

ON THE COMPARISON OF THE SPACES
 $L^1BV(\mathbb{R}^n)$ AND $BV(\mathbb{R}^n)$

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ABSTRACT. The notion of L^1 -variation and the space L^1BV arise in the study of regularity properties of solutions to perturbed conservation laws. In this article we show that this notion is equivalent to variation in the regular sense, and therefore the space L^1BV is the same as the space BV in the sense of Cesari-Tonelli. We also point out some connection between the space L^1BV and the Favard classes for translation semigroups.

1. INTRODUCTION

We recently proposed measuring variation of functions utilizing the L^1 -norm [9]. For a measurable function on \mathbb{R}^n we define

$$L^1Var(f) = \sup_{h \neq 0} \frac{1}{|h|} \int_{\mathbb{R}^n} |f_h - f| \, dx,$$

where $h \in \mathbb{R}^n$ and $f_h(x) = f(x + h)$. We also define

$$L^1BV(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : L^1Var(f) < \infty\}.$$

The norm $\|f\|_{L^1BV} = L^1Var(f) + \|f\|_{L^1}$ equips the space $L^1BV(\mathbb{R}^n)$ with a Banach space structure. Note that the expression for L^1Var is similar to the concept of differential quotient used by Lions and Magenes [10] to study regularity problems in Hilbert spaces.

The motivation to define variation in the L^1 -sense arises from measuring regularity of solutions to perturbed conservation laws of the form

$$(1.1) \quad \begin{aligned} u_t + \operatorname{div}(f(u)) + \frac{g(u)}{H(x)} &= 0, \quad x \in \mathbb{R}^n, \quad t > 0, \\ u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^n; \end{aligned}$$

see [9]. It is well known that solutions to this type of equation lose regularity due to occurrence of shocks [13]. This Cauchy problem can be considered as an ordinary

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differential equation on the Banach space $L^1(\mathbb{R}^n)$

$$(1.2) \quad \begin{aligned} \frac{d}{dt} u(t) &= Au(t), \\ u(0) &= u_0, \end{aligned}$$

where $Au = -\operatorname{div}(f(u)) - \frac{g(u)}{H(x)}$. Under some mild hypothesis the operator A is quasi m -dissipative [3, 8], and it generates an L^1 -quasi contraction semigroup $\{T_t\}_{t \geq 0}$, i.e. $\|T_t\|_{Lip} \leq e^{\omega t}$ for all $t \geq 0$, and some $\omega \in \mathbb{R}$. Therefore the (mild) solution to (1.2) is obtained by evolving initial data via the semigroup, $u(t) = T_t u_0$. Observing that a translated solution satisfies a translated Cauchy problem (1.2), together with the resolvent condition of an m -dissipative operator, it is natural to define quantities such as L^1Var . Using this version of variation to measure regularity of solutions to the Cauchy problem, we were able to estimate the upper bound of L^1 -variation of solutions as time evolves, and therefore showing that the Banach space L^1BV is invariant under the semigroup which governs the Cauchy problem (1.2) (cf. [9]).

For other regularity results in conservation laws, we cite Conway and Smoller [2] for showing that solutions of (unperturbed) single conservation laws in n -spatial dimensions are of bounded variation (in the sense of Cesari-Tonelli). They established the fact that the variation of the solution at time t does not exceed the variation of initial data. This was done in the framework of difference schemes. In the language of semigroups, this means that the space BV is invariant under the semigroup which governs (unperturbed) conservation laws. Another notion of regularity of solutions to conservation laws (in one spatial dimension) was proposed by Schaeffer [12]. He showed that for smooth initial data in the Schwartz space $\mathcal{S}(\mathbb{R})$, the solution of (unperturbed) conservation laws is piecewise smooth, except for initial data in a certain subset of $\mathcal{S}(\mathbb{R})$ of the first (Baire) category. He used singularity theory to obtain this result.

In this paper we will show that the notion of L^1 -variation we defined above is equivalent to the notion of variation in the sense of Cesari-Tonelli. Consequently, the space BV is invariant under the semigroup governing the Cauchy problem (1.2) of perturbed conservation laws. This is done by showing the equivalence between L^1Var and Var for smooth functions. Through regularization and lower semicontinuity of both L^1Var and Var we are able to pass the limits and thus show the equivalence in general.

