\operatorname{Tot} C^{\bullet\bullet}(X, M)$.

Let $\operatorname{diag} C^{\bullet\bullet}(X, M)$ be the cochain complex whose p^{th} group is $C^{p,p}(X, M)$ and whose differential is $d_h d_v$. Then by a result of Dold and Puppe ([2]) we have

$$H^n(\operatorname{Tot} C^{\bullet\bullet}(X, M)) \cong H^n(\operatorname{diag} C^{\bullet\bullet}(X, M)).$$

Now $C^{p,p}(X, M)$ can be thought of as all functions on $N(\Delta_G(X))$ which send a p-simplex $(\sigma_0 \longrightarrow \sigma_1 \longrightarrow \cdots \longrightarrow \sigma_p)$ to an element of $M(v_X(\sigma_0))$, and the differential on $C^{p,p}(X,M)$ is just the differential induced from the face maps of $N_p(\Delta_G(X))$. Hence,

$$H^n(\operatorname{diag} C^{\bullet\bullet}(X,M)) \cong H^n(\Delta_G(X), v_X^* M) = H^n_G(X,M).$$

We now compute the E_1 term of the spectral sequence associated with the p-filtration of the bicomplex $C^{\bullet\bullet}(X,M)$.

Recall that $S_n(X^{(-)}) : \mathcal{O}(G)^{op} \longrightarrow \underline{\text{Sets}} \subset \underline{\text{Cat}}$ is the functor which sends G/H to $S_n(X^H)$.

Let

$$\mathcal{C}_n = \int_{\mathcal{O}(G)} S_n(X^{(-)})$$

be the category obtained by the Grothendieck construction on $S_n(X^{(-)})$.

We can identify $\mathcal{C}_n$ with the category whose objects are equivariant n-simplices of X and whose morphisms between $\sigma : G/H \times \Delta^n \longrightarrow X$ and $\tau : G/K \times \Delta^n \longrightarrow X$ are G-maps $\hat{a} : G/H \longrightarrow G/K$ such that $\tau \circ (\hat{a} \times 1) = \sigma$.

Define a functor

$$M_n : \mathcal{C}_n^{op} \longrightarrow \underline{Ab}$$

as follows: M_n takes $\sigma : G/H \times \Delta^n \longrightarrow X$ to $M(v_X(\sigma))$. If $\hat{a} : G/H \longrightarrow G/K$ is a morphism from σ to τ, then $(\hat{a}, [id])$ is a morphism in $\Pi_G(X)$ from $v_X(\sigma)$ to $v_X(\tau)$ and we define $M_n(\hat{a}) = M((\hat{a}, [id]))$.

Fix a $u = (\underline{n_0} \xrightarrow{u_1} \cdots \xrightarrow{u_p} \underline{n_p}) \in N_p(\Delta)$. Let us denote the composition $u_p \circ \cdots \circ u_1$ by u again. Corresponding to this u there is a functor $F : \mathcal{C}_{n_p} \longrightarrow \mathcal{C}_{n_0}$ which takes an object $\sigma : G/H \times \Delta^{n_p} \longrightarrow X$ of $\mathcal{C}_{n_p}$ to $\sigma \circ (id \times u) : G/H \times \Delta^{n_0} \longrightarrow X$ and a morphism $\hat{a} : G/H \longrightarrow G/K$ between $\sigma : G/H \times \Delta^{n_p} \longrightarrow X$ and $\tau : G/K \times \Delta^{n_p} \longrightarrow X$ to $\hat{a}$. We define a functor M_u on $\mathcal{C}_{n_p}$ to be

$$M_u = M_{n_0} \circ F.$$

Then for all $p \geq 0$,

$$C^{p,q}(X,M) \cong \prod_{u \in N_p(\Delta)} C^q(\mathcal{C}_{n_p}, M_u),$$

the correspondence being given as follows: Let f be an element of $C^{p,q}(X,M)$. Then f induces an element

$$(f_u) \in \prod_{u \in N_p(\Delta)} C^q(\mathcal{C}_{n_p}, M_u),$$

where $f_u \in C^q(\mathcal{C}_{n_p}, M_u)$ is defined as follows: To a simplex

$$v = \sigma_0 \xrightarrow{\hat{a}_1 \times 1} \cdots \xrightarrow{\hat{a}_q \times 1} \sigma_q, \ \sigma_i : G/H_i \times \Delta^{n_p} \longrightarrow X,$$

of the nerve of $\mathcal{C}_{n_p}$, we associate a $(u, \alpha, \sigma) \in \tilde{X}^{p,q}$ where u is given by the choice of the index, $\sigma = \sigma_q$ and $\alpha = (G/H_0 \xrightarrow{a_1} G/H_1 \longrightarrow \cdots \xrightarrow{a_q} G/H_q)$. Then let $f_u(v) = f(u, \alpha, \sigma)$.

