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ON THE STRONG MAXIMUM PRINCIPLE

ARRIGO CELLINA

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ABSTRACT. This paper presents a necessary and sufficient condition on the convex function f in order that continuous solutions to

$$\text{minimize} \int_\Omega f(\|\nabla u(x)\|)\, dx \text{ on } u^0 + W_0^{1,1}(\Omega)$$

satisfy a Strong Maximum Principle on any open connected Ω.

INTRODUCTION

In the present paper we consider properties of solutions to Problem (P):

$$\text{(P)} \qquad \text{minimize} \int_\Omega f(\|\nabla u(x)\|)\, dx \text{ on } u^0 + W_0^{1,1}(\Omega).$$

Our Theorem 2 below presents a necessary and sufficient condition on the convex function f in order that continuous solutions to (P) satisfy a Strong Maximum Principle on any open connected Ω. Theorem 1 (instrumental to the Proof of the main result) is instead a Comparison Principle, or a weak Maximum Principle, for solutions to (P).

MAIN RESULTS

In what follows we shall deal with an extended valued, lower semicontinuous, rotationally invariant, convex integrand $F(\xi) = f(\|\xi\|)$ such that $F(0) = 0$. By a *solution* to Problem (P) we shall always mean a function giving a *finite* value to the integral $\int_\Omega f(\|\nabla u(x)\|)\, dx$. We shall say that the integrand f has the *Strong Maximum Principle Property* if for any open connected Ω, for every *continuous* non-negative solution to Problem (P), $u(x^0) = 0$ for some $x^0 \in \Omega$ implies $u(x) \equiv 0$ on Ω. In this formulation, it is actually a Strong Minimum Principle.

It will simplify our notations to assume that f is extended as 0 for negative values of t. In the case $f(t) = 0$ for $t = 0, = +\infty$ for $t > 0$, any function u giving a finite value to the above integral must be such that a.e. $\nabla u(x) = 0$. In this case the maximum principles are trivially satisfied, on a connected Ω, and we will not consider this case further. The subdifferential of f, ∂f, is a maximal monotone map. Since we have excluded that $\partial f(0) = [0, +\infty)$, $\partial f(t)$ must be defined for t in

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some interval $[0,\tau)$, otherwise we would contradict maximality. We will call *face* a convex extremal subset of $epi(F)$. We shall say that an open set Ω is *regular* if it has the *segment property* [2], [4], i.e. it is such that, for every $x^0 \in \partial\Omega$, there exists a neighborhood U^0 of x^0 and a non-zero vector k such that $x + tk \in \Omega$ whenever $x \in \overline{\Omega} \cap U^0$ and $t \in (0,1)$. We shall say that $u^0(x) \geq w(x)$ (or, equivalently, that $u^0(x) - w(x) \geq 0$) for $x \in \partial\Omega$ in the sense of $W^{1,1}(\Omega)$ if $(u^0(x) - w(x))^- \in W_0^{1,1}(\Omega)$. The Euclidean norm is $\|.\|$ and the scalar product $\langle .,. \rangle$. We shall denote by B_1 the ball $\{x : \|x\| \leq 1\}$ and by $A_{\alpha,\beta}(x^0)$ the annulus $\alpha < \|x - x_0\| < \beta$ for $0 < \alpha < \beta$. Comparison Theorems such as Theorem 1 have been considered by the author in [1], for an integrand of the form $f(p)$, and by Mariconda and Treu, in [3], in the more general setting of an integrand of the form $f(x,u,p)$.

Theorem 1. *Let Ω in $\mathbb{R}^N$ be regular; let α and β be positive constants such that $\Omega \subset A_{\alpha,\beta}(x^0)$. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a (possibly extended valued) lower semicontinuous, convex function, such that $f(0) = 0$ and $f(t) > 0$ for $t > 0$. Let $u^0(x)$ be in $W^{1,1}(\Omega)$, and let w be a solution to problem* (P) *yielding a finite value for $\int_\Omega f(\|\nabla w(x)\|)\, dx$. Let H be a non-negative real and $r \to R_H(r)$, r in $[\alpha,\beta]$, be differentiable and such that $H/r^{N-1} \in \partial f(|\frac{d}{dr} R_H(r)|)$.*

Assume that for x in $\partial\Omega$, we have $u^0(x) \geq R_H(\|x - x^0\|)$ in the sense of $W^{1,1}(\Omega)$. Then, a.e. in Ω, $w(x) \geq R_H(\|x - x^0\|)$.

