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ON SYSTEMS OF BINOMIALS IN THE IDEAL OF A TORIC VARIETY

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Abstract. Let Γ be a toric set in the affine space $\mathbb{A}_k^n$. Given a set of binomials $g_1, \dots , g_r$ in the toric ideal P of Γ, we give a criterion for deciding the equality $\operatorname{rad}(g_1, \dots , g_r) = P$. This criterion extends to arbitrary dimension, and to arbitrary fields, an earlier result which concerned only monomial curves over an algebraically closed field of characteristic zero.

1. Introduction

Let k be any field and D a fixed $m \times n$ matrix with non-negative integer entries d_{ij} and with non-zero columns. The *toric set* Γ determined by the matrix D is the set in the affine space $\mathbb{A}_k^n$ given parametrically by $x_i = t_1^{d_{1i}} \cdots t_m^{d_{mi}}$ for all i. Thus, one has

$$\Gamma = \{(t_1^{d_{11}} \cdots t_m^{d_{m1}}, \dots , t_1^{d_{1n}} \cdots t_m^{d_{mn}}) \in \mathbb{A}_k^n \mid t_1, \dots , t_m \in k\}.$$

Let $R = k[x_1, \dots , x_n]$ and $k[t_1, \dots , t_m]$ be two polynomial rings over k, graded by $\deg(x_i) = d_{1i} + \cdots + d_{mi}$ for all i, and $\deg(t_j) = 1$ for all j, respectively. Let ϕ be the graded homomorphism $\phi\colon R \to k[t_1, \dots , t_m]$, induced by $\phi(x_i) = t^{D_i}$, where $D_i = (d_{1i}, \dots , d_{mi})$ is the transpose of the ith column of D, and where $t^{D_i} = t_1^{d_{1i}} \cdots t_m^{d_{mi}}$. We denote the image of ϕ by $k[\Gamma]$. The kernel of ϕ, denoted by P, is known as the *toric ideal* of $k[\Gamma]$; see [6] for general facts on toric ideals.

If $\alpha = (\alpha_1, \dots , \alpha_n) \in \mathbb{N}^n$ (*resp.* $\alpha \in \mathbb{Z}^n$), we denote by $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$ the corresponding monomial in R (*resp.* in $k(x_1, \dots , x_n)$). Given a binomial $g = x^\alpha - x^\beta$ in R, we let $\widehat{g} = \alpha - \beta \in \mathbb{Z}^n$.

Closely related to the map ϕ is the homomorphism $\psi\colon \mathbb{Z}^n \to \mathbb{Z}^m$, determined by the matrix D in the standard bases of $\mathbb{Z}^n$ and $\mathbb{Z}^m$. Indeed, we have $\phi(x^\alpha) = t^{\psi(\alpha)}$ for all $\alpha \in \mathbb{N}^n$. The kernel of ψ plays an important role in the sequel, so we denote it by K_Γ throughout the paper. As a consequence of the link between the maps ϕ and ψ, we see that *a binomial* $g = x^\alpha - x^\beta$ *belongs to* $P = \ker(\phi)$ *if and only if* $\widehat{g} = \alpha - \beta$ *belongs to* $K_\Gamma = \ker(\psi)$.

Given a subset $I \subset R$ we denote its zero set in $\mathbb{A}_k^n$ by $V(I)$, and given a subset $X \subset \mathbb{A}_k^n$ we denote its vanishing ideal in R by $I(X)$. If $h_1, \dots , h_r \in \mathbb{Z}^n$, we denote by $\langle h_1, \dots , h_r\rangle$ the subgroup of $\mathbb{Z}^n$ generated by these elements.

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Note that if k is infinite and P is the toric ideal of $k[\Gamma]$, then P is in fact equal to the (vanishing) ideal of Γ and $V(P)$ is equal to $\overline{\Gamma}$, the Zariski closure of Γ. Note also that $\dim(\overline{\Gamma})$ is equal to the rank of the matrix D.

