NILPOTENCY DEGREE OF COHOMOLOGY RINGS 
IN CHARACTERISTIC 3 

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Abstract. The purpose of this note is to provide a 3-group $G$ whose mod-3 cohomology ring has a nilpotent element $\xi \in H^* (G)$ satisfying $\xi^3 \neq 0$.

1. Statement of the main result

For every $p$-group $G$, denote by $H^*(G)$ the mod-$p$ cohomology algebra of $G$. We are now interested in the nilpotency degrees of elements of $H^*(G)$. For the case $p = 2$, in [1], [2], it was shown that, given any positive integer $n$, there exists a 2-group whose cohomology ring has elements of nilpotency degree $n + 1$. However, the methods given there do not generalize to the case of odd characteristic $p$. In this case, it seems that, until now, we do not have any example of elements of $H^*(G)$ having nilpotency degrees greater than $p$.

The purpose of this note is to provide a 3-group $G$ whose cohomology ring has a nilpotent element $\xi \in H^* (G)$ satisfying $\xi^3 \neq 0$. The group $G$ is obtained as follows. Let

$$E = \langle a_1, a_2 | a_1^3 = a_2^3 = [a_1, a_2] = [a_2, [a_1, a_2]] = 1 \rangle$$

be the extraspecial 3-group of order 3$^3$ and of exponent 3, and let

$$K = C_3^4 = \langle a_3, \ldots, a_6 | a_i^3 = [a_i, a_j] = 1 \rangle$$

be the elementary abelian 3-group of rank 4. Set $G = E \times K$. Define $u_i \in H^1 (G) = \text{Hom}(G, \mathbb{Z}/3), v_i \in H^2 (G), 1 \leq i \leq 6$, by

$$u_i(a_j) = \delta_{ij},$$

$$v_i = \beta u_i,$$

with $\delta_{ij}$ the Kronecker symbol and $\beta$ the Bockstein homomorphism. For $3 \leq i \leq 6$, consider $u_i, v_i$ as elements of $H^* (K)$ via the restriction map. So, by the Künneth formula,

$$H^*(K) = \Lambda[u_3, \ldots, u_6] \otimes \mathbb{F}_3[v_3, \ldots, v_6],$$

and $H^*(G)$ may be identified with

$$H^*(E) \otimes H^*(K).$$
Consider the central extension

\((G) \quad 0 \to \mathbb{Z}/3 \to G \to 1\)

corresponding to the cohomology class \(z = v_1 + u_3 u_4 + u_5 u_6 \in H^2(G)\). We will prove

**Theorem.** There exists a nilpotent element \(\xi \in H^3(G)\) satisfying \(\xi^3 \neq 0\).

However, for \(p > 3\), our method could not be applied. So, the existence of a \(p\)-group \(G\) having a cohomology class of nilpotency degree greater than \(p > 3\) still remains an open problem. Besides, we could not provide any example of cohomology classes of a 3-group having nilpotency degrees greater than a given \(n\), although it is known that, for any \(p \geq 2\) and for any \(p\)-group \(G\) of order \(p^m\), nilpotency degrees of nilpotent elements of \(H^*(G)\) are bounded above by \(p^{m-1}\) (see [5]).

**2. Proof of the main result**

Via the restriction map, for \(1 \leq i \leq 2\), \(u_i\) and \(v_i\) can be considered as elements of \(H^*(E)\). Set \(c = [a_1, a_2]\) and let \(A_0 = \langle a_1, c \rangle, A_1 = \langle a_1, a_2, c \rangle, A_2 = \langle a_1^2 a_2, c \rangle, A_3 = \langle a_2, c \rangle\) be elementary abelian subgroups of \(E\). Set \(s_0 = \text{Res}^E_{A_0}(u_1), s_i = \text{Res}^E_{A_i}(u_2), 1 \leq i \leq 3\), and let \(t_j = \beta s_j, 0 \leq j \leq 3\). Let \(u\) be a generator of \(H^1(\langle c \rangle)\) and set \(v = \beta u \in H^2(\langle c \rangle)\). It follows that

\[
H^*(\langle c \rangle) = \Lambda[u] \otimes F_3[v].
\]

So, by Künneth formula, we have

\[
H^*(A_i) = \Lambda[s_i, u] \otimes F_3[t_i, v],
\]

\(0 \leq i \leq 3\).

