

**AN ADDITION TO THE  $\cos \pi\rho$ -THEOREM FOR SUBHARMONIC  
 AND ENTIRE FUNCTIONS OF ZERO LOWER ORDER**

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ABSTRACT. We obtain a sharp asymptotic relation between the infimum and the maximum on a circle of a subharmonic function of zero lower order. An example is constructed, which shows the sharpness of the relation in the class of entire functions of zero order such that  $\log M(r, f)/\log^2 r \rightarrow +\infty$ , where  $M(r, f) = \max\{|f(z)| : |z| = r\}$  as  $r \rightarrow +\infty$ .

1. INTRODUCTION

Let  $u(z)$  be a subharmonic function in  $\mathbb{C}$ . We denote  $A(r, u) = \inf\{u(z) : |z| = r\}$ ,  $B(r, u) = \max\{u(z) : |z| = r\}$ ,  $r > 0$ . The classical  $\cos \pi\rho$ -theorem asserts that if  $f(z)$  is an entire function of finite order  $\rho \in [0; 1]$ , then

$$(1.1) \quad A(r_n, \log |f|) > (C_1(\rho) + o(1))B(r_n, \log |f|),$$

where  $C_1(\rho) = \cos \pi\rho$ ,  $r_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ . It is known ([1], Chapter 6) that one can replace  $\log |f|$  by an arbitrary subharmonic in the plane function  $u$  of lower order  $\lambda \leq 1$  in (1.1). In this case,  $C_1(\rho)$  must be replaced by  $C_1(\lambda)$ , and the constant is sharp. Hence, if  $\lambda = 0$ , then (1.1) implies  $A(r_n, u) \sim B(r_n, u)$  as  $r_n \rightarrow +\infty$ . Naturally, the question on improvement of  $\cos \pi\rho$ -theorem in the case of subharmonic or entire functions of zero order arises.

In 1962 P. Barry [2] showed that for an arbitrary subharmonic function  $u$  the inequality  $B(r, u) \leq \psi(r)$  ( $r \geq r_0$ ) implies

$$(1.2) \quad (\forall \varepsilon > 0) \quad A(r_n, u) > B(r_n, u) - (1 + \varepsilon)\frac{\pi^2}{2}\psi_2(r_n), \quad r_n \rightarrow +\infty,$$

where  $\psi \in C^2(\mathbb{R}_+)$  satisfies the following conditions:  $\psi_2(r)$  is continuous and slowly varying on  $[1, +\infty)$  (i.e.  $\psi_2(2r) \sim \psi_2(r)$  as  $r \rightarrow +\infty$ ), here  $\psi_j(r) = \frac{d^j \psi(r)}{(d \log r)^j}$ ,  $j = 1, 2$ . On the other hand, in 1980 P. Fenton [3] proved that for subharmonic function  $u$  the condition  $\liminf_{r \rightarrow +\infty} B(r, u)/\log^{p+1} r = \sigma < +\infty$  implies for  $p > 0$  that  $\forall \varepsilon > 0$  the inequality

$$A(r, u) > B(r, u) - (\sigma + \varepsilon) \operatorname{Re}\{\log^{p+1} r - (\log r + i\pi)^{p+1}\}$$

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holds for all  $r$  outside a set  $E$  satisfying

$$\liminf_{r \rightarrow +\infty} \frac{1}{\log^p r} \int_{E \cap [1, r]} \frac{p \log^{p-1} t}{t} dt \leq \frac{\sigma}{\sigma + \varepsilon}.$$

We remark that

$$\begin{aligned} \operatorname{Re}\{\log^{p+1} r - (\log r + i\pi)^{p+1}\} &= (1 + o(1)) \frac{\pi^2(p+1)p}{2} \log^{p-1} r \\ &= (1 + o(1)) \frac{\pi^2}{2} \frac{d^2 \log^{p+1} r}{(d \log r)^2}, \quad r \rightarrow +\infty. \end{aligned}$$

In the case  $\psi(r) = \sigma(\log^+ r)^2$  ( $r \geq 0$ ) A. A. Gol'dberg [4] and P. Fenton [5, 6] showed that

$$(1.3) \quad \limsup_{r \rightarrow +\infty} \frac{A(r, |f|)}{B(r, |f|)} \geq C_2(\sigma) = \left( \prod_{n=1}^{+\infty} \frac{1 - q^{2n-1}}{1 + q^{2n-1}} \right)^2,$$

where  $q = e^{-\frac{1}{4\sigma}}$ ,  $C_2(0) = 1$ , for every entire function [6] such that  $\liminf_{r \rightarrow +\infty} \frac{B(r, \log |f|)}{\log^2 r} = \sigma < +\infty$  (for every meromorphic function [4] such that  $\liminf_{r \rightarrow +\infty} N(r, 0, \infty, f) \log^{-2} r \leq \sigma < +\infty$ ), here  $N(r, 0, \infty, f) = N(r, 0, f) + N(r, \infty, f)$  is Nevanlinna's counting function of zeros and poles. The constant  $C_2(\sigma)$  in (1.3) is sharp [2], [4].

In 1982 P. Barry [7] established a sharp lower estimate for the remainder  $A(r, u) - B(r, u)$  in the case  $u = \log |f|$ ,  $f$  is an entire function with  $B(r, \log |f|) = O(\log^{p+1} r)$  as  $r \rightarrow +\infty$ ,  $0 < p < 1$ . He also remarked, that a sharp estimate for  $p > 1$  in the class on entire function was unknown. In this connection the following questions arise:

- 1) Is inequality (1.2) valid for subharmonic functions of zero lower order such that  $\liminf_{r \rightarrow +\infty} \frac{B(r, u)}{\psi(r)} \leq 1$ ?
- 2) What is a sharp inequality between  $A(r, \log |f|)$  and  $B(r, \log |f|)$  in the class of entire functions such that  $B(r, \log |f|)/\log^2 r \rightarrow +\infty$  as  $r \rightarrow +\infty$ ?