This paper is organized as follows. In Section 2 we establish some facts about regularization of L^1BV -functions. Section 3 is a reminder of facts regarding BV space theory. We refer to Evans and Garipey [5] for most of it. In Section 4, we prove our main result. The facts we develop in Section 2 and Section 3 show that it suffices for us to work in the smooth case. We include some characterizations of L^1BV in terms of the notion of generalized domain or Favard class in Section 5.

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2. REGULARIZATION

Our objective here is to show that a function with bounded L^1 -variation can be approximated by smooth functions with compact support. Define the C^∞ -function

ϕ as follows:

$$\phi(x) = \begin{cases} c \exp\left(\frac{1}{|x|^2-1}\right), & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1; \end{cases}$$

the constant c is chosen so that

$$\int_{\mathbb{R}^n} \phi(x) dx = 1.$$

Next we define

$$\phi^\epsilon(x) = \frac{1}{\epsilon^n} \phi\left(\frac{x}{\epsilon}\right)$$

for $\epsilon > 0$ and $x \in \mathbb{R}^n$. The sequence $\{\phi^\epsilon\}$ is the standard mollifier. Now let $u \in L^1BV(\mathbb{R}^n)$, and define $u^\epsilon = u * \phi^\epsilon$. This convolution makes u^ϵ become smooth and further, if $u \in L^1(\mathbb{R}^n)$, then $u^\epsilon \rightarrow u$ in L^1 -norm. An immediate computation shows that regularization reduces L^1 -norm,

$$(2.1) \quad \|u^\epsilon\|_1 \leq \|u\|_1.$$

For the rest of the paper, let u_h denotes the translate of u by a fixed vector h , $u_h(x) = u(x + h)$. Observe that (2.1) implies

$$\|u_h^\epsilon - u^\epsilon\|_1 = \|(u_h - u)^\epsilon\|_1 \leq \|u_h - u\|_1.$$

Upon taking the quotient with $|h|$ and taking suprema over nonzero h , we have the fact that regularization also reduces L^1Var

$$(2.2) \quad L^1Var(u^\epsilon) \leq L^1Var(u).$$

In general, suppose we have $u^\epsilon \rightarrow u$ in L^1 as $\epsilon \rightarrow 0$ (not necessarily a regularization). Let us also fix a nonzero vector h in \mathbb{R}^n . By Fatou's lemma

$$|h|^{-1} \|u_h - u\|_1 \leq \liminf_{\epsilon \rightarrow 0} |h|^{-1} \|u_h^\epsilon - u^\epsilon\|_1.$$

Again, by taking suprema we have

$$(2.3) \quad L^1Var(u) \leq \liminf_{\epsilon \rightarrow 0} L^1Var(u^\epsilon),$$

i.e. lower semicontinuity of L^1Var .

Now we can construct a smooth ϵ -approximant to a given u in L^1BV . This will show the density theorem below in the following sense.

Theorem 2.1. *Given $v \in L^1BV(\mathbb{R}^n)$, there exists a sequence $\{v^\epsilon\} \subset C_0^\infty(\mathbb{R}^n) \cap L^1BV(\mathbb{R}^n)$ such that:*

- (1) $v^\epsilon \rightarrow v$, in L^1 -norm, and
- (2) $L^1Var(v^\epsilon) \rightarrow L^1Var(u)$, as $\epsilon \rightarrow 0$.

Notice that we do not state $L^1Var(v^\epsilon - v) \rightarrow 0$, as $\epsilon \rightarrow 0$. In fact, this is not expected since this would imply that the closure of C_0^∞ , relative to L^1BV -norm, is L^1BV . However this is not true since the closure of C_0^∞ , relative to L^1BV -norm, is the Sobolev space $W^{1,1}$, which is not equal to L^1BV . The function

$$f(x) = \begin{cases} C(x), & \text{for } 0 \leq x \leq 1, \\ 1, & \text{for } 1 < x \leq 2, \\ 1 - C(x - 2), & \text{for } 2 < x \leq 3, \\ 0, & \text{otherwise,} \end{cases}$$

where C denotes the Cantor function, gives an example of a function which is in $L^1BV(\mathbb{R})$ but not in $W^{1,1}(\mathbb{R})$. For further inspiration see Giusti [6]. A similar argument is used there to show that (strong) density of C_0^∞ in BV cannot be expected.