Conversely, let $(f_u) \in \prod_{u \in N_p(\Delta)} C^q(\mathcal{C}_{n_p}, M_u)$. Then we get an f in $C^{p,q}(X,M)$ defined as follows: A (p,q)-simplex (u, α, σ) of $\tilde{X}$, where

$$u = (\underline{n_0} \xrightarrow{u_1} \underline{n_1} \longrightarrow \cdots \xrightarrow{u_p} \underline{n_p}) \in N_p(\Delta),$$
$$\alpha = (G/H_0 \xrightarrow{\alpha_1} G/H_1 \longrightarrow \cdots \xrightarrow{\alpha_q} G/H_q) \in N_q(\mathcal{O}(G)),$$
$$\sigma : G/H_q \times \Delta^{n_p} \longrightarrow X \text{ is a } G\text{-map},$$

corresponds to a q-simplex

$$v = \tau_0 \xrightarrow{\alpha_1 \times 1} \tau_1 \longrightarrow \cdots \xrightarrow{\alpha_q \times 1} \tau_q$$

of the nerve of $\mathcal{C}_{n_p}$, where $\tau_q = \sigma$ and $\tau_i = \tau_{i+1}(\alpha_{i+1} \times 1)$. Let

$$f(u, \alpha, \sigma) = f_u(v).$$

Let us denote the differential on $C^\bullet(\mathcal{C}_{n_p}, M_u)$ by d_u. Then $C^{p,\bullet}(X, M)$ is isomorphic to the cochain complex $(\prod_{u \in N_p(\Delta)} C^\bullet(\mathcal{C}_{n_p}, M_u), \prod_{u \in N_p(\Delta)} d_u)$. It follows that

$$H^q(C^{p,\bullet}(X, M)) \cong \prod_{u \in N_p(\Delta)} H^q(\mathcal{C}_{n_p}, M_u).$$

We now compute $H^q(\mathcal{C}_{n_p}, M_u)$.

Let us denote the first vertex of Δ^{n_0} by e_0 and let σ restricted to $u(e_0)$ be σ'. Then M_u is naturally isomorphic to the functor which takes σ to $M(\sigma')$ and hence to M_{n_p}. Thus,

$$H^*(\mathcal{C}_{n_p}, M_u) \cong H^*(\mathcal{C}_{n_p}, M_{n_p}).$$

Now for all $n \geq 0$, $S_n(X)$ is a G-set, the G action induced by the action on X, i.e. for any $\phi : \Delta^n \longrightarrow X$, $g.\phi : \Delta^n \longrightarrow X$ is defined by $t \mapsto g\phi(t)$.

Recall that for the G-set $S = G/H$, the "global section" or the "inverse limit" functor

$$\Gamma : \underline{Ab}\left(\int_{\mathcal{O}(G)} (S)^{(-)}\right) \longrightarrow \underline{Ab}$$

is an exact functor ([4]). Also any G-set S can be written as a union of orbits, say $S = \bigcup_H G/H$, where the union is over conjugacy classes of isotropy subgroups , one representative chosen from each class. If $\mathcal{D} = \int_{\mathcal{O}(G)} S^{(-)}$ and we let

$$\int_{\mathcal{O}(G)} (G/H)^{(-)} = \mathcal{D}_H,$$

then $\mathcal{D}$ is the union of the categories $\mathcal{D}_H$. Also if $M \in \underline{Ab}(\mathcal{D})$ and we denote $M|\mathcal{D}_H = M_H$, then M_H are contravariant functors on $\mathcal{D}_H$ and it is clear from the definition of cohomology of categories that

$$H^q(\mathcal{D}, M) = \bigoplus_H H^q(\mathcal{D}_H, M_H).$$

Also $\Gamma(M) = \bigoplus_H \Gamma(M_H)$. Combining these facts we get for all $n \geq 0$,

$$\begin{aligned} H^q(\mathcal{C}_n, M_n) &= 0 \qquad \text{if } q > 0; \\ H^q(\mathcal{C}_n, M_n) &= \Gamma(M_n) \quad \text{if } q = 0. \end{aligned}$$

Now recall that $\Gamma(M_n)$ consists of all functions ϕ which take an object σ of $\mathcal{C}_n$ to an element of $M_n(\sigma) = M(v_X(\sigma))$ so that if $\hat{a} : G/H \longrightarrow G/K$ is a morphism between $\sigma : G/H \times \Delta^n \longrightarrow X$ and $\tau : G/K \times \Delta^n \longrightarrow X$, i.e. if $\tau \circ (\hat{a} \times 1) = \sigma$, then $M_n(\hat{a})(\phi(\tau)) = \phi(\sigma)$. Hence

$$\Gamma(M_n) = S^n_G(X, M).$$

Thus for each $u = (\underline{n_0} \longrightarrow \cdots \longrightarrow \underline{n_p})$ in $N(\Delta)$ we get a copy of $S_G^{n_p}(X, M)$ which we denote by $S_G^{n(u)}(X, M)$ and we have

$$\begin{array}{rcll} H^q(C^{p,\bullet}(X,M)) & \cong & \prod_{u \in N_p(\Delta)} S_G^{n(u)}(X,M) & \text{if } q = 0, \\ & = & 0 & \text{if } q > 0. \end{array}$$

Thus,

$$\begin{array}{rcl} H^p(\text{Tot } C^{\bullet\bullet}(X,M)) & \cong & H^p(\prod_{u \in N(\Delta)} S_G^{n(u)}(X,M)) \\ & \cong & H^p(\Delta^{op}, S_G^{\bullet}(X,M)), \end{array}$$

where $S_G^{\bullet}(X, M)$ is the cosimplicial group which takes $\underline{n}$ to $S_G^n(X, M)$ with obvious face and degeneracy maps induced from those on Δ. Then we know that ([4])

$$H^p(\Delta^{op}, S_G^{\bullet}(X,M)) \cong H^p(S_G^{\bullet}(X,M)).$$

Hence,

$$H^p(\text{Tot } C^{\bullet\bullet}(X,M)) \cong H_{BI}^p(X,M).$$

□

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