

VIETORIS-BEGLE THEOREM FOR SPECTRAL PRO-HOMOLOGY

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ABSTRACT. Dydak and Kozłowski (1991) obtained a generalization of the Vietoris-Begle theorem for the cohomology theories induced by CW spectra. In this note we prove a dual of their theorem involving the pro-homology theories induced by CW spectra.

Given any CW spectrum E , let E^* and E_* denote the cohomology and homology theories induced by E on the homotopy category \mathbf{HCW}_{spec} of CW spectra. So E^* and E_* are contravariant and covariant functors from \mathbf{HCW}_{spec} to the category \mathbf{Ab}_* of graded abelian groups. These functors extend to the functors $\text{pro-}E^*$ and $\text{pro-}E_*$ from the homotopy category \mathbf{HTop} of pointed topological spaces to the pro-category $\mathbf{pro-Ab}_*$ of graded abelian groups. Let \check{E}^* denote the Čech extension of E^* over \mathbf{HTop} .

Throughout the paper, every topological space X is assumed to be unpointed and is regarded as the pointed space X^+ with a discrete base point.

First, recall the following generalization of the Vietoris-Begle theorem:

Theorem 1 (Dydak and Kozłowski [D-K]). *Let E be a CW spectrum, and let $f : X \rightarrow Y$ be a closed surjective map between paracompact Hausdorff spaces such that $\text{Ind } Y = m < \infty$. If $\check{E}^k(f_y) : \check{E}^k(y) \rightarrow \check{E}^k(f^{-1}(y))$ is an isomorphism for all $y \in Y$ and $k = m_0, m_0 + 1, \dots, m_0 + m$, then $\check{E}^k(f) : \check{E}^k(Y) \rightarrow \check{E}^k(X)$ is an isomorphism for $k = m_0 + m$ and a monomorphism for $k = m_0 + m + 1$.*

Here $\text{Ind } Y$ means the large inductive dimension of Y , and f_y is the restricted map $f|_{f^{-1}(y)} : f^{-1}(y) \rightarrow \{y\}$.

In this note we prove the following dual form:

Theorem A. *Let E be a ring spectrum, and let $f : X \rightarrow Y$ be a surjective map from a compact metric space X to a compact metric space Y with a finite covering dimension such that for each $y \in Y$, $f^{-1}(y)$ has a finite stable shape dimension. If $\text{pro-}E_*(f_y) : \text{pro-}E_*(f^{-1}(y)) \rightarrow \text{pro-}E_*(y)$ is an isomorphism for each $y \in Y$, then the induced morphism $\text{pro-}E_*(f) : \text{pro-}E_*(X) \rightarrow \text{pro-}E_*(Y)$ is an isomorphism.*

Let Σ denote the obvious functor from the homotopy category \mathbf{HCW} of CW complexes to \mathbf{HCW}_{spec} which assigns to each CW complex X the suspension spectrum $\Sigma(X)$. For each topological space X , the *stable shape dimension* $\text{sd}_{spec} X \leq n$

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provided whenever $\mathbf{p} = (p_\lambda) : X \rightarrow \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is an **HCW**-expansion of X , for each $\lambda \in \Lambda$ there exists $\lambda' \geq \lambda$ such that $\Sigma(p_{\lambda\lambda'})$ factors in \mathbf{HCW}_{spec} through a CW spectrum Z with a dimension at most n , where an expansion is in the sense of [Ma-S]. Note that $\text{sd}_{spec} X \leq \text{sd} X$ where $\text{sd} X$ is the shape dimension of X (see [Ma-S]) and there is a strict inequality case (see [Mi-S]).

The main tools are the duality between compact metric spaces and CW spectra in generalized stable shape theory and the Whitehead theorem in the generalized stable shape category, which we quote below:

Lemma 2 (Miyata [Mi, Theorem 5.3]). *For each compact metric space X , there exist a CW spectrum X^* and a natural isomorphism*

$$\check{E}^n(X) \cong E_{-n}(X^*).$$

Lemma 3 (Miyata and Segal [Mi-S, Lemma 4.9]). *Let $n, k \in \mathbb{Z}$ with $k \leq n$, and let $(f_\lambda) : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \rightarrow \mathbf{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ be a level morphism of inverse systems in \mathbf{HCW}_{spec} , which is an n -equivalence. Then every $\lambda \in \Lambda$ admits $\lambda' \geq \lambda$ such that the following two statements hold:*