Notice that when $H = 0$ and Ω is bounded, Ω is contained in some $A_{\alpha,\beta}(x^0)$ and we obtain the same result as in [1], applied to the case of rotationally invariant integrands f. Since the case $H = 0$ is considered in [1], we shall assume $H > 0$.

Proof. The Theorem is a special case of Theorem 3.14 of [3]. Its proof is also a minor modification of the Proof of Theorem 1 of [1] and it will be only sketched here.

Sketch of the Proof. It is convenient to set $S_H(x)$ to be $R_H(\|x - x^0\|)$.

a) Set $E^- = \{x \in \Omega : w(x) < S_H(x)\}$. We wish to prove that E^- has measure zero. Set η^- to be $\min\{w - S_H, 0\}$; then $\eta^- \in W_0^{1,1}(\Omega)$ and we have

$$(w - \eta^-)(x) = \begin{cases} w(x) & \text{if } w(x) \geq S_H(x), \\ S_H(x) & \text{otherwise,} \end{cases}$$

$$\nabla(w - \eta^-)(x) = \begin{cases} \nabla w(x) & \text{if } w(x) \geq S_H(x), \\ \nabla S_H(x) & \text{otherwise,} \end{cases}$$

so that

$$\int_\Omega [f(\|\nabla w(x)\|) - f(\|\nabla(w(x) - \eta^-(x))\|)]\, dx$$
$$= \int_{E^-} [f(\|\nabla w(x)\|) - f(\|\nabla S_H(x)\|)]\, dx.$$

From the convexity of f we have that

$$f(\|\nabla w(x)\|) \geq f(\|\nabla S_H(x)\|) + \langle p(x), \nabla w(x) - \nabla S_H(x) \rangle$$

for any selection p from $x \to \partial F(\nabla S_H(x))$, hence in particular for $p_H(x) = \frac{H}{\|x-x^0\|^{N-1}} \frac{(x-x^0)}{\|x-x^0\|}$. Therefore

$$\int_{E^-} [f(\|\nabla w(x)\|) - f(\|\nabla(w(x) - \eta^-(x))\|)]\, dx \geq \int_{E^-} \langle p_H(x), \nabla w(x) - \nabla S_H(x)\rangle\, dx.$$

b) We claim that

$$\int_{E^-} \langle p_H(x), \nabla w(x) - \nabla S_H(x)\rangle\, dx = 0.$$

Proof of this claim. As in [1], Theorem 1, let (ψ_n), $\psi_n \in C^\infty(\overline{\Omega})$ and $\psi_n(x) \leq 0$ for x in $\partial\Omega$ be a sequence converging to $(w - S_H)$ in $W^{1,1}(\Omega)$ (the assumption of regularity of Ω is used here). Set $w_n = S_H + \psi_n$ and set E_n to be $\{x \in \Omega : w_n(x) - S_H(x) < 0\} = \{x \in \Omega : \psi_n(x) < 0\}$. To prove the claim it is enough to show that $\int_{E_n} \langle p_H(x), \nabla w_n(x) - \nabla S_H(x)\rangle\, dx = 0$. Passing to polar coordinates $\{\omega, r\}$, the intersection of a half line $L_c = \{\omega = c, r \geq 0\}$ with the open set E_n can be described as $\{\omega = c; r \in \bigcup_i (\alpha_i(c), \beta_i(c))\}$ where some or all of the $\{\omega = c; \alpha_i(c)\}$ and of the $\{\omega = c; \beta_i(c)\}$ can belong to $\partial\Omega$. We have

$$\int_{E_n} \langle p_H(x), \nabla(w_n(x) - S_H(x))\rangle\, dx$$

$$= \int_{\|\omega\|=1} (\sum_i \int_{\alpha_i(c)}^{\beta_i(c)} \frac{H}{r^{N-1}} (\frac{d}{dr}(w_n(r,\omega) - v(r))) r^{N-1}\, dr\, d\omega.$$

For each i, when $\{\omega = c; \alpha_i(c)\}$ is in Ω, w_n and S_H coincide and the same is true for $\{\omega = c; \beta_i(c)\}$. At $\partial\Omega$, $w_n(x) \leq S_H(x)$). When $\{\omega = c; \alpha_i(c)\} \in \partial\Omega$, $\{\omega = c; \alpha_i(c)\}$ is the limit of points where $w_n(x) > S_H(x)$. The same remark applies at the points $\{\omega = c; \beta_i(c)\}$. Hence we obtain that the last integral is zero.

