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# AVERAGING DISTANCES IN FINITE DIMENSIONAL NORMED SPACES AND JOHN'S ELLIPSOID

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Abstract. A Banach space $X$ has the average distance property (ADP) if there exists a unique real number $r = r(X)$ such that for each positive integer $n$ and all $x_1, \dots, x_n$ in the unit sphere of $X$ there is some $x$ in the unit sphere of $X$ such that

$$\frac{1}{n}\sum_{k=1}^{n} \|x_k - x\| = r.$$

A theorem of Gross implies that every finite dimensional normed space has the average distance property. We show that, if $X$ has dimension $d$, then $r(X) \leq 2 - 1/d$. This is optimal and answers a question of Wolf (Arch. Math., 1994). The proof is based on properties of the John ellipsoid of maximal volume contained in the unit ball of $X$.

A rendezvous number of a metric space $(M, d)$ is a real number $r$ with the property that for each positive integer $n$ and $x_1, \dots, x_n \in M$ there exists $x \in M$ such that

$$\frac{1}{n}\sum_{k=1}^{n} d(x_k, x) = r.$$

A remarkable theorem of Gross [Gro] states that any compact connected metric space has a unique rendezvous number. A nice survey of this subject is given in [CMY].

A real Banach space $X$ of dimension at least 2 has the average distance property (ADP) if its unit sphere $S(X)$ has a unique rendezvous number $r(X)$. In the following all Banach spaces are real. Gross' theorem implies that finite dimensional normed spaces have the average distance property. It can be easily seen that in this case $r(X) < 2$. Since the functional $X \longrightarrow r(X)$ is continuous on the Minkowski compactum of normed spaces of fixed dimension $d \geq 2$, it follows that

$$r(d) = \sup\{r(X) \ : \ X \text{ is a } d\text{-dimensional normed space}\} < 2.$$

It was conjectured in [Wo1] that $r(d) = r(l_1^d) = 2 - 1/d$. This conjecture has been proved only in the case $d = 2$; see [Wo1]. That $r(X) \leq 2 - 1/d$ has also been shown if $X$ has a 1-unconditional basis (see [Wo2]), and if $X$ is isometrically

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isomorphic to a subspace of $L_1[0,1]$ (see [Wo3]). The general upper bound

$$r(d) \le 2 - \frac{1}{2+(d-1)2^{d+1}}$$

was shown in [BCP].

It is the purpose of this paper to verify Wolf's conjecture.

**Theorem 1.** *If $X$ is a $d$-dimensional Banach space with $d \ge 2$, then*

$$r(X) \le 2 - \frac{1}{d}.$$

Our proof relies on properties of the John ellipsoid. Let us gather what we need. Consider the $d$-dimensional Banach space $X$ as $\mathbb{R}^d$ equipped with a norm $\| \, . \, \|$. Let $B = \{x \in X \ : \ \|x\| \le 1\}$ and $S = \{x \in X \ : \ \|x\| = 1\}$ be the closed unit ball and the unit sphere of $X$, respectively. The John ellipsoid is the unique ellipsoid of maximal volume contained in $B$; see [Joh] and [TJ], ch. 3, for proofs of the properties cited below. By an affine transformation, we may assume that this ellipsoid is the standard euclidean ball $B_2^d = \{x \in X \ : \ |x| \le 1\}$. Here $|x| = \sqrt{(x,x)}$ is the standard euclidean norm and $(x,y)$ denotes the scalar product of $x, y \in \mathbb{R}^d$. Also let $S_2^d$ be the euclidean unit sphere in $\mathbb{R}^d$. Then there exist $m \ge d$ and contact points $u_1, \dots, u_m \in S \cap S_2^d$ and positive real numbers $c_1, \dots, c_m$ such that $|(x,u_i)| \le 1$ for $x \in B$ and $i = 1, \dots, m$ and

$$x = \sum_{i=1}^m c_i (x,u_i) u_i \quad \text{for} \quad x \in \mathbb{R}^d. \tag{1}$$

In particular, it follows by scalar multiplication with $x$ that

$$|x|^2 = \sum_{i=1}^m c_i (x,u_i)^2 \quad \text{for} \quad x \in \mathbb{R}^d \tag{2}$$

and, by adding equations (2) for an orthonormal basis and using $|u_i| = 1$, that

$$\sum_{i=1}^m c_i = d. \tag{3}$$

Moreover, $|(x,u_i)| \le 1$ for $x \in B$ together with (2) and (3) imply

$$|x| \le \sqrt{d} \quad \text{for} \quad x \in B. \tag{4}$$

We will prove the following theorem, which in turn implies Theorem 1.

