POROUS MEASURES ON $\mathbb{R}^n$: LOCAL STRUCTURE AND DIMENSIONAL PROPERTIES

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Abstract. We study dimensional properties of porous measures on $\mathbb{R}^n$. As a corollary of a theorem describing the local structure of nearly uniformly porous measures we prove that the packing dimension of any Radon measure on $\mathbb{R}^n$ has an upper bound depending on porosity. This upper bound tends to $n-1$ as porosity tends to its maximum value.

1. Introduction and preliminaries

Porosity is a quantity that describes irregularities of fractals. The study of dimensional properties featured by porous sets was pioneered by P. Mattila. In [M1] he verified the existence of a non-increasing function which gives an upper bound for Hausdorff dimension of any set in $\mathbb{R}^n$ as a function of porosity. Furthermore, he showed that this upper bound tends to $n-1$ as porosity tends to its maximum value. In [S] A. Salli generalized the corresponding results for packing dimension and established the correct asymptotic behaviour for the upper bound.

For measures the following definition of porosity was introduced in [EJJ].

1.1. Definition. The porosity of a Radon measure $\mu$ on $\mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is defined by

\[
\text{por}(\mu, x) = \lim_{\varepsilon \to 0} \liminf_{r \to 0} \text{por}(\mu, x, r, \varepsilon)
\]

where for all $r, \varepsilon > 0$

\[
\text{por}(\mu, x, r, \varepsilon) = \sup\{p \geq 0 \mid \text{there is } z \in \mathbb{R}^n \text{ such that } B(z, pr) \subset B(x, r) \text{ and } \mu(B(z, pr)) \leq \varepsilon \mu(B(x, r))\}.
\]

The porosity of $\mu$ is

\[
\text{por}(\mu) = \text{ess sup}_{x \in \mathbb{R}^n} \text{por}(\mu, x)
\]

\[
= \inf\{s \geq 0 \mid \text{por}(\mu, x) \leq s \text{ for } \mu\text{-almost all } x \in \mathbb{R}^n\}.
\]

We will relate porosity of measures to packing dimension defined as follows. (For the definition of the packing dimension of a set see [M2], 5.9 and Theorem 5.11.)
1.5. Definition. Let $\mu$ be a Radon measure on $\mathbb{R}^n$. The packing dimension of $\mu$ is defined in terms of upper local dimensions

$$\dim_p(\mu) = \sup\{s \geq 0 \mid \limsup_{i \to \infty} \frac{\log \mu(D_i(x))}{\log 2^{-i}} \geq s \text{ for } \mu\text{-almost all } x \in \mathbb{R}^n\}$$

where $D_i(x)$ is the closed dyadic cube of side-length $2^{-i}$ containing $x$. Equivalently, this definition can be given using packing dimensions of Borel sets with positive $\mu$-measure

$$\dim_p(\mu) = \inf\{\dim_p(A) \mid A \text{ is a Borel set with } \mu(A) > 0\}.$$  

Replacing "liminf" by "limsup" in (1.2) gives the upper porosity of a measure which was studied by M. E. Mera and M. Morán in [MM]. They showed that if $\mu$ satisfies the doubling condition, that is,

$$\limsup_{r \to 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty$$

for $\mu$-almost all $x \in \mathbb{R}^n$, then the upper porosity of $\mu$ is either 0 or 1/2. (Above $B(x, r)$ is the closed ball with radius $r$ and centre $x$.) Furthermore, for any non-doubling measure the upper porosity equals 1. Note that the (lower) porosity may obtain any value between 0 and 1/2 for both doubling and non-doubling measures (see [JJM] (2.17) and [EJJ, Example 4]). The upper porosity is too weak for the purpose of obtaining a non-trivial upper bound for dimension; for any $p = 0, 1/2, 1$ and $0 \leq d \leq n$ there exists a Radon measure $\mu$ with the upper porosity equal to $p$ and with both Hausdorff and packing dimension equal to $d$.

In this paper we will establish a connection between porosity and packing dimension for all Radon measures on $\mathbb{R}^n$. The case $n = 1$ was studied in [JJ]. In [EJJ] the emphasis was given to doubling measures on $\mathbb{R}^n$. For such measures the porosity can be given in terms of porosities of Borel sets with positive measure:

$$\text{por}(\mu) = \sup\{\text{por}(A) \mid A \text{ is a Borel set with } \mu(A) > 0\}.$$  

(The doubling condition is necessary here; see [EJJ] for details.)

