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AN ADDITION TO THE $\cos \pi\rho$-THEOREM FOR SUBHARMONIC AND ENTIRE FUNCTIONS OF ZERO LOWER ORDER

I. E. CHYZHYKOV

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Abstract. We obtain a sharp asymptotic relation between the infimum and the maximum on a circle of a subharmonic function of zero lower order. An example is constructed, which shows the sharpness of the relation in the class of entire functions of zero order such that $\log M(r,f)/\log^2 r \to +\infty$, where $M(r,f) = \max\{|f(z)| : |z| = r\}$ as $r \to +\infty$.

1. Introduction

Let $u(z)$ be a subharmonic function in $\mathbb{C}$. We denote $A(r,u) = \inf\{u(z) : |z| = r\}$, $B(r,u) = \max\{u(z) : |z| = r\}$, $r > 0$. The classical $\cos \pi\rho$-theorem asserts that if $f(z)$ is an entire function of finite order $\rho \in [0;1]$, then

$$A(r_n, \log|f|) > (C_1(\rho) + o(1))B(r_n, \log|f|), \tag{1.1}$$

where $C_1(\rho) = \cos \pi\rho$, $r_n \to +\infty$ as $n \to +\infty$. It is known ([1], Chapter 6) that one can replace $\log|f|$ by an arbitrary subharmonic in the plane function u of lower order $\lambda \le 1$ in (1.1). In this case, $C_1(\rho)$ must be replaced by $C_1(\lambda)$, and the constant is sharp. Hence, if $\lambda = 0$, then (1.1) implies $A(r_n, u) \sim B(r_n, u)$ as $r_n \to +\infty$. Naturally, the question on improvement of $\cos \pi\rho$-theorem in the case of subharmonic or entire functions of zero order arises.

In 1962 P. Barry [2] showed that for an arbitrary subharmonic function u the inequality $B(r,u) \le \psi(r)$ $(r \ge r_0)$ implies

$$(\forall \varepsilon > 0) \quad A(r_n, u) > B(r_n, u) - (1+\varepsilon)\frac{\pi^2}{2}\psi_2(r_n), \quad r_n \to +\infty, \tag{1.2}$$

where $\psi \in C^2(\mathbb{R}_+)$ satisfies the following conditions: $\psi_2(r)$ is continuous and slowly varying on $[1, +\infty)$ (i.e. $\psi_2(2r) \sim \psi_2(r)$ as $r \to +\infty$), here $\psi_j(r) = \frac{d^j \psi(r)}{(d \log r)^j}$, $j = 1, 2$. On the other hand, in 1980 P. Fenton [3] proved that for subharmonic function u the condition $\liminf\limits_{r\to+\infty} B(r,u)/\log^{p+1} r = \sigma < +\infty$ implies for $p > 0$ that $\forall \varepsilon > 0$ the inequality

$$A(r,u) > B(r,u) - (\sigma + \varepsilon)\,\mathrm{Re}\{\log^{p+1} r - (\log r + i\pi)^{p+1}\}$$

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holds for all r outside a set E satisfying

$$\liminf_{r\to+\infty}\frac{1}{\log^p r}\int_{E\cap[1,r]}\frac{p\log^{p-1}t}{t}dt\le\frac{\sigma}{\sigma+\varepsilon}.$$

We remark that

$$\begin{aligned}\operatorname{Re}\{\log^{p+1}r-(\log r+i\pi)^{p+1}\}&=(1+o(1))\frac{\pi^2(p+1)p}{2}\log^{p-1}r\\&=(1+o(1))\frac{\pi^2}{2}\frac{d^2\log^{p+1}r}{(d\log r)^2},\quad r\to+\infty.\end{aligned}$$

In the case $\psi(r)=\sigma(\log^+ r)^2$ $(r\ge 0)$ A. A. Gol'dberg [4] and P. Fenton [5, 6] showed that

$$\limsup_{r\to+\infty}\frac{A(r,|f|)}{B(r,|f|)}\ge C_2(\sigma)=\Big(\prod_{n=1}^{+\infty}\frac{1-q^{2n-1}}{1+q^{2n-1}}\Big)^2,\tag{1.3}$$

where $q=e^{-\frac{1}{4\sigma}}$, $C_2(0)=1$, for every entire function [6] such that $\liminf\limits_{r\to+\infty}\frac{B(r,\log|f|)}{\log^2 r}=\sigma<+\infty$ (for every meromorphic function [4] such that $\liminf\limits_{r\to+\infty}N(r,0,\infty,f)\log^{-2}r\le\sigma<+\infty$), here $N(r,0,\infty,f)=N(r,0,f)+N(r,\infty,f)$ is Nevanlinna's counting function of zeros and poles. The constant $C_2(\sigma)$ in (1.3) is sharp [2], [4].