In this paper we give answers to both questions.

## 2. IMPROVING THE $\cos \pi\rho$ -THEOREM

Let us introduce the class  $\Psi$  of functions  $\psi: [1; +\infty) \rightarrow \mathbb{R}_+$  such that

$$\psi_1(r) = \frac{d\psi(r)}{d \log r} = K_1 \exp\left\{ \int_1^r \frac{\beta(s)}{s} ds \right\},$$

where  $K_1 > 0$  is a constant,  $\beta \in C(\mathbb{R}_+)$ ,  $\beta(r) > 0$ ,  $\beta(r) \rightarrow 0$  as  $r \rightarrow +\infty$ ,  $\int_1^{+\infty} \beta(s)s^{-1} ds$  is divergent and

$$(2.1) \quad \log \frac{1}{\beta(r)} = O\left(\log \frac{1}{\hat{\beta}(r)}\right), \quad r \rightarrow +\infty,$$

where  $\hat{\beta}(r) = \max\{\beta(s) : s \geq r\}$ . We remark that if function  $\beta$  is not required to satisfy condition (2.1), then the class of all such functions  $\psi_1$  coincides with the class of functions that are positive continuously differentiable on  $[1; +\infty)$ , increasing to  $+\infty$  and slowly varying. On the other hand, the condition  $\psi_2(2r) \sim \psi_2(r)$  as  $r \rightarrow +\infty$  implies  $\psi_1(2r) \sim \psi_1(r)$ , and the latter one implies  $\psi(2r) \sim \psi(r)$ .

**Theorem 1.** *Let  $u(z)$  be a subharmonic function in  $\mathbb{C}$ . If there exists  $\psi \in \Psi$  such that  $\liminf_{r \rightarrow +\infty} B(r, u)/\psi(r) \leq 1$ , then  $\forall \varepsilon > 0$  the inequality*

$$(2.2) \quad A(r, u) \geq B(r, u) - (1 + \varepsilon) \frac{\pi^2}{2} \psi_2(r)$$

holds outside a set  $E_\varepsilon$  of values  $r$  satisfying

$$(2.3) \quad \liminf_{r \rightarrow +\infty} \frac{1}{\psi_1(r)} \int_{E_\varepsilon \cap [1, r]} d\psi_1(t) \leq \frac{1}{1 + \varepsilon}.$$

In order to prove Theorem 1 we partially use Barry's method. Let  $\mu_u$  be a Riesz measure associated with a subharmonic function  $u$ ,  $\mu_u^*(t) = \mu_u(\{\zeta : |\zeta| \leq t\})$ . For  $R > 0$  we define the following functions:

$$u_1(z, R) = \int_{|\zeta| < R} \log \left| 1 - \frac{z}{\zeta} \right| d\mu_u(\zeta),$$

$$u_2(z, R) = \int_{|\zeta| < R} \log \left| 1 + \frac{z}{\zeta} \right| d\mu_u(\zeta) = \int_{(0, R)} \log \left| 1 + \frac{z}{t} \right| d\mu_u^*(t),$$

$$u_3(z, R) = u(z) - u_1(z, R).$$

One can write  $u_2(z, R) = \int_0^R \log \left| 1 + \frac{z}{t} \right| d\mu_u^*(t)$  for almost all  $R > 0$ , where the last integral is understood in the Stieltjes sense.

Put  $B_j(r, R) = \max\{u_j(z, R) : |z| = r\}$ ,  $A_j(r, R) = \min\{u_j(z, R) : |z| = r\}$ . It follows from the definitions of  $u_1$  and  $u_2$  that

$$(2.4) \quad A_2(r, R) \leq A_1(r, R) \leq B_1(r, R) \leq B_2(r, R).$$

Without loss of generality, we may assume that  $u(0) = 0$  and  $u(z)$  is harmonic in the unit disk with the center in the origin. We need the following lemma [8].

**Lemma.** *Suppose  $R > 2r$ . Then*

$$B_3(r, R) \leq \frac{4r}{R} B(2R, u), \quad -A_3(r, R) \leq \frac{4r}{R} B(2R, u).$$

The proof of a similar lemma can be found, for example, in [9], Lemma 6.1.

*Proof of Theorem 1.* We deduce from the definition of  $u_2(z, R)$  that

$$\begin{aligned} B_2(t, R) - A_2(t, R) &= \int_0^R \log \left| 1 + \frac{t}{s} \right| d\mu_u^*(s) - \int_0^R \log \left| 1 - \frac{t}{s} \right| d\mu_u^*(s) \\ &= \int_0^R \log \left| \frac{t+s}{t-s} \right| d\mu_u^*(s). \end{aligned}$$

Let us estimate the following integral using Fubini's theorem and the latter relation ( $0 < x < R$ ):

$$\begin{aligned}
 I(x, R) &= \int_0^x \frac{B_2(t, R) - A_2(t, R)}{t} dt \\
 &= \int_0^x \frac{dt}{t} \int_0^R \log \left| \frac{t+s}{t-s} \right| d\mu_u^*(s) \\
 &= \int_0^R d\mu_u^*(s) \int_0^{x/s} \log \left| \frac{1+\tau}{1-\tau} \right| \frac{d\tau}{\tau} \\
 &\leq \int_0^{+\infty} \log \left| \frac{1+\tau}{1-\tau} \right| \frac{d\tau}{\tau} \mu_u^*(R) = \frac{\pi^2}{2} \mu^*(R).
 \end{aligned}
 \tag{2.5}$$

