PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 2, Pages 591–594
S 0002-9939(01)06073-7
Article electronically published on July 25, 2001

ON PERFECTLY MEAGER SETS IN THE TRANSITIVE SENSE

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(Communicated by Carl G. Jockush, Jr.)

Abstract. We prove that assuming $\mathfrak{c} \leq \aleph_2$ one can always find a perfectly meager set, which is not perfectly meager in the transitive sense.

In the paper [NSW] it was shown that the algebraic sum of a strongly meager set and a set of strong measure zero has to be an s_0-set. Going over the proof of this fact one can easily see that it is based on the following property of strongly meager sets.

Definition 1. A set $X \subseteq 2^\omega$ is said to be an AFC' set (or perfectly meager in the transitive sense) if for any perfect $P \subseteq 2^\omega$, there is F, an F_σ set containing X, such that for each $t \in 2^\omega$, $(F+t) \cap P$ is meager in the relative topology of P.

In [N] and [NSW] it was proven that many other well-known special subsets of the reals like γ-sets or wQN-sets are perfectly meager in the transitive sense. The results appearing in those papers show that one can deduce from ZFC alone the existence of an uncountable AFC' set. On the other hand, it is relatively consistent with ZFC that not every perfectly meager set has to be an AFC' set (see also [NW1]). Thus, it was natural to ask if the class AFC of perfectly meager sets can be equal to the class AFC'.

In this paper we prove that the answer is negative if we let $\mathfrak{c} \leq \aleph_2$. We obtain it by showing that if one assumes $\mathfrak{c} \leq \aleph_2$, then AFC' is strictly included in $\overline{AFC}$, where $\overline{AFC}$ denotes some subclass of AFC defined below. Most of the arguments needed to show the latter fact can be found in [R] and [NSW]. Throughout the paper a set of real numbers is identified with a subset of the Cantor set 2^ω. By "+" we denote the usual modulo 2 coordinatewise addition in 2^ω and for $A, B \subseteq 2^\omega$, $A + B = \{a + b : a \in A, b \in B\}$. We assume that the reader is familiar with standard definitions and terminology of special sets of real numbers.

Definition 2. A set $X \subseteq 2^\omega$ belongs to the class $\overline{AFC}$ (of universally meager sets) iff for every $Y \subseteq 2^\omega$ for which there exists a one–to–one Borel measurable function $f : Y \to X$, we have that $Y \in MGR$ (meager sets).

Theorem 1 (Folklore)**.** *$\overline{AFC} \subseteq AFC$.*

Proof. Suppose that $X \in \overline{AFC}$. Let P be a given perfect set and let $h : P \xrightarrow{\text{onto}} 2^\omega$ be a homeomorphism. Clearly, if $X \cap P$ is non-meager in the relative topology of P, then $h[X \cap P]$ is non-meager in 2^ω, but this contradicts the fact that $X \in \overline{AFC}$. $\square$

Received by the editors December 13, 1999 and, in revised form, June 27, 2000.
2000 *Mathematics Subject Classification.* Primary 03E15, 03E20, 28E15.

Theorem 2. *$AFC' \subseteq \overline{AFC}$.*

Proof. See Theorem 2 in [NW1]. $\square$

Theorem 3. *Suppose that there is a universally meager set of cardinality $\mathfrak{c}$. Then $\overline{AFC} \neq AFC'$.*

Proof. Let C, D be disjoint, perfect subsets of 2^ω with the following property (see [NW1]):

$$(+) \qquad (C+C) \cap (D+D) = \{0\}.$$

Choose $X \in \overline{AFC}$ with $|X| = \mathfrak{c}$. Let $f : 2^\omega \overset{\text{onto}}{\longrightarrow} C$ be a homeomorphism and put $Y = f[X]$. Obviously, Y has to be an $\overline{AFC}$ set. From now on we follow I. Reclaw's argument from [R]. Let $\{B_y\}_{y \in Y}$ be an enumeration of all F_σ subsets of 2^ω. For $y \in Y$, take any $z_y \in D + y$ with $z_y \notin B_y$. If this is impossible, let z_y be any element of $D + y$. It is not hard to see (use (+)) that $Z = \{z_y : y \in Y\}$ belongs to $\overline{AFC}$ as a continuous one–to–one inverse image of Y. By the construction, if $Z \subseteq B_y$ for some $y \in Y$, then $D \subseteq B_y + y$. Thus Z is not an AFC' set. $\square$

