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GOOD IDEALS IN GORENSTEIN LOCAL RINGS OBTAINED BY IDEALIZATION

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Abstract. The structure of certain equimultiple good ideals in Gorenstein local rings obtained by idealization is explored.

1. Introduction

Let R be a Cohen-Macaulay local ring with the maximal ideal $\mathfrak{n}$ and $d = \dim R$. Let K_R be the canonical module of R and let $A = R \ltimes \mathrm{K}_R$ be the idealization of K_R. Hence $A = R \oplus \mathrm{K}_R$ as R-modules, the multiplication in A is given by $(a, x) \cdot (b, y) = (ab, ay + bx)$, and A is a Gorenstein local ring with the maximal ideal $\mathfrak{m} = \mathfrak{n} \times \mathrm{K}_R$ ([R]).

Let I be an ideal in A of height s. Then we say that I is a *good* ideal in A if I contains a reduction $Q = (a_1, a_2, \cdots, a_s)$ generated by s elements in A and the associated graded ring $\mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ of I is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(I)) = 1 - s$ ([GIW], [GK]), where $\mathrm{a}(\mathrm{G}(I))$ denotes the a-invariant of $\mathrm{G}(I)$. Let $\mathcal{X}_A^s$ be the set of good ideals I in A of height s.

Let J be an ideal in R and let L be an R-submodule of K_R. Assume that $J\mathrm{K}_R \subseteq L$ and put $I = J \times L$. Then I forms an ideal in A, satisfying the equality $I^2 = JI$. With this notation the main purpose of this paper is to prove the following.

Theorem (1.1). *Let $s \geq 0$ be an integer. Then the following conditions are equivalent*:

(1) $I \in \mathcal{X}_A^s$.
(2) *The following three conditions are satisfied:* (i) $\mathrm{ht}_R J = s$, (ii) *J contains a reduction $\mathfrak{q}$ generated by s elements such that $J^2 = \mathfrak{q}J$ and $L = \mathfrak{q}\mathrm{K}_R :_{\mathrm{K}_R} J$, and* (iii) *$R/J$ is a Cohen-Macaulay ring.*

The analysis of $\mathfrak{m}$-primary good ideals in general Gorenstein local rings $(A, \mathfrak{m})$ started from the paper [GIW] and there was given by [GK] a generalization to the case of equimultiple ideals. In [GK] Goto and Kim showed that the set $\mathcal{X}_A^s$ is infinite whenever $3 \leq s \leq \dim A$, which gave an affirmative answer to a question raised by [GIW]. However, although good ideals in their sense certainly form a distinguished

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class of ideals, little is known about the structure of good ideals except certain special cases (cf. [GIW, Sections 5, 7]). Our Theorem (1.1) is the first attempt to describe the structure of equimultiple good ideals of height s. From this viewpoint the theorem might have some significance in further developments of the theory of good ideals, although it deals with ideals I of the form $I = J \times L$ in Gorenstein local rings obtained by idealization.

As immediate consequences of Theorem (1.1), we have the following.

Corollary (1.2). *Suppose that R is a Gorenstein local ring and let J be an ideal in R of height s. Then $J \times J \in \mathcal{X}^s_{R \ltimes R}$ if and only if $J \in \mathcal{X}^s_R$.*

Corollary (1.3). *Assume that $d = \dim R \geq 1$. Then $\sharp \mathcal{X}^s_A = \infty$ for all $1 \leq s \leq d$.*

Our proof of Theorem (1.1) is based on two facts. The first one is that for each ideal $I = J \times L$ in $A = R \ltimes \mathrm{K}_R$, the associated graded ring $\mathrm{G}(I)$ of I is also obtained by idealization, which we shall discuss in Section 3. The second one is the theory of canonical filtrations developed by [GI]. Let J be an ideal in R and let $\mathrm{R}'(J) = \sum_{n \in \mathbb{Z}} J^n t^n$ ($\subseteq R[t, t^{-1}]$, t an indeterminate over R) denote the extended Rees algebra of J. Then there exists a unique family $\omega = \{\omega_n\}_{n \in \mathbb{Z}}$ of R-submodules of K_R satisfying the following four conditions:

(i) $\omega_n = \mathrm{K}_R$ for $n \ll 0$,
(ii) $\omega_n \supseteq \omega_{n+1}$ for all $n \in \mathbb{Z}$,
(iii) $J^m \omega_n \subseteq \omega_{m+n}$ for all $m, n \in \mathbb{Z}$, and
(iv) $\mathrm{K}_{\mathrm{R}'(J)} \cong \sum_{n \in \mathbb{Z}} \omega_n t^n$ as graded $\mathrm{R}'(J)$-modules, where $\mathrm{K}_{\mathrm{R}'(J)}$ denotes the canonical module of $\mathrm{R}'(J)$ and the sum $\sum_{n \in \mathbb{Z}} \omega_n t^n$ is considered inside $\mathrm{K}_R[t, t^{-1}] = \mathrm{K}_R \otimes_R R[t, t^{-1}]$ (cf. [GI, Theorem 2.1]).

