ASYMPTOTIC PROPERTIES
OF THE VECTOR CARLESON EMBEDDING THEOREM

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Abstract. The dyadic Carleson embedding operator acting on $C^n$-valued functions has norm at least $C \log n$. Thus the Carleson Embedding Theorem fails for Hilbert space valued functions.

Let $T$ be the unit circle in $\mathbb{C}$, and $\{I\}_{I \in D}$ its collection of dyadic arcs. Let $w_I$ be nonnegative real numbers indexed by $I \in D$. For integrable functions $f$ on $T$, denote by $\langle f \rangle_I$ the average $|I|^{-1} \int_I f(y)dy$. The classical Carleson embedding theorem [1] is equivalent to the following dyadic result:

Theorem. If $\sum_{I \subset K} w_I \leq |K|$ for all $K \in D$, then $\sum_{I \in D} w_I \langle f \rangle_I^2 \leq C \|f\|^2$ for all $f \in L^2(T)$.

The converse is also true (up to the placement of constants) and is verified by considering functions of the form $f = \chi_J, J \in D$.

An analogous statement may be made for functions taking values in $\mathbb{C}^n$ with matrix-valued weights $W_I \geq 0$ in the sense of quadratic forms. We wish to consider the following $n$-dimensional embedding theorem:

Proposition. If $\|\sum_{I \subset K} W_I\| \leq |K|$ for all $K \in D$, then $\sum_{I \in D} (W_I \langle f \rangle_I, \langle f \rangle_I) \leq C_n \|f\|^2$ for all $f \in L^2(T; \mathbb{C}^n)$.

The space $\mathbb{C}^n$ here is viewed as a finite-dimensional Hilbert space. One might ask whether a similar result still holds when $f$ takes values in a general Hilbert space $\mathbb{H}$ and $W_I$ are positive selfadjoint operators. This is answered in the negative by [4], which proves that $C_n$ must be bounded from below by $c \log n$. In the current paper we will use the construction in [4] to verify the stronger bound $C_n \geq c(\log n)^2$, which is also proved in [5]. A precise statement is as follows:

Theorem 1. There exist a function $f \in L^2(T; \mathbb{C}^n)$ and matrix weights $W_I \geq 0$ such that $\|\sum_{I \subset K} W_I\| \leq |K|$ and $\sum_{I \in D} (W_I \langle f \rangle_I, \langle f \rangle_I) \geq c(\log n)^2 \|f\|^2$, where $c > 0$ is independent of $n$. 

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I to itself. Thus the first sum is less than A to finite-dimensional subspaces. It is well known [2] that the best possible C_n is bounded above by C(\log n)^2, making these results sharp up to a constant factor.

Proof of Theorem 1. Let e_0, e_1, ..., e_n be the standard basis for \mathbb{C}^{n+1}. Define the Rademacher functions r_j(e^{2\pi it}) = (-1)^{\lfloor 2^j t \rfloor}. For a dyadic interval I, |I| \leq 2^{-i}, r_j is seen to be constant along I. Its value throughout I will be called r_j(I).

Let f(x) = \sum_{j=0}^{n} r_j(x)e_j. Clearly \|f\|^2 = n + 1. The averages of f over dyadic intervals are also easy to compute. When \|I\| = 2^{-i}, i \leq n, \langle f \rangle_I = \sum_{j=0}^{i} r_j(I)e_j.

Let W_I, |I| \geq 2^{-n}, be the rank-one operator satisfying W_I v = |I|(v, \phi_I)\phi_I, where \phi_I = \sum_{j=0}^{i} \frac{1}{1 + 1 - j} r_j(I)e_j. Define \phi_I to be 0 when \|I\| < 2^{-n}. Already we can estimate the sum

\[ \sum_{I \in D} (W_I(f)_I, (f)_I) = \sum_{I \in D} |I|((f)_I, \phi_I)^2 = \sum_{i=0}^{n} \left( \sum_{j=0}^{i} \frac{1}{i + 1 - j} \right)^2 \geq cn(\log n)^2. \]

The only task remaining is to show that \|\sum_{I \subseteq K} W_I\| is controlled by |K|. We will prove the estimate \sum_{I \subseteq K} (W_I v, v) = \sum_{I \subseteq K} |I|(v, \phi_I)^2 \leq C|K||v|^2 for all v \in \mathbb{C}^{n+1}.

For each interval I with |I| = 2^{-i}, split the vector \phi_I into the sum of two parts, \phi_I = \sum_{j=0}^{k} \frac{1}{1 + 1 - j} r_j(K)e_j + \sum_{j=k+1}^{i} \frac{1}{1 + 1 - j} r_j(I)e_j. Denote the first sum, which depends only on the length of I \subseteq K, by g_i. Summing over all I with |I| = 2^{-i}, and exploiting the orthogonality of the Rademacher functions,

\[ \sum_{I \subseteq K \atop |I| = 2^{-i}} |I|(v, \phi_I)^2 = |K|\left(\sum_{i=k}^{n} (v, g_i)^2 + \sum_{j=k+1}^{n} \frac{1}{(i + 1 - j)^2} |v_j|^2 \right). \]

Thus

\[ \sum_{I \subseteq K} (W_I v, v) = |K|\left(\sum_{i=k}^{n} (v, g_i)^2 + \sum_{j=k+1}^{n} |v_j|^2 \sum_{i=j}^{n} \frac{1}{(i + 1 - j)^2} \right). \]

The second sum is less than C|K| \sum_{j=0}^{n} |v_j|^2 = C|K||v|^2. To estimate the first sum, let G represent the \((n - k + 1) \times (k + 1)\) matrix whose \(ij\)th entry is the coefficient of \(e_{i-1}\) in \(g_{i+k-1}\). Then \(\sum_{i=k}^{n} (v, g_i)^2 \leq \|G\|^2 |v|^2.\) Here \(|G|\) is taken as an operator from \(\mathbb{C}^{k+1}\) to \(\mathbb{C}^{n-k+1}\). Under a suitable permutation of indices, however, G is seen to be a restriction of the Hilbert matrix \(A\), \(A_{ij} = \frac{1}{i+j-1})\) to finite-dimensional subspaces. It is well known [2] that \(A\) is bounded from \(\ell^2(\mathbb{N})\) to itself. Thus the first sum is less than \(|K| \|A\|^2 |v|^2 = C|K||v|^2.\) Dividing all weights \(W_I\) by an appropriate constant proves the theorem.

References


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