It follows from the definitions of  $u_1, u_2, u_3$  and the lemma that

$$\begin{aligned}
 A(t, u) &\geq A_1(t, R) + A_3(t, R) \geq A_2(t, R) + A_3(t, R), \\
 B(t, u) &\leq B_2(t, R) + B_3(t, R).
 \end{aligned}$$

Therefore, for  $R > 2x$ , in accordance with Lemma and (2.5) we have

$$\begin{aligned}
 \int_0^x \frac{B(t, u) - A(t, u)}{t} dt &\leq \int_0^x \frac{B_2(t, R) - A_2(t, R)}{t} dt \\
 &+ \int_0^x \frac{B_3(t, R) - A_3(t, R)}{t} dt \leq I(x, R) + \frac{8x}{R} B(2R, u) \\
 &\leq \frac{\pi^2}{2} \mu_u^*(R) + \frac{8x}{R} B(2R, u).
 \end{aligned}
 \tag{2.6}$$

The conditions of the theorem imply that for arbitrary  $\eta > 0$  an arbitrary large  $r$  exists such that,  $B(r, u) < (1+\eta)\psi(r)$ , and, due to Jensen's formula for subharmonic functions [10, Chap. 4]

$$\int_0^r \frac{\mu_u^*(t)}{t} dt \leq B(r, u) < (1+\eta)\psi(r).$$

Define  $r^* = \max\{t \leq r : \mu_u^*(t) \leq (1+2\eta)\psi_1(t)\}$ . The value  $r^*$  is well defined, because, if we assume that there is no  $t$  satisfying  $\mu_u^*(t) \leq (1+2\eta)\psi_1(t)$ , then integrating the inverse inequality divided on  $t$  leads us to a contradiction with the choice of  $r$ .

Thus

$$(1+2\eta)(\psi(r) - \psi(r^*)) = (1+2\eta) \int_{r^*}^r \frac{\psi_1(t)}{t} dt \leq \int_{r^*}^r \frac{\mu_u^*(t)}{t} dt \leq (1+\eta)\psi(r).$$

From the last inequality we find that  $\psi(r^*) \geq \frac{\eta}{1+2\eta}\psi(r)$ . Therefore,

$$B(r^*, u) \leq B(r, u) \leq (1+\eta)\psi(r) \leq \frac{(1+\eta)(1+2\eta)}{\eta} \psi(r^*).$$

In particular, there exists an arbitrary large  $r^*$  which satisfies

$$\mu_u^*(r^*) \leq (1+2\eta)\psi_1(r^*), \quad B(r^*, u) \leq \frac{(1+\eta)(1+2\eta)}{\eta} \psi(r^*).
 \tag{2.7}$$

Choose  $R$  such that inequalities (2.7) hold with  $r^* = 2R$ . Then, from (2.6), using the slow variation of  $\psi_1$  and  $\psi$ , we obtain for sufficiently large  $R$

$$(2.8) \quad \int_0^x \frac{B(t, u) - A(t, u)}{t} dt \leq \frac{\pi^2}{2}(1 + 2\eta)\psi_1(2R) + \frac{8(1 + 2\eta)(1 + \eta)}{\eta} \frac{x}{R} \psi(2R) \\ \leq \frac{\pi^2}{2}(1 + 3\eta)\psi_1(R) + \frac{8(1 + 2\eta)^2}{\eta} \frac{x}{R} \psi(R).$$

In accordance with (2.1) there exists a constant  $K_2 \in (0; +\infty)$  such that  $\hat{\beta}(t) \leq (\beta(t))^{\frac{1}{K_2}}$  as  $t \geq t_0$ . Put  $\gamma(r) = \exp\{(\hat{\beta}(r))^{-\frac{1}{2}}\}$ . Since  $\hat{\beta}(r)$  is nonincreasing function and  $\hat{\beta}(r) \rightarrow 0$  as  $r \rightarrow +\infty$ ,  $\gamma(r)$  is a nondecreasing function, and  $\gamma(r) \rightarrow +\infty$  as  $r \rightarrow +\infty$ . Let  $x$  be defined by the equality  $x\gamma(x) = R$ . Then we find that

$$(2.9) \quad \frac{\psi_1(R)}{\psi_1(x)} = \exp\left\{\int_x^{x\gamma(x)} \frac{\beta(s)}{s} ds\right\} \leq \exp\left\{\int_x^{x\gamma(x)} \frac{\hat{\beta}(s)}{s} ds\right\} \\ \leq \exp\{\hat{\beta}(x) \log \gamma(x)\} = \exp\left\{\sqrt{\hat{\beta}(x)}\right\} \rightarrow 1, \quad x \rightarrow +\infty.$$

Using the slow variation of  $\psi(r)$ , it is easy to check that  $\psi(r)r^{-1/2} \downarrow 0$  as  $r \rightarrow +\infty$ . Therefore

$$(2.10) \quad \frac{x}{R} \psi(R) = \frac{1}{\gamma(x)} \psi(x\gamma(x)) \leq \psi(x)(\gamma(x))^{-\frac{1}{2}}.$$

Applying l'Hospital's rule we obtain

$$(2.11) \quad \lim_{x \rightarrow +\infty} \frac{\psi(x)}{\psi_1(x)\gamma(x)^{\frac{1}{2}}} = \lim_{x \rightarrow +\infty} \frac{\frac{\psi_1(x)}{x}}{\psi_1(x) \frac{\beta(x)}{x} \gamma^{\frac{1}{2}}(x) + \psi_1(x) ((\gamma(x))^{\frac{1}{2}})'} \\ \leq \lim_{x \rightarrow +\infty} \frac{1}{\beta(x)(\gamma(x))^{\frac{1}{2}}} \leq \lim_{x \rightarrow +\infty} \frac{1}{\hat{\beta}^{K_2}(x) \exp\left\{\frac{1}{2\sqrt{\hat{\beta}(x)}}\right\}} = 0.$$

Relations (2.10) and (2.11) imply

$$(2.12) \quad \frac{x}{R} \psi(R) \leq \frac{\psi(x)}{\gamma(x)^{\frac{1}{2}}} = o(\psi_1(x)),$$

on a sequence of values  $x \rightarrow +\infty$ . Combining (2.8), (2.9) and (2.12), we deduce

$$(2.13) \quad \int_0^x \frac{B(t, u) - A(t, u)}{t} dt \leq \frac{\pi^2}{2}(1 + 4\eta)\psi_1(x),$$

for an arbitrary large  $x$ .