If $m = 1$ and the entries of D are relatively prime integers, then Γ is known as a *monomial curve*. The problem of deciding when a given set of binomials defines a monomial curve Γ was studied in [4]. One of our main results (see Theorem 3.1) is a generalization of the following criterion:

Proposition 1.1 ([4]). *Let k be an algebraically closed field of characteristic zero, and let $g_1, \dots, g_r$ be binomials belonging to the toric ideal P of the monomial curve Γ. Then, $\Gamma = V(g_1, \dots, g_r)$ if and only if:*
(a) $\langle \widehat{g}_1, \dots, \widehat{g}_r \rangle = K_\Gamma$,
(b) $V(g_1, \dots, g_r, x_i) = \{0\}$*, for all i.*

The hypothesis that k is of characteristic zero cannot be dropped in this result. As an example (due to D. Eisenbud), consider the monomial curve $\Gamma = (t, t) \in \mathbb{A}^2_k$, where k is a field of characteristic $p > 0$. If g is the binomial $g = x^p - y^p \in k[x, y]$, then $g = (x-y)^p$ and $V(g) = \Gamma$. However, $\widehat{g} = (p, -p)$ and thus $\widehat{g}$ does not generate the subgroup $K_\Gamma = \ker(\psi) = \langle (1, -1) \rangle$ of $\mathbb{Z}^2$.

In order to extend this result to the case where k is of positive characteristic p, we need to replace condition (a) above by the weaker condition that the quotient group $K_\Gamma / \langle \widehat{g}_1, \dots, \widehat{g}_r \rangle$ is a finite p-group. Moreover, our generalized criterion is formulated for deciding the algebraic equality $\mathrm{rad}(g_1, \dots, g_r) = P$ instead of the usually weaker geometric equality $\Gamma = V(g_1, \dots, g_r)$, allowing us to drop as well the assumption that k is algebraically closed. Finally, our new criterion (see Theorem 2.5) concerns monomial sets of arbitrary dimension. It is effective at least in the one-dimensional case.

In contrast with [3, 4] where the base field is assumed algebraically closed and/or of characteristic zero, we will always work here over an arbitrary field.

2. GENERAL CRITERIA

Let $G \subset \mathbb{Z}^n$ be a subgroup. We associate to G an equivalence relation $\sim_G$ on the monomials in $R = k[x_1, \dots, x_n]$ by $x^\alpha \sim_G x^\beta$ if and only if $\alpha - \beta \in G$. Note that this relation is compatible with the product, *i.e.*, $x^\alpha \sim_G x^\beta$ implies $x^\alpha x^\gamma \sim_G x^\beta x^\gamma$ for all $\gamma \in \mathbb{N}^n$.

Definition 2.1. A non-zero polynomial $f = \sum_\alpha \lambda_\alpha x^\alpha$ in R is *simple* (with respect to $\sim_G$) if all its monomials, *i.e.*, those x^α with non-zero coefficient λ_α, are equivalent under $\sim_G$.

Given any $f \in R \setminus \{0\}$, we can group together its monomials by equivalence classes under $\sim_G$, thereby obtaining a decomposition $f = h_1 + \cdots + h_m$ with the property that each summand h_i is simple, and that no monomial in h_i is equivalent to a monomial in h_j if $j \neq i$. Such a decomposition of f as a sum of maximal simple subpolynomials is unique up to order. We will refer to the summands h_i as the *simple components* of f (with respect to $\sim_G$).

We need the following basic lemma, which was proved in [4] in the context of monomial curves.

Lemma 2.2. *Let $g_1, \dots, g_r$ be binomials in R and $G \subset \mathbb{Z}^n$ the subgroup generated by $\widehat{g}_1, \dots, \widehat{g}_r$. If $f \in (g_1, \dots, g_r)$, then every simple component of f with respect to $\sim_G$ also belongs to $(g_1, \dots, g_r)$.*