If $\mathrm{G}(J)$ is a Cohen-Macaulay ring with $a = \mathrm{a}(\mathrm{G}(J))$, then $\omega_n = \mathrm{K}_R \supsetneq \omega_{-a}$ for all $n \leq -a - 1$ and the canonical module $\mathrm{K}_{\mathrm{G}(J)}$ of $\mathrm{G}(J)$ is given by $\mathrm{K}_{\mathrm{G}(J)} = \bigoplus_{n \geq -a} \omega_{n-1}/\omega_n$. We refer to this family ω as the canonical J-filtration of K_R. Together with the first fact, the theory of canonical filtrations will enable us to describe the conditions on ideals $I = J \times L$ in $A = R \ltimes \mathrm{K}_R$, under which the rings $\mathrm{G}(I)$ are Gorenstein (Proposition (3.2)).

We now briefly explain how this paper is organized. We shall summarize in Section 2 some basic results on good ideals in general Gorenstein local rings, which we will need later to prove Theorem (1.1). The proof of Theorem (1.1) will be given in Section 4. Section 3 is devoted to some preliminary steps for the proof of Theorem (1.1), including a few results on the structure of the associated graded rings $\mathrm{G}(I)$ for ideals $I = J \times L$.

Before entering into details, let us fix some standard notation. Let $S = \bigoplus_{n \in \mathbb{Z}} S_n$ be a Noetherian graded ring and assume that S contains a unique graded maximal ideal $\mathfrak{M}$. We denote by $\mathrm{H}^i_{\mathfrak{M}}(*)$ ($i \in \mathbb{Z}$) the $i^{\underline{\mathrm{th}}}$ local cohomology functor of S with respect to $\mathfrak{M}$. For each graded S-module E and $n \in \mathbb{Z}$, let $[\mathrm{H}^i_{\mathfrak{M}}(E)]_n$ denote the homogeneous component of the graded S-module $\mathrm{H}^i_{\mathfrak{M}}(E)$ of degree n. If $S_n = (0)$ for all $n < 0$ and E is a finitely generated graded S-module, we have $[\mathrm{H}^i_{\mathfrak{M}}(E)]_n = (0)$ for all $n \gg 0$ and $i \in \mathbb{Z}$. We put $\mathrm{a}(E) = \sup\left\{n \in \mathbb{Z} \;\middle|\; [\mathrm{H}^p_{\mathfrak{M}}(E)]_n \neq (0)\right\}$ with $p = \dim_S E$ and call it the a-invariant of E ([GW, Definition (3.1.4)]). For each $n \in \mathbb{Z}$ let $E(n)$ stand for the graded S-module, whose underlying S-module coincides with that of E and whose graduation is given by $[E(n)]_i = E_{n+i}$ for all $i \in \mathbb{Z}$. We denote by K_S the canonical module of S.

2. Auxiliary results on good ideals

In this section let A denote a general Gorenstein local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. We are interested in the question of when the set $\mathcal{X}_A^s$ is infinite.

Let I be an ideal in A of height s. We denote by

$$\mathrm{R}(I) = \sum_{n \geq 0} I^n t^n \subseteq A[t],$$
$$\mathrm{R}'(I) = \sum_{n \in \mathbb{Z}} I^n t^n \subseteq A[t, t^{-1}],$$

and

$$\mathrm{G}(I) = \mathrm{R}'(I)/t^{-1}\mathrm{R}'(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$$

the Rees algebra, the extended Rees algebra, and the associated graded ring of I respectively, where t is an indeterminate over A. Then we have the following.