Using the generalized l'Hospital rule, we get

$$(2.14) \quad \frac{\pi^2}{2}(1 + 4\eta) \geq \liminf_{x \rightarrow +\infty} \frac{1}{\psi_1(x)} \int_0^x \frac{B(t, u) - A(t, u)}{t} dt \\ \geq \liminf_{x \rightarrow +\infty} \frac{B(x, u) - A(x, u)}{\psi_2(x)}.$$

This means that on a sequence  $r_n \rightarrow +\infty$ ,

$$A(r_n, u) > B(r_n, u) - \frac{\pi^2}{2}(1 + 5\eta)\psi_2(r_n).$$

Let  $E_\varepsilon$  be the set of those  $r$ , for which (2.2) is not fulfilled. Then taking into account (2.13) we obtain

$$\begin{aligned} \frac{\pi^2}{2}(1 + 4\eta)\psi_1(x) &\geq \int_0^x \frac{B(t, u) - A(t, u)}{t} dt \geq \int_{[0, x] \cap E_\varepsilon} \frac{B(t, u) - A(t, u)}{t} dt \\ &\geq \int_{[0, x] \cap E_\varepsilon} \frac{\pi^2(1 + \varepsilon)\psi_2(t)}{2t} dt = \frac{\pi^2}{2}(1 + \varepsilon) \int_{[0, x] \cap E_\varepsilon} d\psi_1(t). \end{aligned}$$

Now, estimate (2.3) follows from the arbitrariness of  $\eta > 0$ . Theorem 1 is proved. □

**Corollary.** *Let  $f(z)$  be an entire function. If there exists  $\psi \in \Psi$  such that  $\liminf_{r \rightarrow +\infty} B(r, \log |f|)/\psi(r) \leq 1$ , then  $\forall \varepsilon > 0$*

$$(2.15) \quad A(r, \log |f|) \geq B(r, \log |f|) - \frac{\pi^2}{2}(1 + \varepsilon)\psi_2(r)$$

outside a set  $E_\varepsilon$  of values  $r$  satisfying (2.3).

### 3. AN EXAMPLE

Sharpness of inequality (2.2) in the class of subharmonic functions of zero order follows from Barry’s results [2]. However, inequality (2.15) can be improved when  $B(r, \log |f|) = O(\log^2 r)$  ( $r \rightarrow +\infty$ ) (see (1.3) and the following remark).

*Remark 1.* For entire functions  $f$  satisfying  $B(r, \log |f|) \leq \psi(r)$  ( $r \geq r_0$ ),  $\psi(r) = (\log^+ r)^{p+1}$ ,  $0 < p < 1$ , Barry [7] proved that

$$\forall \varepsilon \in (0, 1) \quad \frac{A(r_n, |f|)}{B(r_n, |f|)} > 1 - \exp\left\{-\frac{\varepsilon - 1}{2\psi_2(r_n)}\right\}, \quad r_n \rightarrow +\infty.$$

Using the Wiman-Valiron method, Barry also showed [7] that  $\varepsilon - 1$  cannot be replaced by  $-1$  in the last inequality, and estimated the set through which  $r_n$  may approach  $+\infty$ .

We construct an entire function of zero order, for which in (2.15) the asymptotic equality holds with  $\varepsilon = 0$  on some sequence of  $r = r_n \rightarrow +\infty$  as  $n \rightarrow +\infty$ .

**Theorem 2.** *Let  $\psi(r) = \frac{\sigma}{p+1}(\log^+ r)^{p+1}$ , ( $r \geq 0$ ),  $p > 0$ ,  $0 < \sigma < +\infty$ . There exists an entire function  $f$  such that  $\lim_{r \rightarrow +\infty} B(r, \log |f|)/\psi(r) = 1$ , and on a sequence  $r_n \rightarrow +\infty$*

$$A(r_n, \log |f|) = B(r_n, \log |f|) - \left(\frac{\pi^2}{2} + o(1)\right)\psi_2(r_n), \quad p > 1,$$

$$\frac{A(r_n, |f|)}{B(r_n, |f|)} = 1 - \exp\left\{-\frac{1 + o(1)}{2\psi_2(r_n)}\right\}, \quad 0 < p < 1,$$

$$\frac{A(r_n, |f|)}{B(r_n, |f|)} = C_2(\sigma/2) + o(1), \quad p = 1.$$

*Proof of Theorem 2.* In the notations of Theorem 2 we have

$$\psi_1(t) = \begin{cases} \sigma \log^p t, & t \geq 1, \\ 0, & 0 < t < 1. \end{cases}$$

Let  $\varphi_1(r) = \psi_1^{-1}(r) = \exp\left\{\left(\frac{r}{\sigma}\right)^{\frac{1}{p}}\right\}$  be the inverse to  $\psi_1$  function,  $x_n = \varphi(n)$ . We shall denote by  $[x]$  an integral part of  $x$  and by  $K$  with a subscript a positive constant depending only on  $p$  and  $\sigma$ . Define the entire function  $f(z)$  by the canonical product of zero genus

$$f(z) = \prod_{n=1}^{+\infty} \left(1 + \frac{z}{x_n}\right).$$

This representation becomes, for  $r \in (x_n, x_{n+1})$ ,

$$\begin{aligned} B(r, \log |f|) - A(r, \log |f|) &= \sum_{k=1}^{+\infty} \left(\log \left|1 + \frac{r}{x_k}\right| - \log \left|1 - \frac{r}{x_k}\right|\right) \\ (3.1) \qquad \qquad \qquad &= \left(\sum_{k=1}^n + \sum_{k=n+1}^{+\infty}\right) \log \left|\frac{\frac{r}{x_k} + 1}{\frac{r}{x_k} - 1}\right|. \end{aligned}$$