Proof. First, each generator $g_i = x^{\alpha(i)} - x^{\beta(i)}$ is simple, as its two constituting monomials $x^{\alpha(i)}, x^{\beta(i)}$ are equivalent under $\sim_G$ by construction. Moreover, $x^\gamma g_i$ remains simple for any $\gamma \in \mathbb{N}^n$, since the relation $\sim_G$ on monomials is compatible with the product. Now let f be any element in the ideal $(g_1, \dots, g_r)$. Then, f is a linear combination of polynomials of the form $x^\gamma g_i$, with $i \in \{1, \dots, r\}$ and $\gamma \in \mathbb{N}^n$, which are simple. Hence, every simple component of f is also a linear combination of some $x^\gamma g_i$ and therefore belongs to $(g_1, \dots, g_r)$. ☐

The proof of the next statement provides a first illustration of the use of this lemma. The notation is the same as in the introduction.

Proposition 2.3. *If $g_1, \dots, g_r$ is a set of binomials generating the ideal P of the toric set Γ, then $\widehat{g}_1, \dots, \widehat{g}_r$ generate K_Γ.*

Proof. Set $G = \langle \widehat{g}_1, \dots, \widehat{g}_r \rangle \subset \mathbb{Z}^n$. Let $\gamma \in K_\Gamma$ and write $\gamma = \alpha - \beta$, where $\alpha, \beta \in \mathbb{N}^n$. Proving that $\gamma \in G$ amounts to proving that $x^\alpha \sim_G x^\beta$, *i.e.*, that the binomial $g = x^\alpha - x^\beta$ is simple. Now, g belongs to $P = (g_1, \dots, g_r)$, hence the simple component h of g containing x^α also belongs to P, by Lemma 2.2. Since $x^\alpha \notin P$, it follows that $h = g$, hence that g is simple, as required. ☐

In the sequel, we will denote by $R' = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ the Laurent polynomial ring, that is, the localization of R at the multiplicative set of monomials.

Let $I \subset P$ be a subideal. An obvious necessary condition for the equality $\mathrm{rad}(I) = P$ in R is the equality $\mathrm{rad}(I \cdot R') = P \cdot R'$ in the Laurent polynomial ring R', because taking radicals commutes with localization. If I is generated by binomials $g_1, \dots, g_r$, it turns out that the latter condition can be checked by looking only at the subgroup $\langle \widehat{g}_1, \dots, \widehat{g}_r \rangle$ of K_Γ. Indeed, we have the following result.

Proposition 2.4. *Let $g_1, \dots, g_r$ be a set of binomials in the ideal P of the toric set Γ. Set $I = (g_1, \dots, g_r)$ and $G = \langle \widehat{g}_1, \dots, \widehat{g}_r \rangle \subset K_\Gamma$. If* $\mathrm{char}(k) = p \neq 0$ (resp. $\mathrm{char}(k) = 0$)*, then the following conditions are equivalent:*

(a$_1$) $\mathrm{rad}(I \cdot R') = P \cdot R'$*, where* $R' = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$*,*

(a$_2$) $x^\delta P^N \subset I$ *for some monomial* $x^\delta \in R$ *and some integer* $N \geq 1$*,*

(a$_3$) $p^m K_\Gamma \subset G$ *for some* $m \in \mathbb{N}$ (resp. $K_\Gamma = G$).

Proof. The equivalence of (a$_1$) and (a$_2$) uses the fact that P is finitely generated. It is not hard to prove and is therefore left to the reader.

(a$_2$) ⇒ (a$_3$): We first consider the case $\mathrm{char}(k) = p \neq 0$. The integer N in hypothesis (a$_2$) can be taken of the form $N = p^m$ for some positive integer m. Let $\gamma \in K_\Gamma$, and write it as $\gamma = \alpha - \beta$ with $\alpha, \beta \in \mathbb{N}^n$. Let $g = x^\alpha - x^\beta$. Then, $g \in P$ and $\widehat{g} = \alpha - \beta = \gamma$. By (a$_2$), we have $x^\delta g^{p^m} \in I$, with $x^\delta g^{p^m} = x^{\delta + \alpha p^m} - x^{\delta + \beta p^m}$. Since the simple components of $x^\delta g^{p^m}$ belong to I by Lemma 2.2, and since $x^{\delta + \alpha p^m}$ does not belong to I because it does not belong to P, it follows that $x^\delta g^{p^m}$ is simple, *i.e.*, $x^{\delta + \alpha p^m} \sim_G x^{\delta + \beta p^m}$. Hence $\alpha p^m - \beta p^m = \gamma p^m \in G$. This shows $p^m K_\Gamma \subset G$, as desired.