Assume that G is a family of subsets of 2^ω. For $X \subseteq 2^\omega$, we will say that G is an ω-cover of X iff for any finite set $X' \subseteq X$ there exists $g \in G$, so that $X' \subseteq g$. Let us also recall that by $\mathfrak{b}$ we denote the $\min\{|B| : B$ is an unbounded subset of ω^ω in the quasi-order $\leq^*\}$.

Lemma 1. *Suppose that $\mathfrak{b} = \aleph_1$. Then there is $X \subseteq 2^\omega$, $|X| = \aleph_1$, such that for any sequence $\{G_n\}_{n\in\omega}$ of open ω–covers of X, one can find $A \in [\omega]^\omega$, $\{g_n\}_{n \in A}$ with every $g_n \in G_n$ and a countable $Y \subseteq X$ satisfying $X \setminus Y \subseteq \bigcup_{k\in\omega} \bigcap_{n \geq k} g_n$.*

Proof. See Theorem 5.1 in [JMSS]. $\square$

We call any set with the latter property an $S_1^*(\Omega, \Gamma)$ set and if we assume in the definition of an $X \in S_1^*(\Omega, \Gamma)$ that $A = \omega$ and $Y = \emptyset$, then X is said to be a γ-set.

In the next lemma we show that any $X \in S_1^*(\Omega, \Gamma)$ is an add (meager) – small set, that is, for every sequence $\{G_n\}_{n\in\omega}$ of open covers of X, there exist $\{g_n\}_{n\in\omega}$ with every $g_n \in G_n$ and an increasing function $f \in \omega^\omega$ such that each $x \in X$ belongs to all but finitely many sets of the form $\bigcup_{f(n)\leq j<f(n+1)} g_j$ ([NSW]).

Lemma 2. *$S_1^*(\Omega, \Gamma) \subseteq$ add (meager) – small sets.*

Proof. Suppose $X \in S_1^*(\Omega, \Gamma)$ and let $\{G_n\}_{n\in\omega}$ be a sequence of open covers of X. Assume that $\{A_n\}_{n\in\omega}$ is an infinite partition of ω into infinite subsets. For $n \in \omega$, we define an ω–cover of X in the following way:

$$F_n = \{g_{k_1} \cup \ldots \cup g_{k_r} : k_i \in A_n,\ g_{k_i} \in G_{k_i} \text{ and } k_i < k_{i+1} \quad \text{for} \quad 1 \leq i \leq r\}.$$

Suppose that

$$X \setminus Y \subseteq \bigcup_{k\in\omega} \bigcap_{n\geq k} g'_n,$$

where $g'_n \in F_n$ for $n \in A$, $A \in [\omega]^\omega$ and Y is a countable subset of X. Notice that by taking an appropriate subsequence we may assume that for every $n \in A$:

(1) if $g'_n = g_{k_1} \cup \ldots \cup g_{k_r}$ and $g'_{n+1} = g_{m_1} \cup \ldots \cup g_{m_t}$, then $k_r < m_1$,
(2) there is $h_n = g_{l_1} \cup \ldots \cup g_{l_s}$ with $k_r < l_1$ and $l_s < m_1$ such that $\{y_i\}_{0\leq i\leq n} \subseteq h_n$, where $\{y_i\}_{i\in\omega}$ is an enumeration of a set Y.