We choose  $r = r_n = \sqrt{x_n x_{n+1}}$ . Then

$$\begin{aligned} r_n &= \exp\left\{\frac{1}{2}\left(\left(\frac{n+1}{\sigma}\right)^{\frac{1}{p}} + \left(\frac{n}{\sigma}\right)^{\frac{1}{p}}\right)\right\} \\ (3.2) \qquad \qquad \qquad &= \exp\left\{\left(\frac{n}{\sigma}\right)^{\frac{1}{p}} + \frac{1}{2\sigma^{\frac{1}{p}} p n^{\frac{1}{q}}} + O\left(\frac{1}{n^{1+1/q}}\right)\right\}, \quad n \rightarrow +\infty, \end{aligned}$$

where  $q$  is defined by the equality  $p^{-1} + q^{-1} = 1$ . Let  $t_{-k} = \frac{r_n}{x_{n+1-k}}$  ( $k = 1, \dots, n$ ),  $t_k = \frac{x_{n+k}}{r_n}$  ( $k = 1, 2, \dots$ ). We need asymptotics for  $t_{\pm k}$  as  $k \rightarrow +\infty$ . For arbitrary  $\alpha \in (0, 1)$  and  $1 \leq k \leq n^\alpha$  using (3.2) we find that

$$\begin{aligned} \log t_k &= \left(\frac{n}{\sigma}\right)^{\frac{1}{p}} \left(\left(1 + \frac{k}{n}\right)^{\frac{1}{p}} - \left(1 + \frac{1}{2pn} + O\left(\frac{1}{n^2}\right)\right)\right) \\ (3.3) \qquad \qquad \qquad &= \frac{2k-1}{2p\sigma^{\frac{1}{p}} n^{\frac{1}{q}}} \left(1 + O\left(\frac{k}{n}\right)\right), \quad n \rightarrow +\infty. \end{aligned}$$

It is easy to see that relation (3.3) also holds with  $t_{-k}$  instead of  $t_k$  for the same  $k$ . For  $k > n^\alpha$  we obtain

$$\begin{aligned} \log t_k &= \left(\frac{n+k}{\sigma}\right)^{\frac{1}{p}} - \left(\frac{n}{\sigma}\right)^{\frac{1}{p}} (1 + o(1)) \\ &= \left(\frac{n+k}{\sigma}\right)^{\frac{1}{p}} \left(1 - \left(\frac{n}{n+k}\right)^{\frac{1}{p}} (1 + o(1))\right) \\ (3.4) \qquad \qquad \qquad &\geq \left(\frac{n+k}{\sigma}\right)^{\frac{1}{p}} \left(1 - \frac{1 + o(1)}{(1 + n^{\alpha-1})^{\frac{1}{p}}}\right) \\ &= (1 + o(1)) \left(\frac{n+k}{\sigma}\right)^{\frac{1}{p}} \frac{n^{\alpha-1}}{p} > \frac{(n+k)^{\alpha-\frac{1}{q}}}{2p\sigma^{\frac{1}{p}}}, \quad n \rightarrow +\infty, \end{aligned}$$

and for  $n^\alpha < k \leq n$

$$\begin{aligned} \log t_{-k} &= \left(\frac{n}{\sigma}\right)^{\frac{1}{p}} \left(1 + o(1) - \left(\frac{n+k-1}{n}\right)^{\frac{1}{p}}\right) \\ (3.4') \qquad \qquad \qquad &\geq \left(\frac{n}{\sigma}\right)^{\frac{1}{p}} \left(1 + o(1) - (1 - n^{\alpha-1})^{-\frac{1}{p}}\right) \\ &= (1 + o(1)) \left(\frac{n}{\sigma}\right)^{\frac{1}{p}} \frac{n^{\alpha-1}}{p} > \frac{(n+k)^{\alpha-\frac{1}{q}}}{2^{|\frac{1}{q}-\alpha|+1} p\sigma^{\frac{1}{p}}}, \quad n \rightarrow +\infty. \end{aligned}$$

Put  $\tau_k = \exp\left\{\frac{2k-1}{2p\sigma^{\frac{1}{p}}n^{\frac{1}{q}}}\right\}$  and fix an arbitrary positive  $\alpha$  from the interval  $(\frac{1}{q}, 1)$ . By a similar manner for  $\log \tau_k$  we have

$$\begin{aligned}\log \tau_k &= \frac{(1+o(1))k}{p\sigma^{\frac{1}{p}}n^{\frac{1}{q}}} > \frac{n^{\alpha-\frac{1}{q}}}{2p\sigma^{\frac{1}{p}}} > \frac{(n+k)n^{\alpha-\frac{1}{q}-1}}{4p\sigma^{\frac{1}{p}}}, \quad n^\alpha < k \leq n, \\ \log \tau_k &= \frac{(1+o(1))k}{p\sigma^{\frac{1}{p}}n^{\frac{1}{q}}} > \frac{(n+k)n^{-\frac{1}{q}}}{4p\sigma^{\frac{1}{p}}} \geq \frac{(n+k)n^{\alpha-\frac{1}{q}-1}}{4p\sigma^{\frac{1}{p}}}, \quad k > n.\end{aligned}$$