Suppose now $\mathrm{char}(k) = 0$, for which case the argument is similar to the one given in the proof of [4, Proposition 1]. We only need to show that $K_\Gamma \subset G$. Again, let $\gamma \in K_\Gamma$, and write it as $\gamma = \alpha - \beta$ with $\alpha, \beta \in \mathbb{N}^n$. Letting $g = x^\alpha - x^\beta$, we have $g \in P$ and $\widehat{g} = \alpha - \beta = \gamma$. Let $f = x^\delta g^N$. By (a$_2$), $f \in I$. Consider the expansion of f, $f = \sum_{i=0}^{N} (-1)^i \binom{N}{i} x^{\delta + \alpha i} x^{\delta + \beta (N - i)}$. Assuming, as we may, that N is a prime number, we claim that the two monomials $x^{\delta + \alpha N}$ and $x^{\delta + \beta N}$ of f are equivalent under $\sim_G$.

Indeed, let h be the simple component of f containing $x^{\delta+\alpha N}$. Then, $h \in I$ by Lemma 2.2. Let σ denote the sum of the coefficients in h. Since $h \in I \subset P$, we have $\sigma = 0$. On the other hand, $\sigma = \binom{N}{0} + \sum_S (-1)^i \binom{N}{i}$, for some subset $S \subset \{1, \dots, N\}$. Now, if h does not contain the monomial $x^{\delta+\beta N}$, then S does not contain the index N. But this implies $\sigma \equiv 1 \bmod N$, because N is assumed to be prime, whence $\binom{N}{i} \equiv 0 \bmod N$ for $1 \leq i \leq N-1$. This contradicts the nullity of σ, and proves our claim that $x^{\delta+\alpha N} \sim_G x^{\delta+\beta N}$. Therefore, $N(\alpha - \beta) \in G$.

The same argument shows that $N'(\alpha - \beta) \in G$ for any prime number $N' > N$. By Bezout, it follows that $\gamma = \alpha - \beta \in G$, as desired.

(a_2) $\Leftarrow$ (a_3): First assume that $\mathrm{char}(k) = p \neq 0$. Let $f = x^\mu - x^\nu$ be a binomial in P and write $g_i = x^{\alpha_i} - x^{\beta_i}$. By ($a_3$) there are integers s_i such that

$$(x^{\mu p^m} / x^{\nu p^m}) - 1 = \left(x^{\alpha_1}/x^{\beta_1}\right)^{s_1} \cdots \left(x^{\alpha_r}/x^{\beta_r}\right)^{s_r} - 1,$$

where we may assume $s_i \geq 0$ by replacing x^{α_i}/x^{β_i} by its inverse if necessary. Hence writing $x^{\alpha_i}/x^{\beta_i} = ((x^{\alpha_i}/x^{\beta_i}) - 1) + 1$ and using the binomial theorem, it follows that $x^\gamma f^{p^m}$ is in I, for some monomial x^γ. By the finite generation of P, there is an integer $N \geq 1$ and a monomial x^δ such that $x^\delta P^N \subset I$, as desired. The same argument works in characteristic zero as well, after replacing p^m by 1 throughout. □

Here is one of the main results of this section.

Theorem 2.5. *Let Γ be a toric set and $g_1, \dots, g_r$ a set of binomials in the toric ideal P of $k[\Gamma]$. Set $I = (g_1, \dots, g_r)$ and $G = \langle \widehat{g}_1, \dots, \widehat{g}_r \rangle \subset K_\Gamma$. If* $\mathrm{char}(k) = p \neq 0$ (resp. $\mathrm{char}(k) = 0$)*, then* $\mathrm{rad}(I) = P$ *if and only if:*

(a) $p^m K_\Gamma \subset G$ *for some* $m \in \mathbb{N}$ (resp. $K_\Gamma = G$),

(b) $\mathrm{rad}(I, x_i) = \mathrm{rad}(P, x_i)$*, for all* i.