In 1982 P. Barry [7] established a sharp lower estimate for the remainder $A(r,u)-B(r,u)$ in the case $u=\log|f|$, f is an entire function with $B(r,\log|f|)=O(\log^{p+1}r)$ as $r\to+\infty$, $0<p<1$. He also remarked, that a sharp estimate for $p>1$ in the class on entire function was unknown. In this connection the following questions arise:

1) Is inequality (1.2) valid for subharmonic functions of zero lower order such that $\liminf\limits_{r\to+\infty}\frac{B(r,u)}{\psi(r)}\le 1$?
2) What is a sharp inequality between $A(r,\log|f|)$ and $B(r,\log|f|)$ in the class of entire functions such that $B(r,\log|f|)/\log^2 r\to+\infty$ as $r\to+\infty$?

In this paper we give answers to both questions.

2. Improving the $\cos\pi\rho$-theorem

Let us introduce the class Ψ of functions $\psi\colon[1;+\infty)\to\mathbb{R}_+$ such that

$$\psi_1(r)=\frac{d\psi(r)}{d\log r}=K_1\exp\Big\{\int_1^r\frac{\beta(s)}{s}ds\Big\},$$

where $K_1>0$ is a constant, $\beta\in C(\mathbb{R}_+)$, $\beta(r)>0$, $\beta(r)\to 0$ as $r\to+\infty$, $\int_1^{+\infty}\beta(s)s^{-1}\,ds$ is divergent and

$$\log\frac{1}{\beta(r)}=O\Big(\log\frac{1}{\hat\beta(r)}\Big),\quad r\to+\infty,\tag{2.1}$$

where $\hat\beta(r)=\max\{\beta(s):s\ge r\}$. We remark that if function β is not required to satisfy condition (2.1), then the class of all such functions ψ_1 coincides with the class of functions that are positive continuously differentiable on $[1;+\infty)$, increasing to $+\infty$ and slowly varying. On the other hand, the condition $\psi_2(2r)\sim\psi_2(r)$ as $r\to+\infty$ implies $\psi_1(2r)\sim\psi_1(r)$, and the latter one implies $\psi(2r)\sim\psi(r)$.

Theorem 1. *Let $u(z)$ be a subharmonic function in $\mathbb{C}$. If there exists $\psi \in \Psi$ such that $\liminf_{r\to+\infty} B(r,u)/\psi(r) \le 1$, then $\forall \varepsilon > 0$ the inequality*

$$A(r,u) \ge B(r,u) - (1+\varepsilon)\frac{\pi^2}{2}\psi_2(r) \tag{2.2}$$

holds outside a set E_ε of values r satisfying

$$\liminf_{r\to+\infty} \frac{1}{\psi_1(r)} \int_{E_\varepsilon \cap [1,r]} d\psi_1(t) \le \frac{1}{1+\varepsilon}. \tag{2.3}$$

In order to prove Theorem 1 we partially use Barry's method. Let μ_u be a Riesz measure associated with a subharmonic function u, $\mu_u^*(t) = \mu_u(\{\zeta : |\zeta| \le t\})$. For $R > 0$ we define the following functions:

$$u_1(z,R) = \int_{|\zeta|<R} \log\Big|1 - \frac{z}{\zeta}\Big| d\mu_u(\zeta),$$

$$u_2(z,R) = \int_{|\zeta|<R} \log\Big|1 + \frac{z}{|\zeta|}\Big| d\mu_u(\zeta) = \int_{[0,R)} \log\Big|1 + \frac{z}{t}\Big| d\mu_u^*(t),$$

$$u_3(z,R) = u(z) - u_1(z,R).$$

One can write $u_2(z,R) = \int_0^R \log\Big|1 + \frac{z}{t}\Big| d\mu_u^*(t)$ for almost all $R > 0$, where the last integral is understood in the Stieltjes sense.

Put $B_j(r,R) = \max\{u_j(z,R) : |z| = r\}$, $A_j(r,R) = \inf\{u_j(z,R) : |z| = r\}$. It follows from the definitions of u_1 and u_2 that

$$A_2(r,R) \le A_1(r,R) \le B_1(r,R) \le B_2(r,R). \tag{2.4}$$

Without loss of generality, we may assume that $u(0) = 0$ and $u(z)$ is harmonic in the unit disk with the center in the origin. We need the following lemma [8].

Lemma. *Suppose $R > 2r$. Then*

$$B_3(r,R) \le \frac{4r}{R} B(2R,u), \quad -A_3(r,R) \le \frac{4r}{R} B(2R,u).$$

The proof of a similar lemma can be found, for example, in [9], Lemma 6.1.