We will generalize the results of [JJ] to higher dimensions by verifying that the packing dimension of any Radon measure on $\mathbb{R}^n$ is bounded above by a function that depends on porosity and by showing that this upper bound goes to $n - 1$ as porosity tends to its maximum value 1/2 (see Corollary 2.9). In particular, the packing dimension of any Radon measure on $\mathbb{R}^n$ having porosity close to 1/2 cannot be much larger than $n - 1$.

Our main tools are a dimension estimate obtained from the strong law of large numbers and a description of the local structure of nearly uniformly porous measures. The latter one states that for a given nearly uniformly porous measure any sufficiently small dyadic cube can be divided into three parts, two having small measure and the remaining one being a narrow boundary of a convex set (see Theorem 2.2).

2. Local structure and dimensional properties

We recall the following lemma from [JJ] according to which we may replace any measure by a nearly uniformly porous measure when estimating packing dimension from above.
2.1. Lemma. Assume that $\mu$ is a Radon measure on $\mathbb{R}^n$ such that $\text{por}(\mu) \geq p$. Let $0 < \delta < 1$. Then there is a Radon measure $\mu_\delta$ with compact support $\text{spt}(\mu_\delta) \subseteq \text{spt}(\mu)$ and with $\dim_p(\mu_\delta) \geq \dim_p(\mu)$ such that the following property holds: there exists $\varepsilon_\delta > 0$ such that for all $0 < \varepsilon \leq \varepsilon_\delta$ there are a Borel set $B_{\delta,\varepsilon}$ and $r_{\delta,\varepsilon} > 0$ with $\mu_\delta(\mathbb{R}^n \setminus B_{\delta,\varepsilon}) \leq \delta \mu_\delta(\mathbb{R}^n)$ and
\[
\text{por}(\mu_\delta, x, r, \varepsilon) > p - \frac{\delta}{2}
\]
for all $x \in B_{\delta,\varepsilon}$ and $0 < r \leq r_{\delta,\varepsilon}$.

Proof. See [JJ, Lemma 2.2].

For all positive integers $i$ we use the notation $D_i$ for the family of closed dyadic cubes in $\mathbb{R}^n$ with side-length $2^{-i}$, that is, cubes of the form $\{x \in \mathbb{R}^n \mid k_j 2^{-i} \leq x_j \leq (k_j + 1) 2^{-i} \text{ for all } j = 1, \ldots, n\}$ where $k_j$, $j = 1, \ldots, n$, are integers. For all $Q \in D_i$ and for all positive integers $k$, let $N^k(Q) \subset D_i$ be the family of the $(2k + 1)^n$ neighbouring dyadic cubes of $Q$ with side-length $2^{-i}$ being located symmetrically around $Q$ (including $Q$ itself).

For all $\delta > 0$, a $\delta$-plate is a $\frac{1}{2}$-neighbourhood of an $(n - 1)$-dimensional affine subspace of $\mathbb{R}^n$. An affine $\delta$-boundary of a convex polyhedron $P$ is the union of parts of $\delta$-plates glued on all faces of $P$ such that the union of $P$ and its affine $\delta$-boundary is a polyhedron obtained by magnifying $P$.

The following theorem describes the local structure of porous measures by stating that in all sufficiently small dyadic cubes such measures are essentially concentrated on a narrow boundary of some convex set.

2.2. Theorem. Assume that $\mu$ is a Radon measure on $[0,1]^n$ such that $\text{por}(\mu) \geq \frac{1}{2}(1 - \beta)$ for $0 \leq \beta \leq \frac{1}{4}$. Let $K$ be a positive integer. For all $0 < \delta < \frac{1}{16}$, let $\mu_\delta$ and $\varepsilon_\delta$ be as in Lemma 2.1. Let $0 < \varepsilon < \varepsilon_\delta$. Then there is a positive integer $i_0$ depending on $K$, $\delta$, and $\varepsilon$ such that for all $i \geq i_0$ any cube $Q \in D_i$ can be divided into three disjoint (not necessarily non-empty) parts
\[
Q = E \cup P \cup I
\]
where
\[
\mu_\delta(E) \leq C_Q^K N \varepsilon
\]
for an integer $N$ depending on $K$, $\delta$, and $\beta$ and for $C_Q^K = \max_{D \in N^K(Q)} \mu_\delta(D)$, $P$ is an affine $C_{\beta,\delta} 2^{-i}$-boundary of a convex polyhedron with $C_{\beta,\delta} = 6K(\beta + \delta) + \frac{n}{K(1 - \beta - \delta)}$, and $I \subseteq Q \setminus B_{\delta,\varepsilon}$. Here $B_{\delta,\varepsilon}$ is as in Lemma 2.1.