Proof. If $\mathrm{rad}(I) = P$, then of course $\mathrm{rad}(I, x_i) = \mathrm{rad}(P, x_i)$, for all i. Moreover, we have $\mathrm{rad}(I \cdot R') = P \cdot R'$; hence condition (a) holds by Proposition 2.4.

Conversely, suppose that (a) and (b) hold. By Proposition 2.4, there is a monomial x^δ and an integer $N \geq 1$ such that $x^\delta P^N \subset I$. In order to prove that $\mathrm{rad}(I) = P$, we will show that every prime ideal containing I also contains P. So, let Q be a prime ideal containing I. If Q contains no variable, then the inclusions $x^\delta P^N \subset I \subset Q$ imply that $P \subset Q$, as desired. If Q contains at least one variable, say x_1, then $Q \supset (I, x_1)$; therefore $Q \supset \mathrm{rad}(I, x_1) = \mathrm{rad}(P, x_1)$, whence $Q \supset P$ in this case also. □

Remark 2.6. Since $\mathbb{Z}^n$ is finitely generated and abelian, condition (a) in Theorem 2.5, namely that $p^m K_\Gamma \subset G = \langle \widehat{g}_1, \dots, \widehat{g}_r \rangle$ for some $m \geq 0$, is equivalent to require that the quotient K_Γ / G is a finite p-group. This condition, in turn, can be easily tested by using the fact that K_Γ / G is a finite p-group if and only if $\mathbb{Z}^n / G \simeq \mathbb{Z}^s \times H$, where H is a finite p-group and s is equal to the rank of D.

Now, we want to investigate what it means geometrically for a set of binomials $g_1, \dots, g_r$ in P to satisfy only condition (a) in Theorem 2.5. An answer is easily provided by Proposition 2.4.

Proposition 2.7. *Let $g_1, \dots, g_r$ be a set of binomials in the toric ideal P, and set $G = \langle \widehat{g}_1, \dots, \widehat{g}_r \rangle \subset K_\Gamma$. If k is a field of* $\mathrm{char}(k) = p \neq 0$ (resp. $\mathrm{char}(k) = 0$) *and $p^m K_\Gamma \subset G$ for some $m \geq 0$* (resp. $K_\Gamma = G$)*, then $V(g_1, \dots, g_r) \subset V(P) \bigcup V(x_1 \cdots x_n)$.*

Proof. By Proposition 2.4, there is a monomial x^δ and an integer N such that $x^\delta P^N \subset I$. Hence $V(I) \subset V(P^N) \cup V(x^\delta) \subset V(P) \cup V(x_1 \cdots x_n)$. □

We conclude this section by showing that the above criterion for the equality $\operatorname{rad}(I) = P$ can be framed in quite a general setting.

Definition 2.8. Let I be an ideal of a ring R and $f \in R$. The *saturation* of I with respect to f is $(I\colon f^\infty) = \bigcup_{i\geq 1}(I\colon f^i) = \{r \in R | rf^i \in I, \text{ for some } i \geq 1\}$.

If R is a polynomial ring, then the saturation can be computed using Gröbner bases and the equality $(I\colon f^\infty) = (I, 1 - tf) \cap R$, where t is a new variable.

Proposition 2.9. *Let P be a prime ideal of a commutative ring R and $I \subset P$ an ideal. If $x_1, \dots, x_n \in R \setminus P$, then* $\operatorname{rad}(I) = P$ *if and only if:*
(a) $P = \operatorname{rad}(I\colon (x_1 \cdots x_n)^\infty)$*, and*
(b) $\operatorname{rad}(I, x_i) = \operatorname{rad}(P, x_i)$*, for all i.*

Proof. If J is any ideal of R and $y_1, \dots, y_n \in R$, then by [3, Lemma 3.2] the radical of J satisfies

$$\operatorname{rad}(J) = \operatorname{rad}(J\colon (y_1 \cdots y_n)^\infty) \cap \operatorname{rad}(J, y_1) \cap \cdots \cap \operatorname{rad}(J, y_n).$$