Proof of Theorem 1. We deduce from the definition of $u_2(z,R)$ that

$$\begin{aligned} B_2(t,R) - A_2(t,R) &= \int_0^R \log\Big|1 + \frac{t}{s}\Big| d\mu_u^*(s) - \int_0^R \log\Big|1 - \frac{t}{s}\Big| d\mu_u^*(s) \\ &= \int_0^R \log\Big|\frac{t+s}{t-s}\Big| d\mu_u^*(s). \end{aligned}$$

Let us estimate the following integral using Fubini's theorem and the latter relation $(0 < x < R)$:

$$
\begin{aligned}
I(x,R) &= \int_0^x \frac{B_2(t,R) - A_2(t,R)}{t}\,dt \\
&= \int_0^x \frac{dt}{t} \int_0^R \log\Big|\frac{t+s}{t-s}\Big| d\mu_u^*(s) \\
&= \int_0^R d\mu_u^*(s) \int_0^{x/s} \log\Big|\frac{1+\tau}{1-\tau}\Big| \frac{d\tau}{\tau} \\
&\le \int_0^{+\infty} \log\Big|\frac{1+\tau}{1-\tau}\Big| \frac{d\tau}{\tau} \mu_u^*(R) = \frac{\pi^2}{2}\mu^*(R).
\end{aligned} \tag{2.5}
$$

It follows from the definitions of u_1, u_2, u_3 and the lemma that

$$
\begin{gathered}
A(t,u) \ge A_1(t,R) + A_3(t,R) \ge A_2(t,R) + A_3(t,R), \\
B(t,u) \le B_2(t,R) + B_3(t,R).
\end{gathered}
$$

Therefore, for $R > 2x$, in accordance with Lemma and (2.5) we have

$$
\begin{aligned}
\int_0^x \frac{B(t,u) - A(t,u)}{t} dt &\le \int_0^x \frac{B_2(t,R) - A_2(t,R)}{t} dt \\
&+ \int_0^x \frac{B_3(t,R) - A_3(t,R)}{t} dt \le I(x,R) + \frac{8x}{R} B(2R,u) \\
&\le \frac{\pi^2}{2}\mu_u^*(R) + \frac{8x}{R} B(2R,u).
\end{aligned} \tag{2.6}
$$

The conditions of the theorem imply that for arbitrary $\eta > 0$ an arbitrary large r exists such that, $B(r,u) < (1+\eta)\psi(r)$, and, due to Jensen's formula for subharmonic functions [10, Chap. 4]

$$
\int_0^r \frac{\mu_u^*(t)}{t} dt \le B(r,u) < (1+\eta)\psi(r).
$$

Define $r^* = \max\{t \le r : \mu_u^*(t) \le (1+2\eta)\psi_1(t)\}$. The value r^* is well defined, because, if we assume that there is no t satisfying $\mu_u^*(t) \le (1+2\eta)\psi_1(t)$, then integrating the inverse inequality divided on t leads us to a contradiction with the choice of r.

Thus

$$
(1+2\eta)(\psi(r) - \psi(r^*)) = (1+2\eta) \int_{r^*}^r \frac{\psi_1(t)}{t} dt \le \int_{r^*}^r \frac{\mu_u^*(t)}{t} dt \le (1+\eta)\psi(r).
$$

From the last inequality we find that $\psi(r^*) \ge \frac{\eta}{1+2\eta}\psi(r)$. Therefore,

$$
B(r^*,u) \le B(r,u) \le (1+\eta)\psi(r) \le \frac{(1+\eta)(1+2\eta)}{\eta}\psi(r^*).
$$

In particular, there exists an arbitrary large r^* which satisfies

$$
\mu_u^*(r^*) \le (1+2\eta)\psi_1(r^*), \quad B(r^*,u) \le \frac{(1+\eta)(1+2\eta)}{\eta}\psi(r^*). \tag{2.7}
$$

Choose R such that inequalities (2.7) hold with $r^* = 2R$. Then, from (2.6), using the slow variation of ψ_1 and ψ, we obtain for sufficiently large R

$$\begin{aligned}\int_0^x \frac{B(t,u)-A(t,u)}{t}dt &\le \frac{\pi^2}{2}(1+2\eta)\psi_1(2R) + \frac{8(1+2\eta)(1+\eta)}{\eta}\frac{x}{R}\psi(2R)\\ &\le \frac{\pi^2}{2}(1+3\eta)\psi_1(R) + \frac{8(1+2\eta)^2}{\eta}\frac{x}{R}\psi(R).\end{aligned} \tag{2.8}$$

In accordance with (2.1) there exists a constant $K_2 \in (0;+\infty)$ such that $\hat\beta(t) \le (\beta(t))^{\frac{1}{K_2}}$ as $t \ge t_0$. Put $\gamma(r) = \exp\Big\{(\hat\beta(r))^{-\frac{1}{2}}\Big\}$. Since $\hat\beta(r)$ is nonincreasing function and $\hat\beta(r) \to 0$ as $r \to +\infty$, $\gamma(r)$ is a nondecreasing function, and $\gamma(r) \to +\infty$ as $r \to +\infty$. Let x be defined by the equality $x\gamma(x) = R$. Then we find that

$$\begin{aligned}\frac{\psi_1(R)}{\psi_1(x)} &= \exp\Big\{\int_x^{x\gamma(x)} \frac{\beta(s)}{s}ds\Big\} \le \exp\Big\{\int_x^{x\gamma(x)} \frac{\hat\beta(s)}{s}ds\Big\}\\ &\le \exp\{\hat\beta(x)\log\gamma(x)\} = \exp\Big\{\sqrt{\hat\beta(x)}\Big\} \to 1, \quad x \to +\infty.\end{aligned} \tag{2.9}$$