Proof. Let $\{v^\epsilon\}$ be a regularization for the given $v \in L^1BV(\mathbb{R}^n)$. While part (1) is a standard fact, part (2) follows directly from (2.2) and lower semicontinuity (2.3),

$$L^1Var(v^\epsilon) \leq L^1Var(v) \leq \liminf_{\epsilon \rightarrow 0} L^1Var(v^\epsilon).$$

□

3. A QUICK TOUR OF BV SPACES

A measurable function f on \mathbb{R}^n is said to be of *bounded variation* (in the sense of Cesari-Tonelli) if

$$\sup \left\{ \int_{\mathbb{R}^n} f \operatorname{div} \phi \, dx : \phi \in C_0^1(\mathbb{R}^n, \mathbb{R}^n), |\phi| \leq 1 \right\} < \infty.$$

The supremum itself is called the *variation* of f on \mathbb{R}^n , and is denoted by $Var(f)$. Notice that if $f \in W^{1,1}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} f \operatorname{div} \phi \, dx = - \int_{\mathbb{R}^n} Df \cdot \phi \, dx.$$

In particular, the variation of a constant function is zero. Thus it is possible to define variation for non- $L^1(\mathbb{R}^n)$ -functions. However in later contexts we require f to be in $L^1(\mathbb{R}^n)$. The bounded variation space on \mathbb{R}^n is

$$BV(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : Var(f) < \infty\}.$$

From elementary real analysis we know that *total variation* of f on the real line is

$$TV(f) = \sup_P \sum_P |f(x_i) - f(x_{i-1})|,$$

where the supremum is taken over the set of all countable partitions of the real line. Unfortunately this definition is not compatible with the definition of variation in general; we can modify the value of the function on a set of measure zero and change the total variation. For example, for the characteristic function of rational number $\chi_{\mathbb{Q}}$, we have $TV(\chi_{\mathbb{Q}}) = \infty$ but $TV(\mathbf{0}) = 0$.

The *essential variation* of f on the real line is defined by

$$essVar(f) = \sup_P \sum_P |f(x_i) - f(x_{i-1})|,$$

where the supremum is taken over the set of all countable partitions, for which all the partition points are points of approximate continuity. Recall that f is *approximately continuous* at x_0 if

$$\frac{\mathcal{L}^n(B_r(x_0) \cap \{x : |f(x) - f(x_0)| > \epsilon\})}{\mathcal{L}^n(B_r(x_0))} \rightarrow 0,$$

as $r \rightarrow 0$, for any $\epsilon > 0$. \mathcal{L}^n denotes the n -dimensional Lebesgue measure, and $B_r(x_0)$ is the ball with radius r , centered at x_0 . Roughly speaking the notion of approximate continuity is “continuity by throwing out a set of measure zero”. With

this notion, for example, one can immediately check that $\chi_{\mathbb{Q}}$ is approximately-continuous on any irrational point. This example also shows that the notion of approximate continuity is different from the one of continuity almost everywhere. Clearly, continuity implies approximate-continuity.

The following theorem is from Evans-Gariepy [5], p. 217.

Theorem 3.1. *For $f \in L^1(\mathbb{R})$, $essVar(f) = Var(f)$.*

The next theorem (see Evans-Gariepy [5], p. 220) relates bounded variation on the whole space \mathbb{R}^n and bounded variation on one dimensional subspaces. For this purpose we need to set up some notation. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Fix $k \in \{1, \dots, n\}$. We set $x' = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-1}$, and let $t \in \mathbb{R}$. Write

$$f^k(x', t) = f(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n),$$

and thus we view f^k as a function of one variable t , with x' fixed.

Theorem 3.2. *Assume $f \in L^1(\mathbb{R}^n)$. Then $f \in BV(\mathbb{R}^n)$ if and only if*

$$\int_{\mathbb{R}^{n-1}} essVar(f^k) dx' < \infty$$

for each $k = 1, \dots, n$.