- i) *if Z is a CW spectrum with a dimension at most n , then every morphism $h : Z \rightarrow Y_{\lambda'}$ in \mathbf{HCW}_{spec} admits a morphism $k : Z \rightarrow X_\lambda$ such that $f_{\lambda k} = q_{\lambda\lambda'} h$;*
- ii) *if Z is a CW spectrum with a dimension at most $n - 1$ and $k_1, k_2 : Z \rightarrow X_{\lambda'}$ are morphisms in \mathbf{HCW}_{spec} such that $f_{\lambda' k_1} = f_{\lambda' k_2}$, then $q_{\lambda\lambda'} k_1 = q_{\lambda\lambda'} k_2$.*

Before we prove the theorem, we prove the following lemmas:

Lemma 4. *Let E be a ring spectrum, and let X be a topological space such that $\text{sd}_{spec} X \leq n$ for some $n < \infty$. Then, if $\text{pro-}E_*(X)$ is trivial, $\check{E}^*(X)$ is trivial.*

Proof. Let the ring spectrum E be equipped with the maps $i : S \rightarrow E$ and $\mu : E \wedge E \rightarrow E$, where S is the sphere spectrum. Suppose $\text{sd}_{spec} X \leq n < \infty$, and let $\mathbf{p} = (p_\lambda) : X \rightarrow \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an **HCW**-expansion of X . Suppose $\text{pro-}E_*(X) \approx 0$. Then $0 \approx E_*(\Sigma(\mathbf{X})) = \pi_*(\Sigma(\mathbf{X}) \wedge E) = (\pi_*(\Sigma(X_\lambda) \wedge E), \pi_*(\Sigma(p_{\lambda\lambda'}) \wedge 1_E), \Lambda)$. Let $\lambda \in \Lambda$, and take $\lambda' \geq \lambda$ as in Lemma 3. Since $\text{sd}_{spec} X \leq n$, there exist $\lambda'' \geq \lambda'$, a CW spectrum Z with a dimension at most n and morphisms $g : \Sigma(X_{\lambda''}) \rightarrow Z$ and $h : Z \rightarrow \Sigma(X_{\lambda'})$ in \mathbf{HCW}_{spec} such that $\Sigma(p_{\lambda'\lambda''}) = hg$. Then the composite

$$\Sigma(X_{\lambda''}) \xrightarrow{\Sigma(p_{\lambda'\lambda''})} \Sigma(X_{\lambda'}) \approx \Sigma(X_{\lambda'}) \wedge S \xrightarrow{1_{\Sigma(X_{\lambda'})} \wedge i} \Sigma(X_{\lambda'}) \wedge E \xrightarrow{\Sigma(p_{\lambda\lambda'}) \wedge 1_E} \Sigma(X_\lambda) \wedge E$$

is trivial. For each element $\varphi_\lambda : \Sigma(X_\lambda) \rightarrow E$ of $E^*(\Sigma(X_\lambda))$, there is a commutative diagram:

$$\begin{array}{ccccc} \Sigma(X_{\lambda''}) \wedge E & \xrightarrow{\Sigma(p_{\lambda'\lambda''}) \wedge 1_E} & \Sigma(X_{\lambda'}) \wedge E & \xrightarrow{\Sigma(p_{\lambda\lambda'}) \wedge 1_E} & \\ \uparrow 1_{\Sigma(X_{\lambda''})} \wedge i & & \uparrow 1_{\Sigma(X_{\lambda'})} \wedge i & & \\ \Sigma(X_{\lambda''}) & \xrightarrow{\Sigma(p_{\lambda'\lambda''})} & \Sigma(X_{\lambda'}) & \xrightarrow{\Sigma(p_{\lambda\lambda'})} & \\ & & \Sigma(X_\lambda) \wedge E & \xrightarrow{\varphi_\lambda \wedge 1_E} & E \wedge E \\ & & \uparrow 1_{\Sigma(X_\lambda)} \wedge i & & \downarrow \mu \\ & & \Sigma(X_\lambda) & \xrightarrow{\varphi_\lambda} & E \end{array}$$

This implies that $E^*(\Sigma(p_{\lambda\lambda'}))(\varphi_\lambda) = 0$. Hence $E^*(\Sigma(p_{\lambda\lambda'})) = 0$, which implies $\check{E}^*(X) \approx 0$. \square

Lemma 5. *Let E be a ring spectrum, and let X be a compact metric space. Then, if $\check{E}^*(X)$ is trivial, $\text{pro-}E_*(X)$ is trivial.*