Since  $\log\left|\frac{1+t}{1-t}\right|$  is a decreasing function of  $t$  on  $(1, +\infty)$ , using (3.4), (3.4') and the latter inequalities from (3.1) we have

$$\begin{aligned}B(r, \log |f|) - A(r, \log |f|) &= \sum_{k=1}^n \log\left|\frac{t_{-k}+1}{t_{-k}-1}\right| + \sum_{k=1}^{+\infty} \log\left|\frac{t_k+1}{t_k-1}\right| \\ &= 2 \sum_{k=1}^{+\infty} \log\left|\frac{\tau_k+1}{\tau_k-1}\right| + \sum_{k=1}^n \left(\log\left|\frac{t_{-k}+1}{t_{-k}-1}\right| - \log\left|\frac{\tau_k+1}{\tau_k-1}\right|\right) \\ &\quad + \sum_{k=1}^{+\infty} \left(\log\left|\frac{t_k+1}{t_k-1}\right| - \log\left|\frac{\tau_k+1}{\tau_k-1}\right|\right) - \sum_{k=n+1}^{+\infty} \log\left|\frac{\tau_k+1}{\tau_k-1}\right| \\ &= 2 \sum_{k=1}^{+\infty} \log\left|\frac{\tau_k+1}{\tau_k-1}\right| + \sum_{k=1}^{[n^\alpha]} \left(\int_{\tau_k}^{t_k} + \int_{\tau_k}^{t_{-k}}\right) \frac{2 dx}{x^2-1} \\ (3.5) \quad &+ O\left(\sum_{k=[n^\alpha]+1}^{+\infty} \left(\int_{\tau_k}^{+\infty} + \int_{t_k}^{+\infty}\right) + \sum_{k=[n^\alpha]+1}^n \int_{t_{-k}}^{+\infty}\right) \frac{2 dx}{x^2-1} \\ &= \sum_1 + \sum_2 + O\left(\left(\sum_{k=[n^\alpha]+1}^{+\infty} \int_{\exp\{K_3(n+k)^{\alpha-\frac{1}{q}}\}}^{+\infty} \right. \right. \\ &\quad \left. \left. + \sum_{k=[n^\alpha]+1}^{+\infty} \int_{\exp\{K_3(n+k)n^{\alpha-\frac{1}{q}-1}\}}^{+\infty}\right) \frac{dx}{x^2}\right) \\ &= \sum_1 + \sum_2 + O(\sum_3 + \sum_4).\end{aligned}$$

Let us estimate the last sum in (3.5). Using l'Hospital's rule one can obtain

$$\begin{aligned}(3.6) \quad \sum_3 &\leq \sum_{k=[n^\alpha]+1}^{+\infty} \exp\{-K_3(n+k)^{\alpha-\frac{1}{q}}\} \leq \int_{n+n^\alpha}^{+\infty} \exp\{-K_3t^{\alpha-\frac{1}{q}}\} dt \\ &= O\left(\exp\{-K_3(n+n^\alpha)^{\alpha-\frac{1}{q}}\}(n+n^\alpha)^{1+\frac{1}{q}-\alpha}\right) \\ &= O(\exp\{-K_4n^{\alpha-\frac{1}{q}}\}), \quad n \rightarrow +\infty.\end{aligned}$$

Similarly,

$$\begin{aligned}
 \sum_4 &\leq \sum_{k=[n^\alpha]+1}^{+\infty} \exp\{-K_3(n+k)n^{\alpha-\frac{1}{q}-1}\} \\
 (3.7) \quad &\leq \exp\{-K_3n^{\alpha-\frac{1}{q}}\} \frac{\exp\{-K_3n^{2\alpha-\frac{1}{q}-1}\}}{1-\exp\{-K_3n^{\alpha-\frac{1}{q}-1}\}} \\
 &\leq \frac{\exp\{-K_3(n^{\alpha-\frac{1}{q}}+n^{2\alpha-\frac{1}{q}-1})\}}{K_3n^{-|\alpha-\frac{1}{q}-1|}} = O(\exp\{-K_4n^{\alpha-\frac{1}{q}}\}).
 \end{aligned}$$

$\sum_2$  is estimated dependent on a sign of  $q$ . First, we suppose that  $q > 1$ , i.e.  $p > 1$ . If  $k \leq n^{\frac{1}{q}}$ , then taking into account (3.3) and the inequality  $\log x \leq x-1 \leq K(x_0) \log x$  which holds for  $x \in [1, x_0]$  with some constant  $K(x_0) > 0$ , we find

$$\begin{aligned}
 (3.8) \quad \sum_{k=1}^{[n^{\frac{1}{q}}]} \left| \int_{\tau_k}^{t_k} \frac{2 dx}{x^2-1} \right| &\leq \sum_{k=1}^{[n^{\frac{1}{q}}]} \frac{|t_k - \tau_k|}{(1+o(1)) \log \tau_k} = (1+o(1)) \sum_{k=1}^{[n^{\frac{1}{q}}]} \frac{\tau_k \left| \frac{t_k}{\tau_k} - 1 \right|}{\log \tau_k} \\
 &= O\left( \sum_{k=1}^{[n^{\frac{1}{q}}]} \frac{\log \frac{t_k}{\tau_k}}{\log \tau_k} \right) = O\left( \sum_{k=1}^{[n^{\frac{1}{q}}]} \frac{k}{n} \right) = O(n^{\frac{1}{q}-\frac{1}{p}}).
 \end{aligned}$$