Let $f \in BV(\mathbb{R}^n)$. Suppose $\{f^\epsilon\}$ is a regularization of f obtained from convoluting f with mollifier $\{\phi^\epsilon\}$. We observe that

$$\int_{\mathbb{R}^n} f^\epsilon \operatorname{div} \eta \, dx = \int_{\mathbb{R}^n} f(\operatorname{div} \eta)^\epsilon \, dx = \int_{\mathbb{R}^n} f \operatorname{div} \eta^\epsilon \, dx.$$

Thus

$$\int_{\mathbb{R}^n} f^\epsilon \operatorname{div} \eta = \int_{\mathbb{R}^n} f \operatorname{div} \eta^\epsilon \leq Var(f).$$

Also, after taking the supremum over η we have

$$(3.1) \quad Var(f^\epsilon) \leq Var(f).$$

From the dominated convergence theorem, it follows that

$$\int_{\mathbb{R}^n} f^\epsilon \operatorname{div} \eta \, dx \rightarrow \int_{\mathbb{R}^n} f \operatorname{div} \eta \, dx,$$

and therefore after taking the suprema over η we have

$$(3.2) \quad Var(f) \leq \liminf_{\epsilon \rightarrow 0} Var(f^\epsilon).$$

Lastly we quote a density theorem for C_0^∞ in BV ([5], p. 172). Every BV function can be approximated arbitrarily closely by smooth functions, in the weak sense.

Theorem 3.3. *Assume $u \in BV(\mathbb{R}^n)$. There exist functions $\{u_k\}_{k=1}^\infty \subset BV(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n)$ such that:*

1. $u_k \rightarrow u$ in $L^1(\mathbb{R}^n)$, and
2. $Var(u_k) \rightarrow Var(u)$, as $k \rightarrow \infty$.

4. MAIN RESULT

Our proof of the main result states the equivalence of Var and L^1Var .

Theorem 4.1. $L^1BV(\mathbb{R}^n) = BV(\mathbb{R}^n)$.

Proof. Using facts we developed in previous sections, it suffices to show the equivalence in the case of smooth functions. Let f be a smooth BV function. Hence in this case $Var(f) = \|Df\|_1$. Then

$$\begin{aligned} \|f_h - f\|_1 &= \left\| \int_0^1 h \cdot Df_{th} dt \right\| \\ &\leq |h| \int_0^1 \|Df_{th}\|_1 dt \\ &= |h| \|Df\|_1 \\ &= |h| Var(f). \end{aligned}$$

Taking the supremum over nonzero h we have $L^1Var(f) \leq Var(f)$.

To show the converse, it is sufficient to show that $\|\partial f / \partial x_k\|_1 \leq L^1Var(f)$ for $k = 1, \dots, n$. Let $\{e_1, \dots, e_k\}$ be the standard orthonormal basis for \mathbb{R}^n . Then

$$\begin{aligned} \left\| \frac{\partial f}{\partial x_k} \right\|_1 &= \left\| \lim_{t \rightarrow 0} t^{-1} (f_{te_k} - f) \right\|_1 \\ &\leq \liminf_{t \rightarrow 0} t^{-1} \|f_{te_k} - f\|_1 \\ &\leq L^1Var(f), \end{aligned}$$

therefore $Var(f) \leq nL^1Var(f)$. □

Remarks 4.2. From the proof above, in the case of $n = 1$, we actually proved that $Var(f) = L^1Var(f)$. In higher dimension this may not be true. Consider $u^a = \chi_R$ where $R = [0, 1] \times [0, a]$ is a rectangle in \mathbb{R}^2 . It is straightforward to compute that the ratio $Var(u^a)/L^1Var(u^a)$ is maximized when $a = 1$, with maximum value $\sqrt{2}$. This calculation shows that the ratio depends on a , and the norm L^1Var is not a multiple of the norm Var . Let $K_n = \sup \{Var(u)/L^1Var(u) : u \in L^1BV(\mathbb{R}^n), u \neq 0\}$. Our result shows that $1 \leq K_n \leq n$ and $K_2 \geq \sqrt{2}$. Determining the exact value of K_n is an interesting open question.

5. CHARACTERIZATION VIA FAVARD CLASSES

Let A be a quasi m -dissipative operator, and thus A determines a strongly continuous quasi contraction semigroup $\{T_t\}$ on $\overline{\mathcal{D}(A)}$. The *Favard class* (also called the *generalized domain*) of A , $\widehat{\mathcal{D}}(A)$, is defined by

$$\widehat{\mathcal{D}}(A) = \left\{ f \in \overline{\mathcal{D}(A)} : \|T_t f - f\| \leq M_f t \text{ for some } M_f > 0, 0 < t < 1 \right\}.$$

For further references on Favard class see Butzer and Berens [1] in the linear case, and in the nonlinear case see for example G.R. Goldstein, J.A. Goldstein and Oharu [7].