Proof. Suppose $\check{E}^*(X) \approx 0$. Since $\check{E}^*(X) \approx E_{-*}(X^*)$ for some CW spectrum X^* by Lemma 4, then $0 \approx E_{-*}(X^*) \approx \pi_{-*}(X^* \wedge E)$, which means that $X^* \wedge E$ is contractible. Let $\mathbf{p} = (p_n) : X \rightarrow \mathbf{X} = (X_n, p_{nn+1}, \mathbb{N})$ be an **HCW**-expansion of X such that each X_n is a finite CW complex. Then this induces an **HCW**_{spec}-coexpansion $\mathbf{p}^* = (p_n^*) : \mathbf{X}^* = (X_n^*, p_{nn+1}^*, \mathbb{N}) \rightarrow X^*$ of X^* in the sense of [Mi], where each X_n^* is the Spanier-Whitehead dual of $\Sigma(X_n)$. So for each n , there is $m \geq n$ such that the composite

$$X_n^* \approx X_n^* \wedge S \xrightarrow{1_{X_n^*} \wedge i} X_n^* \wedge E \xrightarrow{p_{nm}^* \wedge 1_E} X_m^* \wedge E$$

is trivial. For each element $\varphi_m : X_m^* \rightarrow E$ of $E^{-*}(X_m^*)$, there is a commutative diagram:

$$\begin{array}{ccccc} X_n^* \wedge E & \xrightarrow{p_{nm}^* \wedge 1_E} & X_m^* \wedge E & \xrightarrow{\varphi_m \wedge 1_E} & E \wedge E \\ \uparrow 1_{X_n^*} \wedge i & & \uparrow 1_{X_m^*} \wedge i & & \downarrow \mu \\ X_n^* & \xrightarrow{p_{nm}^*} & X_m^* & \xrightarrow{\varphi_m} & E \end{array}$$

This implies that $E^{-*}(p_{nm}^*)(\varphi_m) = 0$, and hence $E^{-*}(p_{nm}^*) = 0$. Since $E_*(\Sigma(\mathbf{X})) \approx E^{-*}(\mathbf{X}^*) = (E^{-*}(X_n^*), E^{-*}(p_{nn'}^*), \mathbb{N})$, then we have $\text{pro-}E_*(X) \approx 0$. \square

Lemma 6. *Let E be a ring spectrum, and let $f : X \rightarrow Y$ be a map of topological spaces. Then:*

- i) $\check{E}^*(f) : \check{E}^*(Y) \rightarrow \check{E}^*(X)$ is an isomorphism iff $\check{E}^*(Y \cup_f CX)$ is trivial;
- ii) $\text{pro-}E_*(f) : \text{pro-}E_*(X) \rightarrow \text{pro-}E_*(Y)$ is an isomorphism iff $\text{pro-}E_*(Y \cup_f CX)$ is trivial.

Proof of Theorem A. Suppose $\text{pro-}E_*(f_y)$ is an isomorphism for each $y \in Y$. Then Lemmas 6 ii), 4 and 6 i) imply $\check{E}^*(f_y) : \check{E}^*(y) \rightarrow \check{E}^*(f^{-1}(y))$ is an isomorphism for each $y \in Y$. Since $\text{Ind } Y < \infty$, the Dydak and Kozłowski theorem implies that $\check{E}^*(f) : \check{E}^*(Y) \rightarrow \check{E}^*(X)$ is an isomorphism. This fact, together with Lemmas 6 i), 5 and 6 ii), implies that $\text{pro-}E_*(f) : \text{pro-}E_*(X) \rightarrow \text{pro-}E_*(Y)$ is an isomorphism.

A surjective map $f : X \rightarrow Y$ between compact metric spaces is said to be *stably cell-like* if each fibre has a trivial stable shape type (see [Mi-S]). Theorem A with the spectrum E being the sphere spectrum and [Mi-S, Theorem 6.1] imply

Corollary 7. *Let $f : X \rightarrow Y$ be a stably cell-like map from a compact metric space X with a finite stable shape dimension to a compact metric space Y with a finite covering dimension. Then f is an equivalence in the stable shape category.*

Remark. The “figure eight like continuum” X of Case and Chamberlin [C-C] has a trivial stable shape type but has a nontrivial shape type ([Ma]). For any compact metric space Y , the natural projection $f : X \times Y \rightarrow Y$ is stably cell-like but not cell-like. If Y is finite dimensional, Corollary 7 then implies that f is an equivalence in the stable shape category. The finite dimensionality of Y in Theorem A is essential as pointed out in [D-K] by the cell-like map of Keesling [K] $g : Q \rightarrow Y$ onto

an infinite dimensional compact metric space Y such that $\tilde{K}^*(Y) \not\approx 0$, where \tilde{K}^* denotes the reduced K-theory.

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