The definition of  $\tau_k$  and (3.3) yield and that  $\min\{\log \tau_k, \log t_k\} \geq K_5 > 0$  for  $n^{\frac{1}{q}} < k \leq n^\alpha$ . Therefore, using the mean value theorem, we obtain  $(\xi_k \in [\psi_1(t_k), \psi_1(\tau_k)])$

$$\begin{aligned}
 \sum_{k=[n^{\frac{1}{q}}]+1}^{[n^\alpha]} \left| \int_{\tau_k}^{t_k} \frac{2 dt}{t^2-1} \right| &= \sum_{k=[n^{\frac{1}{q}}]+1}^{[n^\alpha]} \left| \int_{\psi_1(\tau_k)}^{\psi_1(t_k)} \frac{2\varphi_1'(s)}{\varphi_1^2(s)-1} ds \right| \\
 &= \sum_{k=[n^{\frac{1}{q}}]+1}^{[n^\alpha]} \frac{2\varphi_1'(\xi_k)}{\varphi_1^2(\xi_k)-1} |\psi_1(t_k) - \psi_1(\tau_k)| \\
 (3.9) \quad &= O\left( \sum_{k=[n^{\frac{1}{q}}]+1}^{[n^\alpha]} \frac{|\log^p t_k - \log^p \tau_k|}{\xi_k^{\frac{1}{q}} \varphi_1(\xi_k)} \right) \\
 &= O\left( \sum_{k=[n^{\frac{1}{q}}]}^{[n^\alpha]} \frac{|\log \frac{t_k}{\tau_k}| \log^{p-1} \tau_k}{\psi_1(\tau_k)^{\frac{1}{q}} \varphi_1(\xi_k)} \right) \\
 &= O\left( \sum_{k=[n^{\frac{1}{q}}]}^{[n^\alpha]} \exp\left\{-\frac{(1+o(1))k}{p\sigma^{\frac{1}{p}} n^{\frac{1}{q}}}\right\} \frac{k^2}{n^{1+1/q}} \right) \\
 &= O\left( \frac{1}{n^{\frac{1}{p}}} \sum_{k=1}^{+\infty} \exp\left\{-K_6 \frac{k}{n^{\frac{1}{q}}}\right\} \right) = O\left( n^{-\frac{1}{p}} \frac{1}{1-\exp\{-K_6 n^{-\frac{1}{q}}\}} \right) \\
 &= O(n^{\frac{1}{q}-\frac{1}{p}}), \quad n \rightarrow +\infty.
 \end{aligned}$$

In a similar manner,  $\sum_{k=1}^{[n^\alpha]} \left| \int_{\tau_k}^{t-k} \right|$  can be estimated. Hence, (3.8) and (3.9) imply  $\Sigma_2 = O(n^{\frac{1}{q}-\frac{1}{p}})$  as  $n \rightarrow +\infty$  for  $q > 1$ .

Now, suppose that  $0 < p \leq 1$ , i.e.  $q < 0$  or  $q = \infty$ . Then there exists a positive constant  $K_7$  such that  $\tau_k, t_k, t_{-k}$  are greater than  $K_7$  for all  $k$ . Thus, proceeding in a similar fashion as in (3.9) we can see that  $(\zeta_k \in [\psi_1(\tau_k), \psi_1(t_k)])$

$$\begin{aligned}
 \sum_{k=1}^{[n^\alpha]} \left| \int_{\tau_k}^{t_k} \frac{2 dx}{x^2 - 1} \right| &= O\left(\sum_{k=1}^{[n^\alpha]} \frac{\varphi_1'(\zeta_k)}{\varphi_1^2(\zeta_k)} |\psi_1(t_k) - \psi_1(\tau_k)|\right) \\
 &= O\left(\sum_{k=1}^{[n^\alpha]} \frac{|\log^p t_k - \log^p \tau_k|}{\zeta_k^{1/q} \varphi_1(\zeta_k)}\right) \\
 &= O\left(\sum_{k=1}^{[n^\alpha]} \exp\left\{-(1 + o(1)) \frac{2k - 1}{2p\sigma^{\frac{1}{p}} n^{\frac{1}{q}}}\right\} \frac{k^2}{n^{1+1/q}}\right) \\
 (3.10) \quad &= O\left(\frac{1}{n^{\frac{1}{p}}} \sum_{k=1}^{+\infty} \exp\left\{-\frac{3k}{4p\sigma^{\frac{1}{p}} n^{\frac{1}{q}}}\right\}\right) \\
 &= O\left(\frac{n^{-\frac{1}{p}} \exp\left\{-\frac{3}{4p\sigma^{\frac{1}{p}} n^{\frac{1}{q}}}\right\}}{1 - \exp\{-K_7(\sigma, p)n^{-\frac{1}{q}}\}}\right) \\
 &= o\left(\exp\left\{-\frac{3}{4\sigma^{\frac{1}{p}} n^{\frac{1}{q}}}\right\}\right), \quad n \rightarrow +\infty.
 \end{aligned}$$

Recall that due to (3.3) estimate (3.10) also holds for  $\sum_{k=1}^{[n^\alpha]} |\int_{\tau_k}^{t_{-k}}|$ , so  $\sum_2 = o(\exp\{-\frac{3n^{-\frac{1}{q}}}{4p\sigma^{\frac{1}{p}}}\})$  as  $n \rightarrow +\infty$ . Therefore, using (3.6)–(3.10) one can obtain from (3.5) that

$$\begin{aligned}
 (3.11) \quad B(r_n, \log |f|) - A(r_n, \log |f|) &= \sum_{k=1}^{+\infty} \log \left| \frac{1 + \tau_k^{-1}}{1 - \tau_k^{-1}} \right| + \gamma(n) \\
 &= \log \prod_{k=1}^{+\infty} \left| \frac{h^{2k-1} + 1}{h^{2k-1} - 1} \right| + \gamma(n),
 \end{aligned}$$

where

$$h = \exp\left\{-\frac{1}{2p\sigma^{\frac{1}{p}} n^{\frac{1}{q}}}\right\}, \quad \gamma(n) = \begin{cases} O(n^{\frac{1}{q} - \frac{1}{p}}), & q > 1, \\ o\left(\exp\left\{-\frac{3}{4p\sigma^{\frac{1}{p}} n^{\frac{1}{q}}}\right\}\right), & q < 0 \text{ or } q = \infty. \end{cases}$$