Lemma (2.1). *Let $I \in \mathcal{X}_A^s$ with $Q = (a_1, a_2, \cdots, a_s)$ a reduction and extend the sequence $a_1, a_2, \cdots, a_s$ to a system $a_1, a_2, \cdots, a_d$ of parameters in A. Then*:

(1) *The sequence $a_1t, a_2t, \cdots, a_st, a_{s+1}, a_{s+2}, \cdots, a_d$ of elements in $\mathrm{R}(I)$ forms a homogeneous system of parameters for $\mathrm{G}(I)$. Hence $a_{s+1}, a_{s+2}, \cdots, a_d$ is a regular sequence on A/I^n so that A/I^n are Cohen-Macaulay rings for all $n \geq 1$.*
(2) *$I^2 = QI$ but $I \neq Q$.*
(3) *$I^n = Q^n : I$ for all $n \in \mathbb{Z}$ if $s > 0$.*
(4) *$I = (0) : I$ if $s = 0$.*

Proof. Assertions (1) and (2) are well-known; see [GW, Remark (3.1.6)] and notice that $\mathrm{a}(\mathrm{G}(I/Q)) = \mathrm{a}(\mathrm{G}(I)) + s = 1$. Assertions (3) and (4) directly follow from [GI, Proposition 2.4 and Theorem 5.3]. □

Corollary (2.2). (1) *Let $I \in \mathcal{X}_A^s$. Then $IA_{\mathfrak{p}} \in \mathcal{X}_{A_{\mathfrak{p}}}^s$ for all $\mathfrak{p} \in \mathrm{V}(I)$.*

(2) *Let $\mathfrak{p} \in \mathrm{Spec}A$ and assume that $\mathfrak{p} \in \mathcal{X}_A^s$. Then the local ring $A_{\mathfrak{p}}$ is not regular.*

Proof. (1) Notice that $\mathrm{a}(\mathrm{G}(IA_{\mathfrak{p}})) = \mathrm{a}(\mathrm{G}(I))$, since $\mathrm{G}(I)$ is a Gorenstein ring.

(2) We have $\mathfrak{p}A_{\mathfrak{p}} \notin \mathcal{X}_{A_{\mathfrak{p}}}^s$ by Lemma (2.1), Assertions (3) and (4), if $A_{\mathfrak{p}}$ is a regular local ring. □

For a Cohen-Macaulay local ring $(R, \mathfrak{n})$ with $\dim R = n$, let

$$\mathrm{r}(R) = \ell_R\left(\mathrm{Ext}_R^n(R/\mathfrak{n}, R)\right)$$

denote the Cohen-Macaulay type of R.

Proposition (2.3). *Let I be an ideal in A of height s and assume that I contains a reduction Q generated by s elements. Then the following conditions are equivalent*:

(1) *$I \in \mathcal{X}_A^s$.*
(2) *$I^2 = QI$, $I = Q : I$, and the ring A/I is Cohen-Macaulay.*
(3) *$I^2 = QI$, $IA_{\mathfrak{p}} \in \mathcal{X}_{A_{\mathfrak{p}}}^s$ for all $\mathfrak{p} \in \mathrm{Ass}_A A/I$, and the ring A/I is Cohen-Macaulay.*

(4) *The algebra* $\mathrm{R}'(I)$ *is a Gorenstein ring and* $\mathrm{K}_{\mathrm{R}'(I)} \cong \mathrm{R}'(I)(2-s)$ *as graded* $\mathrm{R}'(I)$*-modules.*

If $s > 0$*, we may add the following*:

(5) $I^n = Q^n : I$ *for all* $n \in \mathbb{Z}$ *and the ring* A/I *is Cohen-Macaulay.*

When this is the case, $\mathrm{K}_{A/I} \cong I/Q$*, whence* $\mathrm{r}(A/I) = \mu_A(I) - s \geq 1$.

Proof. (1)⇒(3) See Lemma (2.1) and Corollary (2.2), Assertion (1).

(3)⇒(2) We must show $I = Q : I$. Assume the contrary and choose $\mathfrak{p} \in \mathrm{Ass}_A A/I$ so that $IA_\mathfrak{p} \subsetneq QA_\mathfrak{p} : IA_\mathfrak{p}$. Then $IA_\mathfrak{p} \in \mathcal{X}^s_{A_\mathfrak{p}}$ by Assertion (3) whence $IA_\mathfrak{p} = QA_\mathfrak{p} : IA_\mathfrak{p}$ by Lemma (2.1), Assertions (3) and (4), which is absurd.

(2)⇒(1) Since $I^2 = QI$ and A/I is Cohen-Macaulay, the ring $\mathrm{G}(I)$ is Cohen-Macaulay. Therefore $\mathrm{G}(I)$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(I)) = 1 - s$ by [GI, Proposition 2.4 and Theorem 5.3], because $I = Q : I$.