Using the slow variation of $\psi(r)$, it is easy to check that $\psi(r)r^{-1/2} \downarrow 0$ as $r \to +\infty$. Therefore

$$\frac{x}{R}\psi(R) = \frac{1}{\gamma(x)}\psi(x\gamma(x)) \le \psi(x)(\gamma(x))^{-\frac{1}{2}}. \tag{2.10}$$

Applying l'Hospital's rule we obtain

$$\begin{aligned}\lim_{x\to+\infty} \frac{\psi(x)}{\psi_1(x)\gamma(x)^{\frac{1}{2}}} &= \lim_{x\to+\infty} \frac{\frac{\psi_1(x)}{x}}{\psi_1(x)\frac{\beta(x)}{x}\gamma^{\frac{1}{2}}(x) + \psi_1(x)((\gamma(x))^{\frac{1}{2}})'}\\ &\le \lim_{x\to+\infty} \frac{1}{\beta(x)(\gamma(x))^{\frac{1}{2}}} \le \lim_{x\to+\infty} \frac{1}{\hat\beta^{K_2}(x)\exp\big\{\frac{1}{2\sqrt{\hat\beta(x)}}\big\}} = 0.\end{aligned} \tag{2.11}$$

Relations (2.10) and (2.11) imply

$$\frac{x}{R}\psi(R) \le \frac{\psi(x)}{\gamma(x)^{\frac{1}{2}}} = o(\psi_1(x)), \tag{2.12}$$

on a sequence of values $x \to +\infty$. Combining (2.8), (2.9) and (2.12), we deduce

$$\int_0^x \frac{B(t,u)-A(t,u)}{t}dt \le \frac{\pi^2}{2}(1+4\eta)\psi_1(x), \tag{2.13}$$

for an arbitrary large x.

Using the generalized l'Hospital rule, we get

$$\begin{aligned}\frac{\pi^2}{2}(1+4\eta) &\ge \liminf_{x\to+\infty} \frac{1}{\psi_1(x)}\int_0^x \frac{B(t,u)-A(t,u)}{t}dt\\ &\ge \liminf_{x\to+\infty} \frac{B(x,u)-A(x,u)}{\psi_2(x)}.\end{aligned} \tag{2.14}$$

This means that on a sequence $r_n \to +\infty$,

$$A(r_n,u) > B(r_n,u) - \frac{\pi^2}{2}(1+5\eta)\psi_2(r_n).$$

Let E_ε be the set of those r, for which (2.2) is not fulfilled. Then taking into account (2.13) we obtain

$$\frac{\pi^2}{2}(1+4\eta)\psi_1(x) \geq \int_0^x \frac{B(t,u)-A(t,u)}{t}dt \geq \int_{[0,x]\cap E_\varepsilon} \frac{B(t,u)-A(t,u)}{t}dt$$
$$\geq \int_{[0,x]\cap E_\varepsilon} \frac{\pi^2}{2}\frac{(1+\varepsilon)\psi_2(t)}{t}dt = \frac{\pi^2}{2}(1+\varepsilon)\int_{[0,x]\cap E_\varepsilon} d\psi_1(t).$$

Now, estimate (2.3) follows from the arbitrariness of $\eta > 0$. Theorem 1 is proved. □

Corollary. *Let $f(z)$ be an entire function. If there exists $\psi \in \Psi$ such that $\liminf_{r\to+\infty} B(r,\log|f|)/\psi(r) \leq 1$, then $\forall\varepsilon > 0$*

$$A(r,\log|f|) \geq B(r,\log|f|) - \frac{\pi^2}{2}(1+\varepsilon)\psi_2(r) \tag{2.15}$$

outside a set E_ε of values r satisfying (2.3).

3. An example

Sharpness of inequality (2.2) in the class of subharmonic functions of zero order follows from Barry's results [2]. However, inequality (2.15) can be improved when $B(r,\log|f|) = O(\log^2 r)$ $(r \to +\infty)$ (see (1.3) and the following remark).

Remark 1. For entire functions f satisfying $B(r,\log|f|) \leq \psi(r)$ $(r \geq r_0)$, $\psi(r) = (\log^+ r)^{p+1}$, $0 < p < 1$, Barry [7] proved that

$$\forall\varepsilon \in (0,1) \quad \frac{A(r_n,|f|)}{B(r_n,|f|)} > 1 - \exp\Big\{\frac{\varepsilon-1}{2\psi_2(r_n)}\Big\}, \quad r_n \to +\infty.$$

Using the Wiman-Valiron method, Barry also showed [7] that $\varepsilon - 1$ cannot be replaced by -1 in the last inequality, and estimated the set through which r_n may approach $+\infty$.