Finally, it remains to compute the infinite product in (3.11). If  $p = 1$  ( $q = \infty$ ) we have [4]  $\prod_{k=1}^{+\infty} \left(\frac{h^{2k-1} - 1}{h^{2k-1} + 1}\right)^2 = C_2(\sigma^*)$ , ( $h = \exp\{-\frac{1}{4\sigma^*}\}$ ,  $\sigma = 2\sigma^*$ ). It is known ([11], Chap. IV) that

$$\prod_{k=1}^{+\infty} \left(\frac{h^{2k-1} - 1}{h^{2k-1} + 1}\right)^2 = \frac{\theta_0(0, h)}{\theta_3(0, h)},$$

where  $\theta_0, \theta_3$  are theta-functions. If we associate with  $h$  the value  $\tilde{k}$  of the modulus of a corresponding elliptic integral (see [11], p. 95), then  $(\theta_j(0|\tau) \equiv \theta_j(0, h))$ .

$$(3.12) \quad \tilde{k}^2 = \frac{\theta_2^4(0|\tau)}{\theta_3^4(0|\tau)} = 1 - \frac{\theta_0^4(0|\tau)}{\theta_3^4(0|\tau)} = \left\{ \frac{2h^{\frac{1}{4}} + 2h^{\frac{9}{4}} + \dots}{1 + 2h + 2h^4 + \dots} \right\}^4,$$

where  $h = e^{\pi i\tau}$ . It follows from the latter formula (see also [11], p. 118) that  $\tilde{k}^2 \sim 16h$  as  $h \downarrow 0$ , which is equivalent to

$$(3.13) \quad \tilde{k}'^2 \sim 16h', \quad h' \downarrow 0,$$

where  $\tilde{k}'$  is a complementative modulus ( $\tilde{k}'^2 + \tilde{k}^2 = 1$ ,  $h' = e^{\pi i\tau'}$ ,  $\tau' = -\frac{1}{\tau}$ ). The connections between  $h$ ,  $h'$  and  $\tilde{k}$ ,  $\tilde{k}'$  yield that  $\tilde{k}' \rightarrow 1$  as  $h \uparrow 1$ . Indeed,  $h' = \exp\left\{-\frac{i\pi}{\tau}\right\} = \exp\left\{\frac{\pi^2}{\log h}\right\}$ , so  $k' \rightarrow 0$  as  $h \uparrow 1$ . Hence, using (3.12) and (3.13) we obtain

$$\frac{\theta_0(0, h)}{\theta_3(0, h)} = \sqrt[4]{1 - \tilde{k}^2} \sim \sqrt[4]{16h'} = 2 \exp\left\{\frac{\pi^2}{4 \log h}\right\}, \quad h \uparrow 1.$$

Then if  $q > 1$ ,

$$\begin{aligned} \log \prod_{k=1}^{+\infty} \left(\frac{h^{2k-1} + 1}{h^{2k-1} - 1}\right)^2 &= -\log\left((2 + o(1)) \exp\left\{\frac{\pi^2}{4 \log h}\right\}\right) \\ &= -\log 2 + o(1) - \frac{\pi^2}{4 \log h} = -\log 2 + o(1) + \frac{\pi^2}{2} p\sigma^{\frac{1}{p}} n^{\frac{1}{q}}, \quad n \rightarrow +\infty. \end{aligned}$$

But  $p\sigma^{\frac{1}{p}} n^{\frac{1}{q}} = \sigma p \left(\frac{n}{\sigma}\right)^{\frac{p-1}{p}} \sim \sigma p \log^{p-1} r_n = \psi_2(r_n) \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Therefore, according to (3.11) for  $q > 1$  we have

$$\begin{aligned} B(r_n, \log |f|) - A(r_n, \log |f|) &= \left(\frac{\pi^2}{2} + o(1)\right) \psi_2(r_n), \quad n \rightarrow +\infty, \\ B(r, \log |f|) &\sim N(r, 0, f) \sim \frac{\sigma}{p+1} \log^{p+1} r, \quad r \rightarrow +\infty. \end{aligned}$$

If  $0 < p < 1$  ( $q < 0$ ), then  $\psi_2(r_n) \rightarrow 0$  and  $h = \exp\left\{-\frac{1+o(1)}{2\psi_2(r_n)}\right\} \rightarrow 0$  as  $n \rightarrow +\infty$ . In this case

$$\gamma(n) = o\left(\exp\left\{\frac{-3}{4\sigma^{\frac{1}{p}} p n^{\frac{1}{q}}}\right\}\right) = o\left(\exp\left\{-\frac{1}{2\psi_2(r_n)}\right\}\right), \quad n \rightarrow +\infty.$$

Consequently, taking into account (3.11) as in the previous case,

$$\begin{aligned} \frac{A(r_n, |f|)}{B(r_n, |f|)} &= \sqrt[4]{1 - \tilde{k}^2} e^{-\gamma(n)} = \sqrt[4]{1 - 16h(1 + o(1))} (1 - \gamma(n)(1 + o(1))) \\ &= (1 - 4h(1 + o(1)))(1 - \gamma(n)(1 + o(1))) \\ &= 1 - \exp\left\{-\frac{1 + o(1)}{2\psi_2(r_n)}\right\}, \quad n \rightarrow +\infty. \end{aligned}$$

Theorem 2 is proved. □

*Remark 2.* In [9] (Chapter 6) constructing an example which showed sharpness of inequality (2.15) the author approximated a subharmonic function by the logarithm of modulus of an entire function. But sharpness of (2.15) was shown only under the restriction  $B(r, \log |f|)/\log^{5/2} r \rightarrow +\infty$ .

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