**Theorem 2.** *Let $X$ be a $d$-dimensional normed space such that the John ellipsoid of $X$ is the standard euclidean ball. Let $u_1, \dots, u_m \in S \cap S_2^d$ and $c_1, \dots, c_m > 0$ be as above. Then*

$$\frac{1}{2d} \sum_{i=1}^m c_i \big( \|u_i + x\| + \|u_i - x\| \big) \le 2 - \frac{1}{d}$$

*for all $x \in S$.*

In the following, we always assume that $X$ is a $d$-dimensional Banach space given as $\mathbb{R}^d$ equipped with a norm $\| \, . \, \|$ such that the John ellipsoid is the standard euclidean ball. We fix contact points $u_1, \dots, u_m \in S \cap S_2^d$ and positive real numbers $c_1, \dots, c_m$ as described in the introduction.

Let us first see how Theorem 2 implies Theorem 1. Elton's generalization of Gross' theorem says that, for any regular Borel probability measure $\mu$ on $S$, there exists $x \in S$ such that

$$r(X) = \int_S \|y - x\| d\mu(y);$$

see [CMY]. We define the measure

$$\mu = \frac{1}{2d} \sum_{i=1}^{m} c_i \big(\delta_{u_i} + \delta_{-u_i}\big),$$

where $\delta_y$ is the Dirac measure at $y$. It follows from (3) that this is a Borel probability measure. So Elton's theorem gives us a point $x \in S$ such that

$$r(X) = \frac{1}{2d} \sum_{i=1}^{m} c_i \big(\|u_i + x\| + \|u_i - x\|\big).$$

Now Theorem 1 is an immediate consequence of Theorem 2.

Let us now turn to the proof of Theorem 2. We fix $x \in S$ and consider a new norm $\|\!.\!\|$ on $\mathbb{R}^d$ whose unit ball $B_{\|\,.\,\|}$ is the absolute convex hull of $B_2^d \cup \{x\}$. Since $x \in B$ and $B_2^d \subset B$, we find that $B_{\|\,.\,\|} \subset B$. Hence $\|y\| \le \|\!y\!\|$ for all $y \in \mathbb{R}^d$. So it is enough to show that

$$\frac{1}{2d} \sum_{i=1}^{m} c_i \big(\|\!u_i + x\!\| + \|\!u_i - x\!\|\big) \le 2 - \frac{1}{d}. \tag{5}$$

This will be accomplished in a series of lemmas. We abbreviate $r = |x| \ge 1$. Let $P$ be the orthogonal projection onto $\operatorname{span}\{x\}$, given by $Py = r^{-2}(y, x)x$, and let $Q$ be the complementary projection $Qy = y - Py$. We may assume, interchanging the roles of $u_i$ and $-u_i$ if necessary, that $(u_i, x) \ge 0$. The first lemma gives an explicit description of the norm $\|\,.\,\|$.

**Lemma 1.**

$$\|\!y\!\| = \begin{cases} |y| & \textit{if} \quad |Qy|^2 \ge (r^2 - 1)|Py|^2, \\ \frac{1}{r}\big(|Py| + \sqrt{r^2 - 1}|Qy|\big) & \textit{if} \quad |Qy|^2 \le (r^2 - 1)|Py|^2. \end{cases}$$

*Proof.* Choose two orthonormal vectors $e, f \in \mathbb{R}^d$ such that $x = re$ and $y = ue + vf$. Then $|Py| = |u|$ and $|Qy| = |v|$. Observe that we use $|.|$ to denote both the euclidean norm of a vector in $\mathbb{R}^d$ and the absolute value of a real number. Now the unit ball of $\|\!.\!\|$ in the subspace $\operatorname{span}\{x, y\} = \operatorname{span}\{e, f\}$ is the absolute convex hull of $B_2^2 \cup \{x\}$, which can be seen in Figure 1.

The claimed expression for the norm is now immediately verified observing that the contact points of the tangents on the unit circle passing through $x$ are