Proof. Let $r_{\delta,\varepsilon}$ be as in Lemma 2.1 and let $i_0$ be the smallest integer such that $(1 + K)2^{-i_0} < r_{\delta,\varepsilon}$. Consider an integer $i \geq i_0$. Note that for those $Q \in D_i$ which do not intersect $B_{\delta,\varepsilon}$ equality (2.3) is trivial. Let $Q \in D_i$ such that $Q \cap B_{\delta,\varepsilon} \neq \emptyset$. Setting $r_x = \text{dist}(x, \partial Q) + K2^{-i}$ for all $x \in Q \cap B_{\delta,\varepsilon}$, we have $B(x, r_x) \subset \bigcup_{D \in N^K(Q)} D$. Lemma 2.1 implies that for all $x \in Q \cap B_{\delta,\varepsilon}$ there is a ball $B_x$ with radius $q r_x = \frac{1}{2}(1 - \beta - \delta) r_x$ such that $B_x \subset B(x, r_x)$ and
\[
\mu_\delta(B_x) \leq (2K + 1)^n C_Q^K \varepsilon.
\]
Set $a^{1/n} = 2K(1 - q) n^{1/n}$ where $\alpha = L^n(B(0,1))$ is the Lebesgue measure of the unit ball. Then there are integers $N(a) \geq N(a, Q) \geq 0$ for which there are
$x_1, \ldots, x_{N(a,Q)} \in Q \cap B_{\delta,z}$ such that

$$\mathcal{L}^n\left( (B_{x_j} \cap Q) \setminus \bigcup_{k=1}^{j-1} B_{x_k} \right) \geq a \mathcal{L}^n(Q)$$

for all $j = 1, \ldots, N(a,Q)$ and

$$\mathcal{L}^n\left( (B_x \cap Q) \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k} \right) < a \mathcal{L}^n(Q)$$

for all $x \in Q \cap B_{\delta,z}$ such that $x \neq x_j$ for all $j = 1, \ldots, N(a,Q)$. (We use the interpretation $\bigcup_{k=1}^{0} B_{x_k} = \emptyset$.) In the case $N(a,Q) = 0$ we have $\mathcal{L}^n(B_x \cap Q) < a \mathcal{L}^n(Q)$ for all $x \in Q \cap B_{\delta,z}$.

Define

$$I_1 = \left\{ y \in Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k} \mid \text{dist} \left( y, \partial (Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}) \right) > 2\left( \frac{a}{\alpha} \right)^{1/n} 2^{-i} \right\}.$$

Then

$$\mathcal{L}^n(B_x \cap I_1) = 0$$

for all $x \in I_1 \cap B_{\delta,z}$. In fact, assuming that $B_x = B_{x_j}$ for some $j = 1, \ldots, N(a,Q)$, equality (2.6) holds. If $B_x \cap I_1 \neq \emptyset$ for some $x \in I_1 \cap B_{\delta,z}$ with $B_x \neq B_{x_j}$ for all $j = 1, \ldots, N(a,Q)$, then, as will be indicated shortly, the set $(B_x \cap Q) \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}$ contains a ball with radius $(a/\alpha)^{1/n} 2^{-i}$ contradicting (2.5). To find such a ball, take $z \in B_x \cap I_1$. Then $B(z, 2(a/\alpha)^{1/n} 2^{-i}) \subset Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}$. Since $(a/\alpha)^{1/n} 2^{-i} \leq \frac{1}{2} qr_x$, the ball $B_x$ contains a ball with radius $(a/\alpha)^{1/n} 2^{-i}$ having $z$ on its boundary such that the centre of the ball belongs to the line going through $z$ and the centre of $B_x$. Clearly this ball is a subset of $Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}$. This completes the proof of (2.6).

Set

$$I_2 = \left\{ y \in Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k} \mid \text{dist} \left( y, \partial (Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}) \right) > 3\left( \frac{a}{\alpha} \right)^{1/n} 2^{-i} \right\}.$$

Then $I_2 \subset Q \setminus B_{\delta,z}$. To see this, assume that there exists $x \in I_2 \cap B_{\delta,z}$. From (2.6) we get

$$\text{dist}(x, \partial B_x) > \left( \frac{a}{\alpha} \right)^{1/n} 2^{-i} = 2K(1-2q)2^{-i}.$$

On the other hand $\text{dist}(x, \partial B_x) \leq r_x(1-2q) \leq (K+1/2)(1-2q)2^{-i}$. Hence $I_2 \subset Q \setminus B_{\delta,z}$.