Clearly, if we put $g''_n = g'_n \cup h_n$, then

$$X \subseteq \bigcup_{k\in\omega} \bigcap_{n\geq k} g''_n.$$

□

Lemma 3. *If X is an add (meager) – small set and $Y \in AFC'$, then $X + Y \in AFC'$.*

Proof. This is Theorem 24 in [NSW]. □

Theorem 4. *Assume that $\mathfrak{c} \leq \aleph_2$. Then $\overline{AFC} \neq AFC'$.*

Proof. Let us consider two cases:

1. Suppose that $\mathfrak{b} = \aleph_2$. Then there exists a set

$$X = \{f_\alpha : \alpha \in \aleph_2\} \subseteq \omega^{\omega\uparrow} \subseteq 2^\omega$$

such that

(a) $f_\alpha <^* f_\beta$ for $\alpha < \beta$,
(b) X is unbounded in the quasi - order $\leq^*$.

It is easy to show (see [vD]) that X is universally meager. Hence, by Theorem 3, $\overline{AFC} \neq AFC'$.

2. Let $\mathfrak{b} = \aleph_1$. Suppose that every $X \subseteq 2^\omega$ of cardinality $\aleph_1$ is meager. This implies (see Theorem 1 in [G]) that there exists an $\overline{AFC}$ set of cardinality $\mathfrak{c}$. Thus, by Theorem 3, $\overline{AFC} \neq AFC'$. So, assume that there is a non – meager set X with $|X| = \aleph_1$. Let C, D be disjoint, perfect subsets of 2^ω that satisfy condition (+) from the proof of Theorem 3. Suppose that $f : 2^\omega \xrightarrow{\text{onto}} D$ is a homeomorphism. Put $Y = f[X]$. Notice that $Y \notin \overline{AFC}$. Let $Z \in S_1^*(\Omega, \Gamma)$ and $|Z| = \aleph_1$. We may suppose without loss of generality that $Z \subseteq C$. Define $Z' = \{z + y_z : z \in Z\}$, where $\{y_z\}_{z\in Z}$ is an enumeration of Y. We have that $Z + Z' \supseteq Y$, thus by Lemma 3, $Z' \in \overline{AFC} \setminus AFC'$. □

Applying Theorem 1, we immediately get the main result:

Theorem 5. *Let $\mathfrak{c} \leq \aleph_2$. Then $AFC \neq AFC'$.*

To conclude the paper let us mention that by the above argument we obtain a very simple proof of the following theorem due to A. Nowik, which gives a negative answer to M. Scheeper's question (see problem 3 in [S] and [NW2] for more details).

Theorem 6 (Nowik)**.** *It is consistent with ZFC that there are a strongly measure zero set X and a perfectly meager set Y such that $X + Y$ is not an s_0-set.*

Proof. Assume that $\mathfrak{c} = \aleph_1$ holds. It is well known that there exists a γ-set X of cardinality $\mathfrak{c}$ (see [GM]). Clearly, X is strongly measure zero. Let C, D be disjoint perfect sets as in the proof of Theorem 3. Without loss of generality we may assume that $X \subseteq C$. Suppose that $\{y_x\}_{x\in X}$ is an enumeration of a set D. Define $Y = \{x + y_x : x \in X\}$. Obviously, $Y \in \overline{AFC}$ and we have that $X + Y \supseteq D$. □

Remark. In contrast with the main theorem a parallel fact for the class $\overline{AFC}$ can not be decided by ZFC. This follows from a recent paper by T. Bartoszyński (see [B]) who showed that in Miller's model we have $\mathfrak{c} = \aleph_2$ and $AFC = \overline{AFC}$.

Finally, let us define the cardinal κ to be equal to the least λ such that there are no perfectly meager sets of cardinality λ. Clearly, if either $\kappa = \aleph_2$ or $\mathfrak{c}^+ = \kappa$, then we can argue as before to show that $AFC \neq AFC'$. It seems that for the other cases different methods have to be developed if we want to show that $AFC \neq AFC'$. Thus we end with the following question.

Problem. Suppose that $\mathfrak{c} \geq \aleph_3$. Is it possible to prove on the basis of ZFC that $AFC \neq AFC'$?

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