(1)⇔(4) Recall that t^{-1} is regular on $\mathrm{R}'(I)$ and $\mathrm{G}(I) = \mathrm{R}'(I)/t^{-1}\mathrm{R}'(I)$.

(1)⇒(5) See Lemma (2.1), Assertion (3).

(5)⇒(2) The equality $I^2 = QI$ immediately follows from the proof of the implication (4)⇒(2) in [GIW, Proposition (2.2)].

Let us check the last assertion. We have $I/Q \cong \mathrm{K}_{A/I}$ by [BH, Theorem 3.3.7], since $I = Q : I$. Consequently, $\mathrm{r}(A/I) = \mu_A(I) - s \geq 1$, because Q is a minimal reduction of I. □

Let us give a few consequences of the characterization (2.3).

Corollary (2.4). *Let* $I \in \mathcal{X}^s_A$ *with* $s < d$ *and choose* $f \in \mathfrak{m}$ *so that* f *is* A/I*-regular. Then*:

(1) $I + (f) \in \mathcal{X}^{s+1}_A$.

(2) $(I + (f))/(f) \in \mathcal{X}^s_{A/(f)}$.

Proof. Because A/I^n is a Cohen-Macaulay ring with $\dim A/I^n = d - s$ for all $n \geq 1$ (Lemma (2.1), Assertion (1)), the element f is $\mathrm{G}(I)$-regular and $\mathrm{G}(I)/f\mathrm{G}(I) \cong \mathrm{G}((I + (f))/(f))$ as graded A-algebras. Hence $\mathrm{G}((I + (f))/(f))$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}((I + (f))/(f))) = \mathrm{a}(\mathrm{G}(I)) = 1 - s$ ([GW, Remark (3.1.6)]). Thus $(I + (f))/(f) \in \mathcal{X}^s_{A/(f)}$. Let Q be a reduction of I generated by s elements in A and put $J = Q + (f)$. Then $(I + (f))^2 = J(I + (f))$, because $I^2 = QI$. On the other hand, by Lemma (2.1), Assertion (2) we get $J : (I + (f)) = I + (f)$, since $(I+(f))/(f) \in \mathcal{X}^s_{A/(f)}$ and $J/(f)$ is a reduction of $(I+(f))/(f)$. Thus $I+(f) \in \mathcal{X}^{s+1}_A$ by Proposition (2.3). □

Corollary (2.5). *Suppose that* $\mathcal{X}^s_A \neq \emptyset$ *for some* $0 \leq s < d$*. Then* $\sharp\mathcal{X}^i_A = \infty$ *for all* $s < i \leq d$.

Proof. It suffices to show that $\sharp\mathcal{X}^{s+1}_A = \infty$. Let $f \in \mathfrak{m}$ be A/I-regular. Then $I + (f^n) \in \mathcal{X}^{s+1}_A$ for all $n \geq 1$, whence the assertion. □

Let R be a Cohen-Macaulay local ring with the canonical module K_R and assume that $A = R \ltimes \mathrm{K}_R$ is the idealization of K_R over R. Let $I = (0) \times \mathrm{K}_R$ in A. Then $I^2 = (0)$ and $I = (0) : I$ (recall that $(0) :_R \mathrm{K}_R = (0)$; cf. [BH, Proposition 3.3.11]), whence $I \in \mathcal{X}^0_A$. Therefore by Corollaries (2.4) and (2.5) we get the following.

Corollary (2.6). *With the same notation as above, let* $\mathfrak{q} = (a_1, a_2, \cdots, a_s)$ *be an ideal in* R *generated by a subsystem* $a_1, a_2, \cdots, a_s$ *of parameters in* R*. Then* $\mathfrak{q} \times \mathrm{K}_R \in \mathcal{X}^s_A$*. Hence* $\sharp\mathcal{X}^s_A = \infty$ *if* $1 \leq s \leq \dim R$.

3. The associated graded rings $\mathrm{G}(J \times L)$

Let R be a Cohen-Macaulay local ring with the canonical module $K = \mathrm{K}_R$ and let $A = R \ltimes K$ be the idealization. Let J be an ideal in R and let L be an R-submodule of K. Let $I = J \times L$. Then I is an ideal in A if and only if $JK \subseteq L$.