$$p_\pm = \frac{1}{r}\big(e \pm \sqrt{r^2 - 1} f\big).$$

□

**Lemma 2.** *$r|Pu_i| \le 1$ for $i = 1, \dots, m$.*

*Proof.* Assume that $r|Pu_i| > 1$ for some $i$. Then

$$|Qu_i|^2 = |u_i|^2 - |Pu_i|^2 = 1 - |Pu_i|^2 < (r^2 - 1)|Pu_i|^2.$$

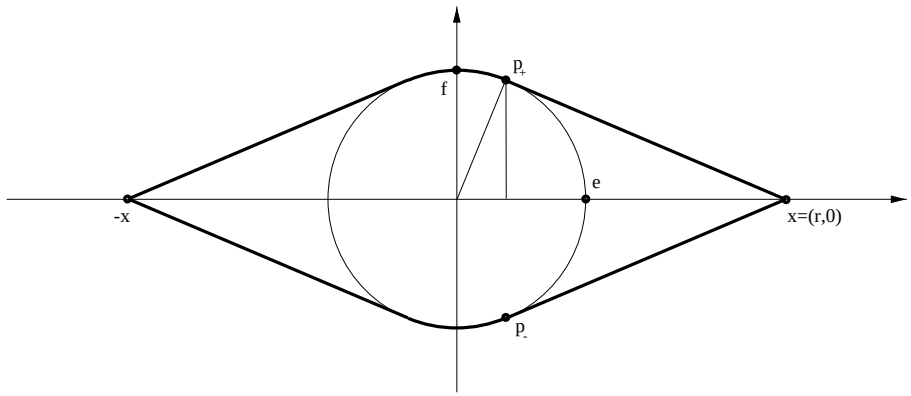


FIGURE 1.

Now it follows from Lemma 1 and $|Pu_i|^2 + |Qu_i|^2 = 1$ that

$$\|u_i\| \le |\!|\!|u_i|\!|\!| = \frac{1}{r}\left(|Pu_i| + \sqrt{r^2-1}|Qu_i|\right) < 1,$$

a contradiction to $u_i \in S$. □

We let $t_i = r|Pu_i|$ for $i = 1, \dots, m$, so that the preceding lemma implies $0 \le t_i \le 1$.

**Lemma 3.** *If $r^2 \ge 2$ and $0 \le t \le 1$, then*

$$1 - \frac{t^2}{r^2} \le (r^2-1)\left(r \pm \frac{t}{r}\right)^2.$$

*Proof.* It is clearly enough to show that $s - t^2 \le (s-1)(s-t)^2$ whenever $s \ge 2$. But this inequality is equivalent to $2 - s \le (s-t-1)^2$ which is certainly true for $s \ge 2$. □

Set $\gamma = \frac{\sqrt{5}+1}{2}$.

**Lemma 4.** *If $\gamma \le r^2 \le 2$ and $0 \le t \le 1$, then*

$$1 - \frac{t^2}{r^2} \le (r^2-1)\left(r + \frac{t}{r}\right)^2.$$

*Proof.* Here it is enough to show that $s - t^2 \le (s-1)(s+t)^2$ whenever $s \ge \gamma$ and $t \ge 0$. If $s \ge \gamma$, we have $s^2 - s - 1 \ge 0$, which in turn implies $\sqrt{2-s} + 1 - s \le 0 \le t$. This is equivalent to the above inequality. □

**Lemma 5.** *If $\gamma \le r^2 \le 2$ and $0 \le t \le 1$, then*

$$\frac{t}{r^2} + \sqrt{r^2+1-2t} \le \sqrt{r^2+1}.$$

*Proof.* Using the inequality $\sqrt{1-\xi} \le 1 - \xi/2$ for $\xi = \frac{2t}{r^2+1} \le 1$, we see that it is enough to show that $\sqrt{r^2+1} \le r^2$. But this is true for $r^2 \ge \gamma$. □

**Lemma 6.**

$$\sum_{i=1}^{m} c_i \sqrt{1 - \frac{t_i^2}{r^2}} \le \sqrt{d(d-1)}.$$

*Proof.* Observe that, by (2) and $|x| = r$,

$$\sum_{i=1}^{m} \frac{c_i t_i^2}{r^2} = \sum_{i=1}^{m} c_i |Pu_i|^2 = \frac{1}{r^4} \sum_{i=1}^{m} c_i (u_i, x)^2 |x|^2 = 1.$$

An application of the Cauchy–Schwarz inequality and (3) then gives

$$\sum_{i=1}^{m} c_i \sqrt{1 - \frac{t_i^2}{r^2}} \leq \sqrt{\sum_{i=1}^{m} c_i} \sqrt{\sum_{i=1}^{m} c_i (1 - r^{-2} t_i^2)} = \sqrt{d(d-1)}.$$

□

We can now finish the proof of (5) which also proves Theorem 2. Observe that

$$|Q(u_i \pm x)|^2 = |Qu_i|^2 = 1 - |Pu_i|^2 = 1 - \frac{t_i^2}{r^2} \tag{6}$$

and

$$|P(u_i \pm x)| = |x \pm Pu_i| = r \pm |Pu_i| = r \pm \frac{t_i}{r}. \tag{7}$$

We distinguish the cases $r^2 \geq 2$, $\gamma \leq r^2 \leq 2$, and $1 \leq r^2 \leq \gamma$. In the latter two cases we only treat $d \geq 3$. It is also possible to derive (5) if $d = 2$, but since our main objective is in showing Theorem 1, and this is already known for $d = 2$, we omit the somewhat tedious technical details for this case.