c) From a) and b) we have

$$\int_{E^-} [f(\|\nabla w(x)\|) - f(\|\nabla S_H(x)\|)]\, dx \geq 0.$$

On the other hand, since w is a solution,

$$\int_{E^-} [f(\|\nabla w(x)\|) - f(\|\nabla S_H(x)\|)]\, dx = \int_\Omega [f(\|\nabla w(x)\|) - f(\|\nabla(w(x) - \eta^-(x))\|)]\, dx \leq 0,$$

so that we infer

$$\int_{E^-} [f(\|\nabla w(x)\|) - f(\|\nabla S_H(x)\|)]\, dx = 0.$$

d) From the previous results we have

$$\int_{E^-} \{f(\|\nabla w(x)\|) - [f(\|\nabla S_H(x)\|) + \langle p_H(x), \nabla w(x) - \nabla S_H(x)\rangle]\}\, dx = 0$$

so that, being the integrand non-negative, we obtain that a.e. in E^-,

$$f(\|\nabla w(x)\|) = f(\|\nabla S_H(x)\|) + \langle p_H(x), \nabla w(x) - \nabla S_H(x)\rangle$$

and we recall that $p_H(x) \in \partial F(\nabla S_H(x))$. Hence, $\nabla w(x)$ belongs a.e., to the same face of the convex function $\xi \to f(\|\xi\|)$ that contains $\nabla S_H(x)$. These faces are either the extremal points or the one-dimensional faces corresponding to intervals where f is affine, i.e. where f' is constant. The function H/r^{N-1}, a selection from $\partial f(|\frac{d}{dr}R_H(r)|)$, is strictly monotonic, so it can coincide at most on a set Z of $\mathbb{R}$ of measure zero with the (at most countable) set of values of the derivative of f corresponding to intervals where f is affine. Hence, $\nabla w(x)$ and $\nabla S_H(x)$ can differ only when x belongs to the set of measure zero $\Omega \cap \{x : \|x - x^0\| \in Z\}$. Hence, $\nabla w(x)$ and $\nabla S_H(x)$ for a.e. x are such that $(\nabla w(x), f(\|\nabla w(x)\|))$ and $(\nabla S_H(x), f(\|\nabla S_H(x)\|))$ are the same extremal point, i.e. $\nabla w(x) = \nabla S_H(x)$ a.e. in E^-. Hence $\nabla \eta^- = 0$ a.e. in Ω and, since $\eta^- \in W_0^{1,1}(\Omega)$, E^- has measure zero.

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}^+$ be a (possibly extended valued) lower semicontinuous, convex function such that $f(0) = 0$. Then f has the Strong Maximum Principle Property if and only if both conditions* i) $\partial f(0) = \{0\}$ *and* ii) $(\partial f)^{-1}(0) = \{0\}$ *hold true.*

Proposition. *Let f be as in Theorem* 2; *then conditions* i) *and* ii) *are equivalent to:*

j) $\lim_{t\to 0^+} f(t)/t = 0$,

jj) $f(t) > 0$ *for* $t > 0$.

Proof. From the assumptions on f, $\partial f(0) \supset \{0\}$ and $(\partial f)^{-1}(0) \supset \{0\}$.

Assume j) and jj). By jj), $(\partial f)^{-1}(0)$ does not contain t for $t > 0$. By j), $\partial f(0) = \{0\}$.

Assume i) and ii). By convexity, $\lim_{t\to 0^+} f(t)/t$ exists. If it is strictly positive, $\partial f(0)$ does not reduce to $\{0\}$. If for some $t > 0$, $f(t) = 0$, then $(\partial f)^{-1}(0) \ni t$.