In what follows we shall summarize some basic results on the graded rings associated to ideals I in A of the form $I = J \times L$. Let $V_n = K$ $(n \leq 0)$ and $V_n = J^{n-1}L$ $(n \geq 1)$. Then $I^n = J^n \times V_n$ in A for all $n \in \mathbb{Z}$ and the family $V = \{V_n\}_{n\in\mathbb{Z}}$ of R-submodules of K satisfies the following two conditions: (i) $V_n \supseteq V_{n+1}$ and (ii) $J^m V_n \subseteq V_{m+n}$ for all $m, n \in \mathbb{Z}$. Let t be an indeterminate over R and put $\mathrm{R}'(V) = \sum_{n\in\mathbb{Z}} V_n t^n$ $(\subseteq K[t, t^{-1}] = K \otimes_R R[t, t^{-1}])$. Then the above conditions (i) and (ii) guarantee that $\mathrm{R}'(V)$ is a finitely generated graded $\mathrm{R}'(J)$-module with the graduation $\{V_n t^n\}_{n\in\mathbb{Z}}$. Let $\mathrm{G}(V) = \mathrm{R}'(V)/t^{-1}\mathrm{R}'(V)$. We furthermore have the following.

Lemma (3.1). $\mathrm{G}(I) \cong \mathrm{G}(J) \ltimes \mathrm{G}(V)$ *as graded rings.*

Proof. Let $f : R \to A$ and $p : A \to R$ denote the homomorphisms of rings defined by $f(a) = (a, 0)$ and $p(a, x) = a$. Since $f(J) \subseteq I$ and $p(I) = J$, these homomorphisms f and p induce the homomorphisms $\psi : \mathrm{G}(J) \to \mathrm{G}(I)$ and $\varphi : \mathrm{G}(I) \to \mathrm{G}(J)$ of the associated graded rings, so that $\varphi\cdot\psi = 1_{\mathrm{G}(J)}$. Let $\mathcal{K} = \mathrm{Ker}\varphi$. Then $\mathrm{G}(I) = \mathrm{Im}\psi \oplus \mathcal{K}$. We have $\mathcal{K}^2 = (0)$, because $\mathcal{K}_n = \left\{\overline{(0, y)t^n} \;\middle|\; y \in V_n\right\}$ for all $n \in \mathbb{Z}$ (here t denotes an indeterminate over A and $\overline{*}$ the reduction mod $t^{-1}\mathrm{R}'(I)$). Therefore, thanks to the isomorphism $\rho : \mathrm{G}(V) \to \mathcal{K}$ of graded $\mathrm{G}(J)$-modules defined by $\rho(\overline{yt^n}) = \overline{(0, y)t^n}$ $(y \in V_n,\ n \in \mathbb{Z})$, we readily get the required isomorphism $\eta : \mathrm{G}(J) \ltimes \mathrm{G}(V) \to \mathrm{G}(I)$, $\eta(a, x) = \psi(a) + \rho(x)$ of graded rings. □

The next result will play a key role in our proof of Theorem (1.1).

Proposition (3.2). *The following conditions are equivalent*:

(1) $\mathrm{G}(I)$ *is a Gorenstein ring.*
(2) $\mathrm{G}(J)$ *is a Cohen-Macaulay ring and* $\mathrm{K}_{\mathrm{G}(J)} \cong \mathrm{G}(V)(n)$ *for some* $n \in \mathbb{Z}$.

When this is the case, the integer n *is uniquely determined and we have the equality*

$$n = \mathrm{a}(\mathrm{G}(I)) = \mathrm{a}(\mathrm{G}(J)) + \min\{i \in \mathbb{Z} \mid V_i \neq V_{i+1}\}.$$

Proof. The equivalence of conditions (1) and (2) directly follows from [R], since $\mathrm{G}(I) = \mathrm{G}(J) \ltimes \mathrm{G}(V)$ by Lemma (3.1). To see the last assertion, let $d = \dim R$ and $\mathfrak{M} = \mathfrak{n}\mathrm{G}(J) + \mathrm{G}(J)_+$, where $\mathfrak{n}$ denotes the maximal ideal in R. Then $n = \mathrm{a}(\mathrm{G}(V))$, because $\mathrm{H}^d_{\mathfrak{M}}(\mathrm{K}_{\mathrm{G}(J)}) \cong \mathrm{H}^d_{\mathfrak{M}}(\mathrm{G}(V))(n)$ and $\mathrm{a}(\mathrm{K}_{\mathrm{G}(J)}) = 0$. On the other hand we have

$$\mathrm{a}(\mathrm{G}(I)) = \max\{\mathrm{a}(\mathrm{G}(J)), \mathrm{a}(\mathrm{G}(V))\},$$