We construct an entire function of zero order, for which in (2.15) the asymptotic equality holds with $\varepsilon = 0$ on some sequence of $r = r_n \to +\infty$ as $n \to +\infty$.

Theorem 2. *Let $\psi(r) = \frac{\sigma}{p+1}(\log^+ r)^{p+1}$, $(r \geq 0)$, $p > 0$, $0 < \sigma < +\infty$. There exists an entire function f such that $\lim_{r\to+\infty} B(r,\log|f|)/\psi(r) = 1$, and on a sequence $r_n \to +\infty$*

$$A(r_n,\log|f|) = B(r_n,\log|f|) - \Big(\frac{\pi^2}{2} + o(1)\Big)\psi_2(r_n), \quad p > 1,$$

$$\frac{A(r_n,|f|)}{B(r_n,|f|)} = 1 - \exp\Big\{-\frac{1+o(1)}{2\psi_2(r_n)}\Big\}, \quad 0 < p < 1,$$

$$\frac{A(r_n,|f|)}{B(r_n,|f|)} = C_2(\sigma/2) + o(1), \quad p = 1.$$

Proof of Theorem 2. In the notations of Theorem 2 we have

$$\psi_1(t) = \begin{cases} \sigma\log^p t, & t \geq 1, \\ 0, & 0 < t < 1. \end{cases}$$

Let $\varphi_1(r) = \psi_1^{-1}(r) = \exp\{(\frac{r}{\sigma})^{\frac{1}{p}}\}$ be the inverse to ψ_1 function, $x_n = \varphi(n)$. We shall denote by $[x]$ an integral part of x and by K with a subscript a positive constant depending only on p and σ. Define the entire function $f(z)$ by the canonical product of zero genus

$$f(z) = \prod_{n=1}^{+\infty}\Big(1 + \frac{z}{x_n}\Big).$$

This representation becomes, for $r \in (x_n, x_{n+1})$,

$$\begin{aligned} B(r, \log|f|) - A(r, \log|f|) &= \sum_{k=1}^{+\infty}\Big(\log\Big|1 + \frac{r}{x_k}\Big| - \log\Big|1 - \frac{r}{x_k}\Big|\Big) \\ &= \Big(\sum_{k=1}^{n} + \sum_{k=n+1}^{+\infty}\Big)\log\Big|\frac{\frac{r}{x_k}+1}{\frac{r}{x_k}-1}\Big|. \end{aligned} \tag{3.1}$$

We choose $r = r_n = \sqrt{x_n x_{n+1}}$. Then

$$\begin{aligned} r_n &= \exp\Big\{\frac{1}{2}\Big(\Big(\frac{n+1}{\sigma}\Big)^{\frac{1}{p}} + \Big(\frac{n}{\sigma}\Big)^{\frac{1}{p}}\Big)\Big\} \\ &= \exp\Big\{\Big(\frac{n}{\sigma}\Big)^{\frac{1}{p}} + \frac{1}{2\sigma^{\frac{1}{p}} p n^{\frac{1}{q}}} + O\Big(\frac{1}{n^{1+1/q}}\Big)\Big\},\ n \to +\infty, \end{aligned} \tag{3.2}$$

where q is defined by the equality $p^{-1} + q^{-1} = 1$. Let $t_{-k} = \frac{r_n}{x_{n+1-k}}$ $(k = 1, \dots, n)$, $t_k = \frac{x_{n+k}}{r_n}$ $(k = 1, 2, \dots)$. We need asymptotics for $t_{\pm k}$ as $k \to +\infty$. For arbitrary $\alpha \in (0, 1)$ and $1 \le k \le n^\alpha$ using (3.2) we find that

$$\begin{aligned} \log t_k &= \Big(\frac{n}{\sigma}\Big)^{\frac{1}{p}}\Big(\Big(1 + \frac{k}{n}\Big)^{\frac{1}{p}} - \Big(1 + \frac{1}{2pn} + O\Big(\frac{1}{n^2}\Big)\Big)\Big) \\ &= \frac{2k-1}{2p\sigma^{\frac{1}{p}} n^{\frac{1}{q}}}\Big(1 + O\Big(\frac{k}{n}\Big)\Big), \quad n \to +\infty. \end{aligned} \tag{3.3}$$

It is easy to see that relation (3.3) also holds with t_{-k} instead of t_k for the same k.