Proof of Theorem 2. *Necessity.* Assume that the Strong Maximum Principle property holds.

a) From the convexity of f we infer that $\lim_{t\to 0^+} f(t)/t$ exists. We claim that it cannot be that this limit ℓ is positive. Assume the contrary. Then, $\partial f(0) = [0, \ell]$ while for $F(\xi) = f(\|\xi\|)$ we have that $\partial F(0) = \ell B_1$ and $\partial F(\xi) = \partial f(\|\xi\|)\frac{\xi}{\|\xi\|}$ for $\|\xi\| \neq 0$. Certainly, it cannot be that the function f is $f(t) = \ell t$, for $t \geq 0$. In this case, in fact, we have that the map $v(x) = x_1\chi_{\{x_1 \geq 0\}}(x)$ is a (non-negative) solution to the problem of minimizing

$$\int_C \ell\|\nabla u(x)\|\, dx \text{ on: } v + W_0^{1,1}(C),$$

where C is the cube $\{x : |x_i| \leq 1\}$, and the Strong Maximum Principle is violated. To check that v is actually a solution, consider that the vector $p = (\ell, 0, ..., 0)$ belongs to the subdifferential of the function $\xi \to \ell\|\xi\|$ both at $\xi = 0$ and at $\xi = (1, 0, ..., 0)$, i.e. p is a (constant) selection from $x \to \partial(\ell\|\nabla v(x)\|)$, so that, for any $\eta \in W_0^{1,1}(C)$, one has

$$\int_C \ell\|\nabla v(x) + \nabla\eta(x)\|\, dx \geq \int_C (\ell\|\nabla v(x)\| + \langle p, \nabla\eta(x)\rangle)\, dx = \int_C \ell\|\nabla v(x)\|\, dx.$$

So the case $f(t) = \ell t$ $(t \geq 0)$ is excluded. This fact implies that the range of the subdifferential of f must contain the interval $[0, \ell + \varepsilon]$ for some positive ε.

Set α such that $1/(\alpha)^{N-1} = \ell + \varepsilon$; set r_1 such that $1/r_1^{N-1} = \ell$; set $\beta = 2r_1$. We have $\alpha < r_1 < \beta$. On $[\alpha, \beta]$, the map $1/r^{N-1}$ takes its values on $(0, \ell + \varepsilon]$

and $(\partial f)^{-1}(1/r^{N-1})$ is defined. Set $-d(r)$ to be a (measurable, non-negative) selection from $(\partial f)^{-1}(1/r^{N-1})$, so that $1/r^{N-1}$ is a selection from $\partial f(-d(r))$. Since $1/r^{N-1} < \ell$ for $r \in [r_1, \beta]$, we have that $-d(r) = 0$ on $[r_1, \beta]$. Since $1/r^{N-1} > \ell$ on $[\alpha, r_1)$, we have that $-d(r) > 0$ there.

Set $v(r)$ to be the non-increasing function

$$v(r) = 0 + \int_\beta^r d(\rho)\, d\rho.$$

Then $v(r) = 0$ for $r \in [r_1, \beta]$ and $v(\alpha) = \int_\alpha^{r_1} -d(r)\, dr > 0$. Moreover, $1/r^{N-1} \in \partial f(-v'(r)) = \partial f(|v'(r)|)$. Set A to be the annulus $A_{\alpha,\beta}(0)$ and $V(x) = v(\|x\|)$, so that $\nabla V(x) = v'(\|x\|)\frac{x}{\|x\|} = |v'(\|x\|)|\frac{-x}{\|x\|}$. Then $\frac{1}{\|x\|^{N-1}}\frac{-x}{\|x\|}$ is a selection from $\partial F(\nabla V(x)), x \in A$.

Consider the minimization problem

$$(P_A) \qquad \text{minimize} \int_A f(\|\nabla u(x)\|)\, dx \text{ on } u^0 + W_0^{1,1}(A),$$

where $u^0(x) = \int_\alpha^{r_1} -d(r)\, dr$ for $\|x\| = \alpha$ while $u^0(x) = 0$ for $\|x\| = \beta$. We claim that V is a solution to this minimum problem. In fact, let u be in $u^0 + W_0^{1,1}(A)$. From the convexity of f, for any selection p from $\partial F(\nabla V(x))$ (in particular, for $p(x) = \frac{1}{\|x\|^{N-1}}\frac{-x}{\|x\|}$), we have

$$\int_A f(\|\nabla u(x)\|)\, dx \geq \int_A (f(\|\nabla V(x)\|) + \langle p(x), \nabla(u(x) - V(x))\rangle)\, dx,$$

and, passing to polar coordinates,

$$\int_A \langle p(x), \nabla(u(x) - V(x))\rangle\, dx = \int_{\|\omega\|=1}\int_\alpha^\beta \frac{1}{r^{N-1}}(\frac{d}{dr}(v(r) - u(r,\omega)))r^{N-1}\, dr\, d\omega = 0.$$