because $\mathrm{G}(I) = \mathrm{G}(J) \oplus \mathrm{G}(V)$ as graded $\mathrm{G}(J)$-modules. Let $a = \mathrm{a}(\mathrm{G}(J))$. Then since $\mathrm{K}_{\mathrm{G}(J)} \cong \mathrm{G}(V)(n)$ and $a = -\min\{i \in \mathbb{Z} \mid [\mathrm{K}_{\mathrm{G}(J)}]_i \neq (0)\}$ (cf. [GW, Definition (2.1.2)]), we get $n - a = \min\{i \in \mathbb{Z} \mid [\mathrm{G}(V)]_i \neq (0)\} \geq 0$. Therefore $n \geq a$ whence $\mathrm{a}(\mathrm{G}(I)) = \max\{a, n\} = n$. □

4. Proof of Theorem (1.1)

The purpose of this section is to prove Theorem (1.1) and its Corollaries (1.2) and (1.3). Similarly as in Section 3 let R be a Cohen-Macaulay local ring with the canonical module $K = \mathrm{K}_R$. Let $A = R \ltimes K$. Let J be an ideal in R of height s and

let L be an R-submodule of K. Assume that $JK \subseteq L$ and put $I = J \times L$. Then I is an ideal in A with $\mathrm{ht}_A I = \mathrm{ht}_R J = s$.

Proof of Theorem (1.1). If $I \in \mathcal{X}_A^s$, then by definition our ideal I contains a reduction $Q = (f_1, f_2, \cdots, f_s)$ generated by s elements $f_i{}'s$ in A so that $I^2 = QI$ by Lemma (2.1), Assertion (2). Let $p : A \to R$ be the projection map $p(a, x) = a$ and we put $J = p(I)$ and $\mathfrak{q} = p(Q)$. Then $J^2 = \mathfrak{q}J$ and $\mathfrak{q}$ is generated by s elements in R. Therefore, in order to prove Theorem (1.1), we may assume that $J^2 = \mathfrak{q}J$ for some reduction $\mathfrak{q} = (a_1, a_2, \cdots, a_s)$ generated by a subsystem $a_1, a_2, \cdots, a_s$ of parameters in R. We begin with the following.

Lemma (4.1). *Assume that either $I \in \mathcal{X}_A^s$ or $L = \mathfrak{q}K :_K J$. Then the following conditions are equivalent*:

(1) $K = L$.
(2) $J = \mathfrak{q}$.

Proof. First let $L = \mathfrak{q}K :_K J$ and we will check the implication (1)⇒(2). If $JK \subseteq \mathfrak{q}K$, then $J \subseteq \mathfrak{q}$, because $K/\mathfrak{q}K = \mathrm{K}_{R/\mathfrak{q}}$ is a faithful $R/\mathfrak{q}$-module (cf. [BH, Proposition 3.3.11]). Hence $J = \mathfrak{q}$. Next assume that $I \in \mathcal{X}_A^s$ and let $K = L$. To see $J = \mathfrak{q}$, we have only to show $JR_\mathfrak{p} = \mathfrak{q}R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_R R/\mathfrak{q}$. Let $\mathfrak{p} \in \mathrm{Ass}_R R/\mathfrak{q}$. Then $\mathrm{ht}_R \mathfrak{p} = s$ and $\mathfrak{p} \supseteq J$. Because $A_\mathfrak{p} = R_\mathfrak{p} \ltimes K_\mathfrak{p}$ and $IA_\mathfrak{p} = JR_\mathfrak{p} \times K_\mathfrak{p}$, we have $\ell_{A_\mathfrak{p}}(A_\mathfrak{p}/IA_\mathfrak{p}) = \ell_{R_\mathfrak{p}}(R_\mathfrak{p}/JR_\mathfrak{p})$. Hence by [GIW, Proposition (2.2)] we get

$$\begin{aligned}\ell_{R_\mathfrak{p}}(R_\mathfrak{p}/JR_\mathfrak{p}) &= \frac{1}{2}\ell_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}) \\ &= \frac{1}{2}\{\ell_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) + \ell_{R_\mathfrak{p}}(K_\mathfrak{p}/\mathfrak{q}K_\mathfrak{p})\},\end{aligned}$$