Let $B_{x_k} = B(z_k, qr_{x_k})$. Since $\frac{a}{\sqrt{K}} 2^{-i}$ is an upper bound for the height of a segment of any ball with radius $qr_x$ having chord with length at most $\sqrt{m} 2^{-i}$, the intersections of each of the annuli $B(z_k, qr_x + 3(a/\alpha)^{1/n} 2^{-i}) \setminus B_{x_k}$ and $Q$ can be included in a $C_{\beta,\delta} 2^{-i}$-plate. Adding parts of the affine $C_{\beta,\delta}$-boundary of $Q$ if necessary concludes the construction of $P$. Setting

$$I = I_2 \setminus P$$
and
\[ E = Q \setminus (P \cup I) \subset Q \cap \bigcup_{k=1}^{N(a,Q)} B_{x_k}, \]
equality (2.3) follows since
\[ \mu_\delta(Q \cap \bigcup_{k=1}^{N(a,Q)} B_{x_k}) \leq N(a)(2K + 1)^n C Q \varepsilon \]
by (2.4).

Let \( \mu \) be a Radon probability measure on \([0, 1]^n\) such that \( \mu(V) = 0 \) for all affine hyperplanes \( V \subset \mathbb{R}^n \). Letting \( k \) be a positive integer, set \( I = \{1, \ldots, 2^{kn}\} \). For all positive integers \( j \), denote by \( \mathbf{P} \) the set of all \( j \)-term sequences of integers belonging to \( I \) and by \( \mathbf{I}^\infty \) the corresponding set of infinite sequences, that is,
\[ \mathbf{P} = \{(i_1, \ldots, i_j) \mid i_l \in I \text{ for all } l = 1, \ldots, j\} \]
and
\[ \mathbf{I}^\infty = \{(i_1, i_2, \ldots) \mid i_l \in I \text{ for all } l = 1, 2, \ldots\}. \]
Divide \([0, 1]^n\) into \( 2^{kn} \) dyadic subcubes, enumerate them and denote them by \( D_j \), \( j = 1, \ldots, 2^{kn} \). Define \( f : [0, 1]^n \to [0, 1]^n \) by setting \( f(x) = 2^k x \text{ mod } 1 \). For each \( x \in [0, 1]^n \) we define a sequence \( \mathbf{i}^x = (i_1, i_2, \ldots) \in \mathbf{I}^\infty \) such that \( f^{l-1}(x) \in D_{i_l} \) for all \( l = 1, 2, \ldots \). Note that for all \( x = (x_1, \ldots, x_n) \) the sequence \( \mathbf{i}^x \) is unique unless \( x_j \) is a dyadic rational for some \( j = 1, \ldots, n \). If \( f^l(x) \) is a dyadic rational for some \( l \) and \( i \), one can choose between indices corresponding to "left" and "right" cube. If one chooses left, then \( f^{l+1}(x)_i = 1 \) and otherwise \( f^{l+1}(x)_i = 0 \). For a positive integer \( l \) and \( i = (i_1, i_2, \ldots) \in \mathbf{I}^\infty \) let \( |i| = (i_1, \ldots, i_l) \in \mathbf{I}^l \) be the sequence of the first \( l \) digits of \( i \) and for all \( j = 1, \ldots, 2^{kn} \), let \( n_j(|i|) \) be the number of \( j \)'s in \( |i| \).

We can attach a sequence \( (P_1^\mu) \) of probability measures on \( I \) such that for all \( j = 1, \ldots, 2^{kn} \), \( P_1^\mu(\{j\}) \) gives the probability that the \( l \)th digit (in the above representation) of a random number (with respect to \( \mu \)) in \([0, 1]^n\) equals \( j \), that is,
\[ P_1^\mu(\{j\}) = \sum_{(i_1, \ldots, i_l) \in I^l} \mu(D_{i_1, \ldots, i_l}) \]
where \( D_{i_1, \ldots, i_l} \) is the closed dyadic subcube of \([0, 1]^n\) of side-length \( 2^{-kl} \) consisting of points whose expansion begins with \((i_1, \ldots, i_l)\). The measures \( P_1^\mu \) are well-defined since \( \mu(V) = 0 \) for all affine hyperplanes \( V \subset \mathbb{R}^n \). We use the notation \( P^\mu \) for the product measure \( \prod_{l=1}^\infty P_1^\mu \) on the code space \( \mathbf{I}^\infty \).