This proves that V is a (continuous, non-negative) solution to the minimization problem (P_A). Since $V(x) = 0$ for $\|x\| \in [r_1, \beta]$ and $v(\alpha) > 0$, we have reached a contradiction to the validity of the strong Maximum Principle on A. Hence $\lim_{t\to 0^+} f(t)/t = 0$.

b) It cannot be that $f(t) = 0$ on some interval $[0, \lambda]$. If such a $\lambda > 0$ exists, set Ω to be an annulus $A_{\alpha,\beta}(0)$, set $v'_\lambda(r)$ to be $-\lambda\, \chi_{[\alpha,(\alpha+\beta)/2]}(r)$ on $A_{\alpha,\beta}(0)$, and $v_\lambda(r)$ to be $\lambda(\alpha+\beta)/2 + \int_\alpha^r v'_\lambda(s)\, ds$; consider the Problem (P_A) where $u^0(x) = 0$ for $\|x\| = \beta$ and $u^0(x) = \lambda(\alpha+\beta)/2$ for $\|x\| = \alpha$. Set $V(x) = v_\lambda(\|x\|)$. Then $f(\|\nabla V(x)\|) = 0$ a.e.; hence V is a continuous non-negative solution to (P_A), while $V(x) = 0$ for $\|x\| = (\alpha+\beta)/2$ and $V(x) > 0$ for $\|x\| = \alpha$.

Sufficiency. Let Ω be any connected open set. Let w be a continuous solution to (P), non-negative on Ω, and assume that both $E = \{x \in \Omega : w(x) = 0\}$ and $\Omega \setminus E$ are non-empty. Then, being $\Omega = E \cup (\Omega \setminus E)$ connected, there exists $x^* \in \Omega$ such that $w(x^*) = 0$ and x^* is the limit of points where w is positive. Let $\rho > 0$ be such that $B_{2\rho}(x^*) \subset \Omega$ and let x^0 be such that $w(x^0) > 0$ and $\|x^0 - x^*\| < \rho$. From the continuity of w, there exists $0 < \alpha < \|x^0 - x^*\|$ and $k^* > 0$ such that on $\{x : \|x - x^0\| = \alpha\}$, we have $w(x) \geq k^*$. Set $\beta = \rho$, so that $A_{\alpha,\beta}(x^0) \subset \Omega$, and set ε to be $k^*/(\beta - \alpha)$.

By assumption, $(\partial f)^{-1}(0) = \{0\}$. Since $z \to (\partial f)^{-1}(z)$ is upper semicontinuous at 0, there exists $\delta > 0$ such that ($(\partial f)^{-1}(z)$ is defined and) $(\partial f)^{-1}(z) \subset [0, \varepsilon]$ for $z \in [0, \delta]$. Choose $H > 0$ sufficiently small so that $0 < \frac{H}{r^{N-1}} < \delta$ for $r \in [\alpha, \beta]$. Hence $(\partial f)^{-1}(\frac{H}{r^{N-1}}) \in [0, \varepsilon]$.

By assumption, $\partial f(0) = \{0\}$. Since $\frac{H}{r^{N-1}} > 0$, we have $(\partial f)^{-1}(\frac{H}{r^{N-1}}) > 0$.

Consider the function $r \to R_H(r) = -\int_\beta^r (\partial f)^{-1}(\frac{H}{r^{N-1}})\, dr$. We have that $R_H(\beta) = 0$ and $0 < R_H(\alpha) \leq (\beta - \alpha)\varepsilon = k^*$. Hence, for x in $\partial A_{\alpha,\beta}(x^0)$, $w(x) \geq R_H(\|x - x^0\|)$. Moreover, $R_H(r)$ is strictly decreasing, so in particular $R_H(\|x^* - x^0\|) > R_H(\beta) = 0$. Apply Theorem 1 with $\Omega = A_{\alpha,\beta}(x^0)$ to infer that, for all x in $A_{\alpha,\beta}(x^0)$, $w(x) \geq R_H(\|x - x^0\|)$. In particular, $0 = w(x^*) \geq R_H(\|x^* - x^0\|) > 0$. Hence the assumption that both E and $\Omega \setminus E$ are non-empty has led to a contradiction.

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Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, Viale Sarca 202, 20126 Milano, Italy

E-mail address: cellina@matapp.unimib.it