We consider the translation group $\{T_t\}$ acting on $L^1(\mathbb{R})$ as

$$(T_t f)(x) = f(x + t),$$

for $x \in \mathbb{R}$, $f \in L^1(\mathbb{R})$, and $t \in \mathbb{R}$. It is well known that the infinitesimal generator of this group is $A = \frac{d}{dx}$, with domain $W^{1,1}(\mathbb{R})$. From the definition of Favard class

above, it is evident that $f \in \widehat{\mathcal{D}}(A)$ if and only if $f \in L^1BV(\mathbb{R})$. It is also known (see [4]) that for the translation group on $L^1(\mathbb{R})$, its Favard class is

$$\widehat{\mathcal{D}}(A) = BV(\mathbb{R}).$$

So from this point of view, again we have

$$L^1BV(\mathbb{R}) = \widehat{\mathcal{D}}(A) = BV(\mathbb{R}).$$

Suppose e is a vector in \mathbb{R}^n with length one. Consider the translation group in the direction of e , acting on $L^1(\mathbb{R}^n)$, as

$$(T_t^e f)(x) = f(x + te)$$

for $x \in \mathbb{R}^n$, $f \in L^1(\mathbb{R}^n)$, and $t \in \mathbb{R}$. By the isometry $L^1(\mathbb{R}^n) = L^1(\mathbb{R}, L^1(\mathbb{R}^{n-1}))$, we may view the operator T_t^e as Banach space-valued translation in the direction of e . This group is generated by the directional derivative operator $A_e = \frac{\partial}{\partial e} = \nabla \cdot e$, with a suitable domain. In this case, it is known that the Favard class is

$$\widehat{\mathcal{D}}(A_e) = BV(\mathbb{R}, L^1(\mathbb{R}^{n-1}))$$

(see [11]). However, for $L^1BV(\mathbb{R}^n)$ we need translation in every direction. We have this immediate proposition.

Proposition 5.1. *Suppose $f \in L^1(\mathbb{R}^n)$. Then $f \in L^1BV(\mathbb{R}^n)$ if and only if:*

1. $f \in \widehat{\mathcal{D}}(A_e) = BV(\mathbb{R}, L^1(\mathbb{R}^{n-1}))$ for any $e \in \mathbb{R}^n$ with $\|e\| = 1$,
2. $\sup_{\|e\|=1} \sup_{t \neq 0} \frac{1}{|t|} \|T_t^e f - f\|_1 < \infty$.

It turns out that we need only to consider translations in the canonical directions. This result is parallel to Theorem 3.2. Again, suppose $\{e_1, \dots, e_n\}$ is the standard orthonormal basis for \mathbb{R}^n .

Theorem 5.2. *Suppose $f \in L^1(\mathbb{R}^n)$. Then $f \in L^1BV(\mathbb{R}^n)$ if and only if*

$$f \in \bigcap_{k=1}^n \widehat{\mathcal{D}}(A_{e_k}).$$

Proof. If $f \in L^1BV(\mathbb{R}^n)$, then it is clear that f is in the Favard class $\widehat{\mathcal{D}}(A_{e_k})$ for any canonical direction e_1, \dots, e_n . We will show the converse. For notational simplicity, without loss of generality we let $n = 2$. Let f be in the intersection. Suppose for $0 < t \leq 1$,

$$\|A_{te_k} f - f\|_1 \leq M_k |t|,$$

where $k = 1, 2$. Let h be a nonzero vector in \mathbb{R}^2 , and without loss of generality we let $\|h\| \leq 1$. Suppose $h = h_1 e_1 + h_2 e_2$ for some numbers h_1 and h_2 . Then

$$\begin{aligned} \|fh - f\|_1 &\leq \|f_{h_1 e_1 + h_2 e_2} - f_{h_1 e_1}\|_1 + \|f_{h_1 e_1} - f\|_1 \\ &\leq M_2 |h_2| + M_1 |h_1| \\ &\leq (M_2 + M_1) |h|. \end{aligned}$$

Taking the supremum over nonzero h , we get $f \in L^1BV(\mathbb{R}^2)$. \square

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