For $k > n^\alpha$ we obtain

$$\begin{aligned} \log t_k &= \Big(\frac{n+k}{\sigma}\Big)^{\frac{1}{p}} - \Big(\frac{n}{\sigma}\Big)^{\frac{1}{p}}(1 + o(1)) \\ &= \Big(\frac{n+k}{\sigma}\Big)^{\frac{1}{p}}\Big(1 - \Big(\frac{n}{n+k}\Big)^{\frac{1}{p}}(1 + o(1))\Big) \\ &\ge \Big(\frac{n+k}{\sigma}\Big)^{\frac{1}{p}}\Big(1 - \frac{1 + o(1)}{(1 + n^{\alpha-1})^{\frac{1}{p}}}\Big) \\ &= (1 + o(1))\Big(\frac{n+k}{\sigma}\Big)^{\frac{1}{p}}\frac{n^{\alpha-1}}{p} > \frac{(n+k)^{\alpha-\frac{1}{q}}}{2p\sigma^{\frac{1}{p}}}, \quad n \to +\infty, \end{aligned} \tag{3.4}$$

and for $n^\alpha < k \le n$

$$\begin{aligned} \log t_{-k} &= \Big(\frac{n}{\sigma}\Big)^{\frac{1}{p}}\Big(1 + o(1) - \Big(\frac{n+k-1}{n}\Big)^{\frac{1}{p}}\Big) \\ &\ge \Big(\frac{n}{\sigma}\Big)^{\frac{1}{p}}\Big(1 + o(1) - (1 - n^{\alpha-1})^{-\frac{1}{p}}\Big) \\ &= (1 + o(1))\Big(\frac{n}{\sigma}\Big)^{\frac{1}{p}}\frac{n^{\alpha-1}}{p} > \frac{(n+k)^{\alpha-\frac{1}{q}}}{2^{|\frac{1}{q}-\alpha|+1}p\sigma^{\frac{1}{p}}}, \quad n \to +\infty. \end{aligned} \tag{3.4$'$}$$

Put $\tau_k = \exp\Big\{\frac{2k-1}{2p\sigma^{\frac{1}{p}}n^{\frac{1}{q}}}\Big\}$ and fix an arbitrary positive α from the interval $(\frac{1}{q},1)$. By a similar manner for $\log\tau_k$ we have

$$\log\tau_k = \frac{(1+o(1))k}{p\sigma^{\frac{1}{p}}n^{\frac{1}{q}}} > \frac{n^{\alpha-\frac{1}{q}}}{2p\sigma^{\frac{1}{p}}} > \frac{(n+k)n^{\alpha-\frac{1}{q}-1}}{4p\sigma^{\frac{1}{p}}}, \quad n^\alpha < k \le n,$$
$$\log\tau_k = \frac{(1+o(1))k}{p\sigma^{\frac{1}{p}}n^{\frac{1}{q}}} > \frac{(n+k)n^{-\frac{1}{q}}}{4p\sigma^{\frac{1}{p}}} \ge \frac{(n+k)n^{\alpha-\frac{1}{q}-1}}{4p\sigma^{\frac{1}{p}}}, \quad k > n.$$

Since $\log\Big|\frac{1+t}{1-t}\Big|$ is a decreasing function of t on $(1,+\infty)$, using (3.4), (3.4′) and the latter inequalities from (3.1) we have

$$\begin{aligned}
B(r,\log|f|) - A(r,\log|f|) &= \sum_{k=1}^{n}\log\Big|\frac{t_{-k}+1}{t_{-k}-1}\Big| + \sum_{k=1}^{+\infty}\log\Big|\frac{t_k+1}{t_k-1}\Big| \\
&= 2\sum_{k=1}^{+\infty}\log\Big|\frac{\tau_k+1}{\tau_k-1}\Big| + \sum_{k=1}^{n}\Big(\log\Big|\frac{t_{-k}+1}{t_{-k}-1}\Big| - \log\Big|\frac{\tau_k+1}{\tau_k-1}\Big|\Big) \\
&\quad + \sum_{k=1}^{+\infty}\Big(\log\Big|\frac{t_k+1}{t_k-1}\Big| - \log\Big|\frac{\tau_k+1}{\tau_k-1}\Big|\Big) - \sum_{k=n+1}^{+\infty}\log\Big|\frac{\tau_k+1}{\tau_k-1}\Big| \\
&= 2\sum_{k=1}^{+\infty}\log\Big|\frac{\tau_k+1}{\tau_k-1}\Big| + \sum_{k=1}^{[n^\alpha]}\Big(\int_{\tau_k}^{t_k} + \int_{\tau_k}^{t_{-k}}\Big)\frac{2\,dx}{x^2-1} \\
&\quad + O\Big(\sum_{k=[n^\alpha]+1}^{+\infty}\Big(\int_{\tau_k}^{+\infty} + \int_{t_k}^{+\infty}\Big) + \sum_{k=[n^\alpha]+1}^{n}\int_{t_{-k}}^{+\infty}\Big)\frac{2\,dx}{x^2-1} \\
&= \sum\nolimits_1 + \sum\nolimits_2 + O\Big(\Big(\sum_{k=[n^\alpha]+1}^{+\infty}\int_{\exp\{K_3(n+k)^{\alpha-\frac{1}{q}}\}}^{+\infty} \\
&\qquad\qquad + \sum_{k=[n^\alpha]+1}^{+\infty}\int_{\exp\{K_3(n+k)n^{\alpha-\frac{1}{q}-1}\}}^{+\infty}\Big)\frac{dx}{x^2}\Big) \\
&= \sum\nolimits_1 + \sum\nolimits_2 + O\Big(\sum\nolimits_3 + \sum\nolimits_4\Big).
\end{aligned} \tag{3.5}$$