It follows from [4] that \(H^*(E)\) is detected by the \(A_i's\). According to [3], we have

**Lemma 1.** (i) There exist elements \(U_1, U_2 \in H^2(E)\) such that:

(a) \(v_1, v_2, U_1, U_2\) is a basis of \(H^2(E)\). Furthermore, we have

\[
\begin{align*}
\begin{array}{l}
u_1 | A_0 = s_0, \quad u_1 | A_1 = s_1, \quad u_1 | A_2 = 2s_2, \quad u_1 | A_3 = 0, \\
u_1 | A_0 = t_0, \quad v_1 | A_1 = t_1, \quad v_1 | A_2 = 2t_2, \quad v_1 | A_3 = 0, \\
u_2 | A_0 = 0, \quad u_2 | A_1 = s_1, \quad u_2 | A_2 = s_2, \quad u_2 | A_3 = s_3, \\
u_2 | A_0 = 0, \quad v_2 | A_1 = t_1, \quad v_2 | A_2 = t_2, \quad v_2 | A_3 = t_3, \\
U_1 | A_0 = s_0 u, \quad U_1 | A_1 = s_1 u + t_1, \quad U_1 | A_2 = 2s_2 u + t_2, \quad U_1 | A_3 = 0, \\
U_2 | A_0 = 0, \quad U_2 | A_1 = 2s_1 u + t_1, \quad U_2 | A_2 = 2s_2 u + 2t_2, \quad U_2 | A_3 = s_3 u;
\end{array}
\end{align*}
\]

(b) \(v_1^2, v_1 v_2, v_2^2, v_1 U_1, v_1 U_2, v_2 U_1, v_2 U_2\) is a basis of \(H^4(E)\) and \(U_1^2 = v_1 U_2, U_2^2 = v_2 U_1, U_1 U_2 = v_1 v_2\).

(ii) \(v_1 u_1, v_1 u_2\) is linearly independent in \(H^3(E)\).

**Lemma 2.** Let \(X\) be an element of \(H^*(G)\) of degree \(n \leq 2\). Then \(X = 0\) provided that one of the following conditions is satisfied:

(i) \(X v_1 = 0\);

(ii) \(X z = 0\).
Proof. The case \( n = 1 \) is trivial. For \( n = 2 \), write \( X = a_{2,0} + a_{1,1} + a_{0,2} \) with \( a_{i,j} \in H^i(\mathbb{E}) \otimes H^j(K) \). If \( X v_1 = 0 \), then \( a_{2,0}v_1 = a_{1,1}v_1 = a_{0,2}v_1 = 0 \), hence \( a_{0,2} = 0 \), and, by Lemma 1, \( a_{2,0} = a_{1,1} = 0 \); so \( X = 0 \). If \( Xz = 0 \), then

\[
\begin{align*}
 a_{2,0}v_1 &= 0, \\
 a_{2,0}(u_3u_4 + u_5u_6) + a_{0,2}v_1 &= 0, \\
 a_{1,1}v_1 &= a_{0,2}(u_3u_4 + u_5u_6) = a_{1,1}(u_3u_4 + u_5u_6) = 0,
\end{align*}
\]

hence, it follows from what we just proved that \( a_{2,0} = a_{1,1} = a_{0,2} = 0 \), so \( X = 0 \). The lemma follows.

Lemma 3. The cup-product with \( v_1^2 \) is an injective map from \( H^2(\mathbb{E}) \) to \( H^6(\mathbb{E}) \).

Proof. If \( v_1^2(\lambda_1 v_1 + \lambda_2 v_2 + \mu_1 u_1 + \mu_2 u_2) = 0 \) with \( \lambda_1, \mu_1 \in \mathbb{Z}/3 \), then restricting to \( A_j, 0 \leq j \leq 3 \) yields \( \lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0 \). The lemma follows.

For elements \( X, Y, \ldots \) of \( H^*(G) \), denote by \( (X, Y, \ldots) \) the ideal of \( H^*(G) \) generated by \( X, Y, \ldots \). We have

Lemma 4. Let \( X \) be an element of \( H^1(G) \) with \( X \beta z \in (z) \). Then \( X = 0 \).

Proof. Write \( X \beta z = Yz \) with \( Y \in H^2(G) \). Since \( X \) and \( \beta z \) are free of \( v_1 \), it follows that \( Yv_1 = 0 \). By Lemma 2, \( Y = 0 \), so \( X \beta z = 0 \). Write \( X = \sum_{i=1}^6 \lambda_i u_i \). A direct verification shows that \( \lambda_1 = \cdots = \lambda_6 = 0 \). The lemma follows.