**Case $r^2 \geq 2$.** In this case, it follows from (6), (7), Lemma 3, and Lemma 1 that

$$\begin{aligned} &\|u_i + x\| + \|u_i - x\| \\ &= \frac{1}{r}\Big(|P(u_i + x)| + |P(u_i - x)| + \sqrt{r^2 - 1}(|Q(u_i + x)| + |Q(u_i - x)|)\Big) \\ &= 2 + 2\sqrt{\left(1 - \frac{1}{r^2}\right)\left(1 - \frac{t_i^2}{r^2}\right)}. \end{aligned}$$

Multiplication by $c_i$, summation over $i = 1, \ldots, m$ and application of Lemma 6 gives

$$\frac{1}{2d} \sum_{i=1}^{m} c_i \big(\|u_i + x\| + \|u_i - x\|\big) \leq 1 + \sqrt{\left(1 - \frac{1}{r^2}\right)\left(1 - \frac{1}{d}\right)}.$$

Since $x \in B$, we conclude from (4) that $r = |x| \leq \sqrt{d}$ which proves (5) in this case.

**Case $\gamma \leq r^2 \leq 2$.** Here we use (6), (7), Lemmas 1 and 4, and $\|u_i - x\| \leq |u_i - x|$ to obtain

$$\|u_i + x\| + \|u_i - x\| \leq 1 + \frac{t_i}{r^2} + \sqrt{\left(1 - \frac{1}{r^2}\right)\left(1 - \frac{t_i^2}{r^2}\right)} + \sqrt{r^2 + 1 - 2t_i}.$$

Multiplication by $c_i$, summation over $i = 1, \ldots, m$, and using Lemmas 6 and 5 yields

$$\begin{aligned} \frac{1}{2d} \sum_{i=1}^{m} c_i \big(\|u_i + x\| + \|u_i - x\|\big) &\leq \frac{1}{2} + \frac{1}{2}\sqrt{r^2 + 1} + \frac{1}{2}\sqrt{\left(1 - \frac{1}{r^2}\right)\left(1 - \frac{1}{d}\right)} \\ &\leq \frac{\sqrt{3} + 1}{2} + \frac{1}{4}\sqrt{2 - \frac{2}{d}}. \end{aligned}$$

It is elementary to check that the latter is bounded by $2 - 1/d$ for $d \geq 3$, so that (5) is also verified in this case.

**Case** $1 \leq r^2 \leq \gamma$**.** In this case, we use that $\|u_i \pm x\| \leq |u_i \pm x|$ which gives

$$\begin{aligned} \|u_i + x\| + \|u_i - x\| &\leq \sqrt{2\big(|u_i + x|^2 + |u_i - x|^2\big)} = 2\sqrt{|u_i|^2 + |x|^2} = 2\sqrt{r^2 + 1} \\ &\leq 2\sqrt{\gamma + 1} = 2\gamma < 4 - \frac{2}{d}, \end{aligned}$$

provided that $d \geq 3$. Averaging as in the previous cases again leads to (5).

## References


[BCP] M. Baronti, E. Casini, P. L. Papini, On the average distance property and the size of the unit sphere, Atti Sem. Mat. Fis. Univ. Modena, **46**, 427–446, 1998. MR **99m:**46028

[CMY] J. Cleary, S. A. Morris, D. Yost, Numerical geometry–numbers for shapes, Amer. Math. Monthly, **93**, 260–275, 1986. MR **87h:**51043

[Gro] O. Gross, The rendezvous value of a metric space, Ann. of Math. Stud., **52**, 49–53, 1964. MR **28:**5841

[Joh] F. John, Extremum problems with inequalities as subsidiary conditions, Courant Anniversary Volume, Interscience, New York (1948), 187–204. MR **10:**719b

[TJ] N. Tomczak-Jaegermann, Banach-Mazur distances and finite dimensional operator ideals, Longman, Harlow, 1989. MR **90k:**46039

[Wo1] R. Wolf, On the average distance property of spheres in Banach spaces, Arch. Math., **62**, 338–344, 1994. MR **95c:**46027

[Wo2] R. Wolf, On the average distance property in finite dimensional real Banach spaces, Bull. Austral. Math. Soc., **51**, 87–101, 1994. MR **96a:**52005

[Wo3] R. Wolf, Averaging distances in real quasihypermetric Banach spaces of finite dimension, Isr. J. Math., **110**, 125–151, 1999. CMP 2000:11



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