2.7 Proposition. Let \( \mu \) be a Radon probability measure on \([0, 1]^n\) such that \( \mu(V) = 0 \) for all affine hyperplanes \( V \subset \mathbb{R}^n \). Let \( p \leq 2^{-kn} \) and \( L \in I \). Assume that
\[ \limsup_{l \to \infty} \frac{1}{l} \sum_{j=1}^{L} P_1^\mu(\{j\}) \leq p \text{ for all } j = 1, \ldots, L. \]
Then
\[ \dim_p \mu \leq \frac{1}{\log 2} \left( Lp \log p + (1 - Lp) \log \left( \frac{1 - Lp}{2^{kn} - L} \right) \right) =: \alpha(p, L). \]

Proof. The strong law of large numbers [Fe X.7.1] gives for all \( j = 1, \ldots, L \) that
\[ \limsup_{l \to \infty} \frac{1}{l} n_{j,l}(|i|) \leq p \]
for $P^n$-almost all $i \in \mathbb{I}^n$. Defining

$$E_{p,L} = \{x \in [0,1]^n \mid \limsup_{l \to \infty} \frac{1}{l} \sum_{j=1}^{L} n_j(x_i^l) \leq p \text{ for all } j = 1, \ldots, L \},$$

this implies that $\mu(E_{p,L}) = 1$. Since $E_{p,L}$ is a Borel set it is enough to prove that $\dim_p(E_{p,L}) \leq \alpha(p,L)$.

Let $\rho$ be a probability measure on $I$ such that $\rho(\{j\}) = p$ for all $j = 1, \ldots, L$ and $\rho(\{j\}) = (1-Lp)/(2^{kn} - L)$ for all $j = L+1, \ldots, 2^{kn}$. Let $\nu$ be the image of the infinite product of the measures $\rho$ under the map $\pi : \mathbb{I}^{\infty} \to [0,1]^n$. Note that since $p \leq 2^{-kn}$ we have

$$-u \log p - (1-u) \log \left(\frac{1-Lp}{2^{kn} - L}\right) \leq \alpha(p,L) \log 2^k$$

for all $u \leq Lp$. Let $x \in E_{p,L}$. The equality

$$\log \nu(D_{kl}(x)) = \log p \sum_{j=1}^{L} n_j(x_i^l) + \log \left(\frac{1-Lp}{2^{kn} - L}\right) \sum_{j=L+1}^{2^{kn}} n_j(x_i^l)$$

gives

$$\liminf_{l \to \infty} \frac{1}{l} \log \left(\frac{\nu(D_{kl}(x))}{2^{-kl}}\right) \geq -\log 2^k \alpha(p,L) + t \log 2^k$$

where $D_{kl}(x)$ is the dyadic cube of side-length $2^{-kl}$ containing $x$. Thus if $t > \alpha(p,L)$, then $\liminf_{l \to \infty} \frac{\nu(D_{kl}(x))}{2^{-kl}} = \infty$, implying

$$\limsup_{l \to \infty} \frac{\log \nu(D_{kl}(x))}{\log 2^{-kl}} \leq t.$$  

By [23], Proposition 2.3 (d) we get $\dim_p(E_{p,L}) \leq \alpha(p,L)$.

Let $k$ and $i$ be positive integers. Dyadic cubes in $D_{ki}$ form a brood if they belong to the same dyadic cube belonging to $D_{k(i-1)}$. Note that each brood consists of $2^{kn}$ dyadic cubes. Given a measure $\mu$ on $\mathbb{R}^n$, order the cubes of every brood such that $\mu(D_j) \leq \mu(D_{j+1})$ for all $j = 1, \ldots, 2^{kn}$. Let $D_{kl}^j(\mu)$ be the set of the $j$th cubes of all broods.