Lemma 5. \( v_1^2 v_2 \notin (z, \beta z) \).

Proof. Assume that \( v_1^2 v_2 \in (z, \beta z) \). Then there exist elements \( a_{i,j}, b_{i,j} \in H^i(\mathbb{E}) \otimes H^j(K) \) satisfying

\[
\begin{align*}
 v_1^2 v_2 &= \sum a_{i,j}z + \sum b_{i,j}\beta z \\
 &= \sum a_{i,j}(v_1 + u_3u_4 + u_5u_6) + \sum b_{i,j}(v_3u_4 - v_4u_3 + v_5u_6 - v_6u_5).
\end{align*}
\]

By decomposing (1), we get

\[
\begin{align*}
 v_1^2 v_2 &= a_{4,0}v_1, \\
 0 &= a_{4,0}(u_3u_4 + u_5u_6) + a_{2,2}v_1, \\
 0 &= a_{2,2}(u_3u_4 + u_5u_6) + a_{0,4}v_1 + b_{2,1}(v_3u_4 - v_4u_3 + v_5u_6 - v_6u_5).
\end{align*}
\]

It follows from (3) and Lemma 2 that \( a_{4,0} \) contains \( v_1 \) as a factor. By Lemma 3, \( a_{4,0} = v_2v_1 \). Hence, by (3) and Lemma 2, \( a_{2,2} = -v_2(u_3u_4 + u_5u_6) \). So, from (4), we get

\[
2v_2u_3u_4u_5u_6 = a_{0,4}v_1 + b_{2,1}(v_3u_4 - v_4u_3 + v_5u_6 - v_6u_5),
\]

a contradiction. The lemma follows.

With some abuse of notation, we consider elements of \( H^*(G) \) as elements of \( H^*(G) \) via the inflation map. Set \( \xi = U_1 \in H^2(G) \). The first part of the theorem follows from

Lemma 6. \( \xi \) is nilpotent.

Proof. It follows from Lemma 1 that \( \xi^2 = v_1 U_2 \); hence

\[
\xi^3 = v_1 U_1 U_2 = v_1^2 v_2.
\]

So \( \xi^9 = (v_1^2 v_2)^3 \). Since, in \( H^*(G) \), \( z = 0 \), we have \( v_1^3 = (-u_3u_4 - u_5u_6)^3 = 0 \). It follows that \( \xi^9 = 0 \). Therefore \( \xi \) is nilpotent. The lemma is proved.
The proof of the theorem is completed by the following.

**Lemma 7.** \( \xi^3 \neq 0 \) in \( H^*(G) \).

**Proof.** Assume that \( \zeta = \xi^3 = v_1^2 v_2 = 0 \) in \( H^*(G) \). So \( \zeta \in \text{ImInf}_G^G \). Set \( Z = i(Z/3) \subset G \). Denote by \( \{E_r, d_r\} \) the Hochschild-Serre spectral sequence corresponding to the extension \( (G) \). So

\[
E_2 = H^*(G) \otimes H^*(Z).
\]

Let \( s \) be a generator of \( H^1(Z) \) and set \( t = \beta s \in H^2(Z) \). \( s \) can be chosen such that \( d_2(s) = z \). \( t \) is then transgressive and \( d_3(t) = \beta z \). So, by Lemma 5, \( \zeta \neq 0 \) in \( E_4^{6,0} \).

Since \( \zeta \in \text{ImInf}_G^G \), it must be hit by \( d_4 \) or \( d_5 \). If \( \zeta = d_4(a \otimes st) \) with \( a \otimes st \in E_4^{2,3} \), then

\[
d_2(a \otimes st) = (a \otimes t)d_2(s)
= az \otimes t
= 0
\]

in \( E_2^{1,2} \); so \( az = 0 \) in \( H^*(G) \), hence \( a = 0 \) by Lemma 2, a contradiction. If \( \zeta = d_5(b \otimes t^2) \) with \( b \otimes t^2 \in E_5^{3,4} \), then

\[
d_3(b \otimes t^2) = (b \otimes 1)d_3(t^2)
= (b \otimes 1)(2\beta z \otimes t)
= 2b\beta z \otimes t
= 0
\]

in \( E_3^{4,2} \); so \( b\beta z \in (z) \), hence \( b = 0 \) by Lemma 4, a contradiction. Therefore \( \zeta \neq 0 \). The lemma follows.

**References**


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