Hence the result follows by applying this formula to I and P, together with the fact that $x_1 \cdots x_n$ is regular on R/P. □

3. The case of monomial curves

In this section, we consider the special case where the toric set Γ is a monomial curve. Thus, we assume that the matrix D has only one row, namely $D = (d_1, \dots, d_n)$, and that $d_1, \dots, d_n$ are relatively prime. In this case, Theorem 2.5 can be strengthened, in the sense that we can replace the algebraic condition (b) in Theorem 2.5, namely $\operatorname{rad}(I, x_i) = \operatorname{rad}(P, x_i)$ for all i, by the weaker geometric condition $V(I, x_i) = \{0\}$ for all i.

The main result of this section is:

Theorem 3.1. *Let $g_1, \dots, g_r$ be a set of binomials in the ideal P of the monomial curve Γ. Set $I = (g_1, \dots, g_r)$ and $G = \langle \widehat{g}_1, \dots, \widehat{g}_r \rangle \subset K_\Gamma$. If* $\operatorname{char}(k) = p \neq 0$ (resp. $\operatorname{char}(k) = 0$)*, then* $\operatorname{rad}(I) = P$ *if and only if:*
(a) $p^m K_\Gamma \subset G$ *for some* $m \in \mathbb{N}$ (resp. $K_\Gamma = G$)*,*
(b) $V(g_1, \dots, g_r, x_i) = \{0\}$*, for all i.*

Proof. If $\operatorname{rad}(I) = P$, then (a) follows from Theorem 2.5. Moreover, the hypothesis implies $\operatorname{rad}(I, x_i) = \operatorname{rad}(P, x_i)$, whence $V(g_1, \dots, g_r, x_i) = V(P, x_i) = V(P) \cap V(x_i) = \Gamma \cap V(x_i) = \{0\}$ for all i.

Conversely, suppose that (a) and (b) hold. By Proposition 2.4, we have $x^\delta P^N \subset I$ for some monomial x^δ and some integer $N \geq 1$. In order to prove that $\operatorname{rad}(I) = P$, we will show that every prime ideal Q containing I also contains P. If Q contains no variable, then the inclusions $x^\delta P^N \subset I \subset Q$ imply that $P \subset Q$, as desired. If Q contains at least one variable, then we claim that Q contains *all* the variables, implying $Q = (x_1, \dots, x_n) \supset P$. (This is the point in the proof where we need a special argument, compensating for the fact that condition (b) above is weaker than in Theorem 2.5.) Indeed, up to a renumbering of the variables, we may assume that $\{x_1, \dots, x_s\}$ is the list of all the variables contained in Q, for some integer s, $1 \leq s \leq n$. Let $a = (a_i)$ be the point in affine space defined by

$a_1 = \cdots = a_s = 0$, $a_{s+1} = \cdots = a_n = 1$. We claim that $a \in V(I)$. Indeed, let $g = x^\alpha - x^\beta$ be any binomial belonging to I. Since g belongs to Q, it easily follows that $x^\alpha \in (x_1, \ldots, x_s)$ if and only if $x^\beta \in (x_1, \ldots, x_s)$. As a consequence, $a^\alpha = a^\beta$, this common value being 0 if $x^\alpha \in (x_1, \ldots, x_s)$, or 1 otherwise. In either case, we have $g(a) = 0$. It follows that $a \in V(I)$, as claimed. But $a_1 = 0$, hence a belongs to $V(I, x_1)$, which is $\{0\}$ by condition (b). Thus $a = 0$, which by definition of a implies that $s = n$. Hence, Q contains all the variables and consequently contains the ideal P, as desired. □

Remark 3.2. Let Γ be an arbitrary toric set and P the toric ideal of $k[\Gamma]$. The proof above shows that condition (b) in Theorem 3.1 implies that $\mathrm{rad}(P, x_i) = (x_1, \ldots, x_n)$ and consequently $\dim(k[\Gamma]) = \dim(R/P) = 1$. Thus the result above makes sense only for curves.