2.8. Theorem. Let $\mu$ be a Radon probability measure on $[0,1]^n$ such that $\text{por}(\mu) \geq \frac{1}{2}(1-\beta)$ for $0 \leq \beta \leq \frac{1}{18}$ and $\mu(V) = 0$ for all affine hyperplanes $V \subset \mathbb{R}^n$. For all $0 < \delta < \frac{1}{18}$ let $\mu_\delta$ be as in Lemma 2.1. Then there is an integer $L$ with $2^{k(\beta,\delta)n} \geq L \geq 2^{k(\beta,\delta)n} - c2^{k(\beta,\delta)(n-1)}$ where $c$ is a constant depending only on $n$ and $k(\beta,\delta) \to \infty$ as $\delta \to 0$ and $\beta \to 0$ such that the following inequality is valid: for all $j = 1, \ldots, L$ we have

$$\limsup_{l \to \infty} \frac{1}{l} \sum_{i=1}^{l} \sum_{D \in D_{kl}^i(\mu_\delta)} \mu_\delta(D) \leq \delta \mu_\delta([0,1]^n).$$

Proof. Let $k_0$ be the largest integer such that $\beta + \delta \leq 2^{-2k_0}$. Set $K = 2^{k_0}$. Let $0 < \varepsilon < \varepsilon_\delta$ and let $i \geq i_0$ where $\varepsilon_\delta$ and $i_0$ are as in Theorem 2.2. Let $k$ be the largest integer such that $2^{-k} > 4(6 + 4n)2^{-k_0}$. Then $2^{-k} > 4C_{\beta,\delta}$ where $C_{\beta,\delta}$ is as in Theorem 2.2. Consider $Q \in D_{ki}$. Let $Q = E_Q \cup P_Q \cup I_Q$ be the partition of $Q$ given in Theorem 2.2. Take any $x \in P_Q$. Then $D_{k(i+1)}(x)$ and its neighbouring cubes in $D_{k(i+1)}$ cover a part of $P_Q$ such that the $L^{-n}$-measure of the covered part of both inner and outer boundary of $P_Q$ is at least $2^{-(n-1)}2^{-k(i+1)(n-1)}$. By
the convexity of $PQ$ the $C^n$-measure of the outer boundary of $PQ$ is less than $2n2^{-kl(n-1)}$. Hence we need at most $3^22n2^{n-1}2^{k(n-1)}$ cubes from $D_{k(i+1)}$ to cover $PQ$. Thus there are $L \geq 2^{kn} - 3^{2n}2^{n-1}2^{k(n-1)}$ cubes in $D_{k(i+1)}$ which belong to $E_Q \cup I_Q$. Clearly $\mu_\delta(D_j) \leq \mu_\delta(E_Q \cup I_Q)$ for all $j = 1, \ldots, L$. Theorem 2.2 and Lemma 2.1 give

$$\limsup_{l \to \infty} \frac{1}{l} \sum_{i=1}^l \sum_{D \in D_{k,(\delta)}} \mu_\delta(D) \leq \limsup_{l \to \infty} \frac{1}{l} \sum_{i=1}^l \sum_{Q \in D_{k,L}} \mu_\delta(E_Q \cup I_Q)$$

$$\leq \limsup_{l \to \infty} \frac{1}{l} \sum_{i=1}^l \left( (2K + 1)^n N_\varepsilon + \delta \right) \mu_\delta([0, 1]^n) \to \delta \mu_\delta([0, 1]^n).$$

Since $k_0 \to \infty$ as $\delta$ and $\beta$ tend to zero we may let $k$ tend to infinity when $\delta \to 0$ and $\beta \to 0$.

**2.9. Corollary.** Assume that $\mu$ is a Radon measure on $\mathbb{R}^n$. If $0 < \beta \leq 1$ such that $\text{por}(\mu) \geq \frac{1}{2}(1 - \beta)$, then $\dim_\beta(\mu) \leq d(\beta)$ where $d(\beta) \to n - 1$ as $\beta \to 0$.

**Proof.** The claim follows from Theorem 2.8 and from the obvious generalization of [JJ] Lemma 3.3) (see [JJ] Corollary 3.4]) with $d(\beta) = \lim_{\delta \to 0} \alpha(\delta, L)$ where $L$ is as in Theorem 2.8. Note that by the choices of $k$ and $k_0$ in Theorem 2.8 we have $C_n(\beta + \delta)^{-1/2} \leq 3^k \leq C_n(\beta + \delta)^{-1/2}$ for constants $C_n$ and $\tilde{C}_n$ depending only on $n$. By the lower and upper bounds given in Theorem 2.8 for $L$ we obtain that $d(\beta) \to n - 1$ when $\beta \to 0$.

**2.10. Remark.** After finishing this paper we obtained the preprint [BS] from D. B. Beliaev and S. K. Smirnov where similar dimension results have been proved using different methods.

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