Let us estimate the last sum in (3.5). Using l'Hospital's rule one can obtain

$$\begin{aligned}
\sum\nolimits_3 &\le \sum_{k=[n^\alpha]+1}^{+\infty}\exp\{-K_3(n+k)^{\alpha-\frac{1}{q}}\} \le \int_{n+n^\alpha}^{+\infty}\exp\{-K_3t^{\alpha-\frac{1}{q}}\}\,dt \\
&= O\Big(\exp\{-K_3(n+n^\alpha)^{\alpha-\frac{1}{q}}\}(n+n^\alpha)^{1+\frac{1}{q}-\alpha}\Big) \\
&= O(\exp\{-K_4n^{\alpha-\frac{1}{q}}\}), \quad n\to+\infty.
\end{aligned} \tag{3.6}$$

Similarly,

$$
\begin{aligned}
\sum\nolimits_4 &\le \sum_{k=[n^\alpha]+1}^{+\infty} \exp\{-K_3(n+k)n^{\alpha-\frac{1}{q}-1}\} \\
&\le \exp\{-K_3 n^{\alpha-\frac{1}{q}}\}\frac{\exp\{-K_3 n^{2\alpha-\frac{1}{q}-1}\}}{1-\exp\{-K_3 n^{\alpha-\frac{1}{q}-1}\}} \\
&\le \frac{\exp\{-K_3(n^{\alpha-\frac{1}{q}}+n^{2\alpha-\frac{1}{q}-1})\}}{K_3 n^{-|\alpha-\frac{1}{q}-1|}} = O(\exp\{-K_4 n^{\alpha-\frac{1}{q}}\}).
\end{aligned} \tag{3.7}
$$

$\sum_2$ is estimated dependent on a sign of q. First, we suppose that $q>1$, i.e. $p>1$. If $k \le n^{\frac{1}{q}}$, then taking into account (3.3) and the inequality $\log x \le x-1 \le K(x_0)\log x$ which holds for $x \in [1, x_0]$ with some constant $K(x_0)>0$, we find

$$
\begin{aligned}
\sum_{k=1}^{[n^{\frac{1}{q}}]}\Big|\int_{\tau_k}^{t_k}\frac{2\,dx}{x^2-1}\Big| &\le \sum_{k=1}^{[n^{\frac{1}{q}}]}\frac{|t_k-\tau_k|}{(1+o(1))\log\tau_k} = (1+o(1))\sum_{k=1}^{[n^{\frac{1}{q}}]}\frac{\tau_k\Big|\frac{t_k}{\tau_k}-1\Big|}{\log\tau_k} \\
&= O\Big(\sum_{k=1}^{[n^{\frac{1}{q}}]}\frac{\log\frac{t_k}{\tau_k}}{\log\tau_k}\Big) = O\Big(\sum_{k=1}^{[n^{\frac{1}{q}}]}\frac{k}{n}\Big) = O(n^{\frac{1}{q}-\frac{1}{p}}).
\end{aligned} \tag{3.8}
$$

The definition of τ_k and (3.3) yield and that $\min\{\log\tau_k, \log t_k\} \ge K_5 > 0$ for $n^{\frac{1}{q}} < k \le n^\alpha$. Therefore, using the mean value theorem, we obtain ($\xi_k \in [\psi_1(t_k), \psi_1(\tau_k)]$)

$$
\begin{aligned}
\sum_{k=[n^{\frac{1}{\alpha}}]+1}^{[n^{\frac{1}{\alpha}}]}\Big|\int_{\tau_k}^{t_k}\frac{2\,dt}{t^2-1}\Big| &= \sum_{k=[n^{\frac{1}{q}}]+1}^{[n^\alpha]}\Big|\int_{\psi_1(\tau_k)}^{\psi_1(t_k)}\frac{2\varphi_1'(s)}{\varphi_1^2(s)-1}ds\Big| \\
&= \sum_{k=[n^{\frac{1}{q}}]+1}^{[n^\alpha]}\frac{2\varphi_1'(\xi_k)}{\varphi_1^2(\xi_k)-1}|\psi_1(t_k)-\psi_1(\tau_k)| \\
&= O\Big(\sum_{k=[n^{\frac{1}{q}}]+1}^{[n^\alpha]}\frac{|\log^p t_k - \log^p\tau_k|}{\xi_k^{\frac{1}{q}}\varphi_1(\xi_k)}\Big) \\
&= O\Big(\sum_{k=[n^{\frac{1}{q}}]}^{[n^\alpha]}\frac{|\log\frac{t_k}{\tau_k}|\log^{p-1}\tau_k}{\psi_1(\tau_k)^{\frac{1}{q}}\varphi_1(\xi_k)}\Big) \\
&= O\Big(\sum_{k=[n^{\frac{1}{q}}]}^{[n^\alpha]}\exp\Big\{-\frac{(1+o(1))k}{p\sigma^{\frac{1}{p}}n^{\frac{1}{q}}}\Big\}\frac{k^2}{n^{1+1/q}}\Big) \\
&= O\Big(\frac{1}{n^{\frac{1}{p}}}\sum_{k=1}^{+\infty}\exp\Big\{-K_6\frac{k}{n^{\frac{1}{q}}}\Big\}\Big) = O\Big(n^{-\frac{1}{p}}\frac{1}{1-\exp\{-K_6 n^{-\frac{1}{q}}\}}\Big) \\
&= O(n^{\frac{1}{q}-\frac{1}{p}}), \quad n\to+\infty.
\end{aligned} \tag{3.9}
$$