Remark 3.3. The underlying hypothesis $\gcd(d_1, \ldots, d_n) = 1$ in the definition of a monomial curve is not used in the above proof.

Lemma 3.4. *Let k be any field and I the ideal generated by the set of binomials $\{x_i^{d_j} - x_j^{d_i} \mid 1 \le i < j \le n\}$, where $\gcd(d_1, \ldots, d_n) = 1$. Then $\Gamma = V(I)$.*

Proof. It is enough to prove the inclusion $V(I) \subset \Gamma$, because clearly one has $\Gamma \subset V(I)$. Let $a = (a_1, \ldots, a_n)$ be an element in $V(I)$. One may assume $a_i \neq 0$ for all i, for otherwise $a = 0$. We must show that there is $t \in k$ with $a_i = t^{d_i}$ for all i. One can write $1 = c_1 d_1 + \cdots + c_n d_n$, for some $c_i \in \mathbb{Z}$. From the equalities

$$(a_1^{c_1} \cdots a_n^{c_n})^{d_i} = a_1^{c_1 d_i} \cdots a_i^{c_i d_i} \cdots a_n^{c_n d_i} = a_i^{c_1 d_1} \cdots a_i^{c_i d_i} \cdots a_i^{c_n d_n} = a_i,$$

one derives that $t = a_1^{c_1} \cdots a_n^{c_n}$ is the required element. □

Remark 3.5. If k is algebraically closed the conclusion of Lemma 3.4 remains valid even if we drop the assumption $\gcd(d_1, \ldots, d_n) = 1$. More generally, if X is the curve in $\mathbb{A}_k^n$ given parametrically by $x_i = f_i(t)$, where $f_i(t) \in k[t] \setminus k$ for $i = 1, \ldots, n$, then by the extension theorem (see, e.g., [2, Chapter 3]) it follows that $V(P) = X$, where P is the toric ideal of $k[f_1(t), \ldots, f_n(t)]$.

Remark 3.6. Let $I \subset P$ be a binomial ideal. In general the algebraic equality $\mathrm{rad}(I) = P$ and the geometric equality $\Gamma = V(I)$ are not equivalent (see Example 3.8), but they are related as follows. If k is any field and $\mathrm{rad}(I) = P$, then $V(I) = V(P) = \Gamma$. On the other hand if k is algebraically closed and $V(I) = \Gamma$, then by the Nullstellensatz $I(V(I)) = \mathrm{rad}(I) = I(\Gamma) = P$.

Corollary 3.7. *If $I = (\{x_i^{d_j} - x_j^{d_i} \mid 1 \le i < j \le n\})$ with $\gcd(d_1, \ldots, d_n) = 1$, then $\mathrm{rad}(I) = P$.*

Proof. Using the method of proof of [1, Proposition 2.2] it follows that K_Γ is generated by the set $\{d_j e_i - d_i e_j \mid 1 \le i < j \le n\}$, where $e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{Z}^n$. Thus, $\mathrm{rad}(I) = P$ by Theorem 3.1. □

Example 3.8. Let P be the toric ideal of $k[t^6, t^8, t^9]$ and $I = (g_1, g_2)$, where $g_1 = x_1^3 - x_3^2$, $g_2 = x_2^9 - x_3^8$. As $\mathbb{Z}^3/\langle \widehat{g}_1, \widehat{g}_2 \rangle \simeq \mathbb{Z} \times \mathbb{Z}_3$, by Theorem 3.1 one has:

(i) $P = (g_1, x_2^3 - x_1 x_3^2)$ for any field k (see [5]),

(ii) $\mathrm{rad}(I) = P$ if $\mathrm{char}(k) = 3$, and $\mathrm{rad}(I) \neq P$ if $\mathrm{char}(k) = 2$,

(iii) $\Gamma = \{(t^6, t^8, t^9) \mid t \in k\} = \{(0,0,0), (1,1,1)\} = V(I)$ if $k = \mathbb{Z}_2$.

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