because $IA_\mathfrak{p} \in \mathcal{X}_{A_\mathfrak{p}}^s$ by Corollary (2.2), Assertion (1), and $\mathfrak{q}A_\mathfrak{p}$ is a reduction of $IA_\mathfrak{p}$. Therefore $\ell_{R_\mathfrak{p}}(R_\mathfrak{p}/JR_\mathfrak{p}) = \ell_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p})$, since $\ell_{R_\mathfrak{p}}(K_\mathfrak{p}/\mathfrak{q}K_\mathfrak{p}) = \ell_{R_\mathfrak{p}}(\mathrm{K}_{R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}}) = \ell_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p})$. Thus $JR_\mathfrak{p} = \mathfrak{q}R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_R R/\mathfrak{q}$ so that we have $J = \mathfrak{q}$. Conversely, assume $J = \mathfrak{q}$. Then $I = \mathfrak{q}A : I$ by Lemma (2.1), because $\mathfrak{q}A$ is a reduction of I. Let $x \in K$ and $(a, y) \in I = \mathfrak{q} \times L$. Then because $(0, x) \cdot (a, y) = a(0, x) \in \mathfrak{q}A$, we have $(0, x) \in \mathfrak{q}A : I = I$ whence $x \in L$. Thus $K = L$. □

Let us continue the proof of Theorem (1.1). Because $\mathfrak{q} \times K \in \mathcal{X}_A^s$ by Corollary (2.6), we may assume that $J \neq \mathfrak{q}$ whence $K \neq L$ by Lemma (4.1). By Proposition (3.2) $I \in \mathcal{X}_A^s$ if and only if $\mathrm{G}(J)$ is a Cohen-Macaulay ring with $\mathrm{a}(\mathrm{G}(J)) = 1 - s$ and $\mathrm{K}_{\mathrm{G}(J)} \cong \mathrm{G}(V)(1 - s)$, where $V = \{V_n\}_{n \in \mathbb{Z}}$ is the filtration of $K = \mathrm{K}_R$ given in Section 3. The ring $\mathrm{G}(J)$ is Cohen-Macaulay if and only if R/J is Cohen-Macaulay, because $J^2 = \mathfrak{q}J$. When this is the case, we have $\mathrm{a}(\mathrm{G}(J)) = 1 - s$, since $J \neq \mathfrak{q}$. Consequently, we conclude that $I \in \mathcal{X}_A^s$ if and only if the ring R/J is Cohen-Macaulay and $\mathrm{K}_{\mathrm{R}'(J)} = \mathrm{R}'(V)(2-s)$ (recall that $\mathrm{K}_{\mathrm{G}(J)} = [\mathrm{K}_{\mathrm{R}'(J)}/t^{-1}\mathrm{K}_{\mathrm{R}'(J)}](-1)$). Thus by [GI, Theorem 2.1], $I \in \mathcal{X}_A^s$ if and only if R/J is Cohen-Macaulay and $V = \omega(s - 2)$, where $\omega = \{\omega_n\}_{n \in \mathbb{Z}}$ denotes the canonical J-filtration of K_R. Let us now assume R/J is Cohen-Macaulay. Hence $I \in \mathcal{X}_A^s$ if and only if $V_n = \omega_{n+s-2}$ for all $n \in \mathbb{Z}$. On the other hand we have by [GI, Lemma 2.3 (1) and Lemma 5.1 (1)] that $\omega_{n+s-2} = \mathfrak{q}^n K :_K J$ for all $n \in \mathbb{Z}$ (resp. $\omega_{n-2} = (0) :_K J^{2-n}$ for all $n \in \mathbb{Z}$) if $s \geq 1$ (resp. $s = 0$), because $J^2 = \mathfrak{q}J$ but $J \neq \mathfrak{q}$. Therefore by [GI, Lemma 5.1 (2)] $I \in \mathcal{X}_A^s$ if and only if $L = \mathfrak{q}K :_K J$, which completes the proof of Theorem (1.1). □

Proof of Corollaries (1.2) *and* (1.3). Assume R is a Gorenstein local ring and take $\mathrm{K}_R = R$. Let J be an ideal in R of height s. Then by Proposition (2.3) $J \in \mathcal{X}_A^s$ if and only if R/J is Cohen-Macaulay and J contains a reduction $\mathfrak{q} = (a_1, a_2, \cdots, a_s)$ with $J^2 = \mathfrak{q}J$ and $J = \mathfrak{q} : J$. The latter conditions are exactly the same as Condition (2) in Theorem (1.1), whence $J \in \mathcal{X}_R^s$ if and only if $J \times J \in \mathcal{X}_A^s$, which proves Corollary (1.2). Corollary (1.3) is now clear. □

To conclude this paper let us note a remark about the characterization (2.3) of good ideals. Let I be an ideal of height s in a general Gorenstein local ring A and assume that I contains a reduction Q generated by s elements in A. Then $\mathrm{depth} A/I > 0$, once $I = Q : I$ and $s < d$. Therefore $I \in \mathcal{X}_A^s$ if $I^2 = QI$ and $I = Q : I$, provided $s \geq d-1$. This is no longer true unless $s \geq d-1$. We actually have the following.

Proposition (4.2). *Let $s, d \in \mathbb{Z}$ be integers with $0 \leq s \leq d-2$. Then there exists an ideal I of height s in a Gorenstein local ring A of dimension d which satisfies the following two conditions*:

(1) *The ideal I contains a reduction Q generated by s elements such that $I^2 = QI$ and $I = Q : I$.*
(2) *The ring A/I is not Cohen-Macaulay.*

Hence $I \notin \mathcal{X}_A^s$.

Proof. Passing to the ideal $IC + (X_1, X_2, \cdots, X_s)$ in the formal power series ring $C = A[[X_1, X_2, \cdots, X_s]]$ in s variables, we may assume that $s = 0$ and $d \geq 2$. First we take a Cohen-Macaulay complete local domain R with $d = \dim R$ and let $0 \to Z \to F \to \mathfrak{n} \to 0$ be a minimal free presentation of the maximal ideal $\mathfrak{n}$ in R. We put $B = R \ltimes F$ and $J = (0) \times Z$. Then B is a Cohen-Macaulay local ring and $B/J = R \ltimes \mathfrak{n}$. Hence B/J is not Cohen-Macaulay, because $\mathrm{depth}_R \mathfrak{n} = 1 < d$. We have $J^2 = (0)$ and $\mathrm{Ass}_B B/J = \mathrm{Ass} B$. Let $K = \mathrm{K}_B$ be the canonical module of B and put $A = B \ltimes K$. Let $L = (0) :_K J$ and $I = J \times L$. Then A is a Gorenstein local ring of dimension d, I is an ideal in A, and the ring $A/I = B/J \ltimes K/L$ is not Cohen-Macaulay, since B/J is not Cohen-Macaulay. We certainly have $I^2 = J^2 \times JL = (0)$. However,

Claim (4.3). $I = (0) : I$.

Proof. Assume that $I \subsetneq (0) : I$ and choose $P \in \mathrm{Ass}_A A/I$ so that $IA_P \subsetneq (0) : IA_P$. Let $P = \mathfrak{p} \times K$ with $\mathfrak{p} \in \mathrm{Spec} B$. Then $A_P = A_\mathfrak{p}$ and we have either $\dim B_\mathfrak{p} = 0$ or $\mathrm{depth}_{B_\mathfrak{p}} K_\mathfrak{p}/L_\mathfrak{p} = 0$, since $P \in \mathrm{Ass}_A A/I$. If $\dim B_\mathfrak{p} = 0$, then the ideal $JB_\mathfrak{p}$ of $B_\mathfrak{p}$ and the $B_\mathfrak{p}$-submodule $L_\mathfrak{p}$ of $K_\mathfrak{p}$ satisfy the conditions: $(JB_\mathfrak{p})^2 = (0)$ and $L_\mathfrak{p} = (0) :_{K_\mathfrak{p}} JB_\mathfrak{p}$, so that by Theorem (1.1) the ideal $I_\mathfrak{p} = JB_\mathfrak{p} \times L_\mathfrak{p}$ of $A_\mathfrak{p} = B_\mathfrak{p} \ltimes K_\mathfrak{p}$ must be in $\mathcal{X}^0_{B_\mathfrak{p} \ltimes K_\mathfrak{p}}$, whence $(0) : IA_\mathfrak{p} = IA_\mathfrak{p}$ by Lemma (2.1), Assertion (4). Therefore $\mathrm{ht}_B \mathfrak{p} > 0$ and $\mathfrak{p} \in \mathrm{Ass}_B K/L$. Let us choose an element $x \in K$ so that $\mathfrak{p} = L :_B x$. Then since $\mathfrak{p}x \subseteq L$, we have $\mathfrak{p} \cdot Jx = (0)$. Let $f \in \mathfrak{p}$ be a regular element in B. Then f is regular on $K = \mathrm{K}_B$ so that $Jx = (0)$ whence $x \in L$, which is impossible. Thus $I = (0) : I$ which completes the proof of Proposition (4.2). □

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