In a similar manner, $\sum_{k=1}^{[n^\alpha]}|\int_{\tau_k}^{t_{-k}}|$ can be estimated. Hence, (3.8) and (3.9) imply $\Sigma_2 = O(n^{\frac{1}{q}-\frac{1}{p}})$ as $n\to+\infty$ for $q>1$.

Now, suppose that $0 < p \le 1$, i.e. $q < 0$ or $q = \infty$. Then there exists a positive constant K_7 such that τ_k, t_k, t_{-k} are greater than K_7 for all k. Thus, proceeding in a similar fashion as in (3.9) we can see that ($\zeta_k \in [\psi_1(\tau_k), \psi_1(t_k)]$)

$$\begin{aligned}
\sum_{k=1}^{[n^\alpha]}\Big|\int_{\tau_k}^{t_k}\frac{2\,dx}{x^2-1}\Big| &= O\Big(\sum_{k=1}^{[n^\alpha]}\frac{\varphi_1'(\zeta_k)}{\varphi_1^2(\zeta_k)}|\psi_1(t_k)-\psi_1(\tau_k)|\Big)\\
&= O\Big(\sum_{k=1}^{[n^\alpha]}\frac{|\log^p t_k - \log^p \tau_k|}{\zeta_k^{1/q}\varphi_1(\zeta_k)}\Big)\\
&= O\Big(\sum_{k=1}^{[n^\alpha]}\exp\Big\{-(1+o(1))\frac{2k-1}{2p\sigma^{\frac{1}{p}}n^{\frac{1}{q}}}\Big\}\frac{k^2}{n^{1+1/q}}\Big)\\
&= O\Big(\frac{1}{n^{\frac{1}{p}}}\sum_{k=1}^{+\infty}\exp\Big\{-\frac{3k}{4p\sigma^{\frac{1}{p}}n^{\frac{1}{q}}}\Big\}\Big)\\
&= O\Big(\frac{n^{-\frac{1}{p}}\exp\Big\{-\frac{3}{4p\sigma^{\frac{1}{p}}n^{\frac{1}{q}}}\Big\}}{1-\exp\{-K_7(\sigma,p)n^{-\frac{1}{q}}\}}\Big)\\
&= o\Big(\exp\Big\{-\frac{3}{4\sigma^{\frac{1}{p}}n^{\frac{1}{q}}}\Big\}\Big), \quad n\to+\infty.
\end{aligned} \tag{3.10}$$

Recall that due to (3.3) estimate (3.10) also holds for $\sum_{k=1}^{[n^\alpha]}|\int_{\tau_k}^{t_{-k}}|$, so $\sum_2 = o\big(\exp\{-\frac{3n^{-\frac{1}{q}}}{4p\sigma^{\frac{1}{p}}}\}\big)$ as $n\to+\infty$. Therefore, using (3.6)–(3.10) one can obtain from (3.5) that

$$\begin{aligned}
B(r_n,\log|f|)-A(r_n,\log|f|) &= \sum_{k=1}^{+\infty}\log\Big|\frac{1+\tau_k^{-1}}{1-\tau_k^{-1}}\Big| + \gamma(n)\\
&= \log\prod_{k=1}^{+\infty}\Big|\frac{h^{2k-1}+1}{h^{2k-1}-1}\Big| + \gamma(n),
\end{aligned} \tag{3.11}$$

where

$$h = \exp\Big\{-\frac{1}{2p\sigma^{\frac{1}{p}}n^{\frac{1}{q}}}\Big\}, \quad \gamma(n) = \begin{cases} O(n^{\frac{1}{q}-\frac{1}{p}}), & q>1,\\ o\Big(\exp\Big\{-\frac{3}{4p\sigma^{\frac{1}{p}}n^{\frac{1}{q}}}\Big\}\Big), & q<0 \text{ or } q=\infty.\end{cases}$$

Finally, it remains to compute the infinite product in (3.11). If $p=1$ ($q=\infty$) we have [4] $\prod_{k=1}^{+\infty}\Big(\frac{h^{2k-1}-1}{h^{2k-1}+1}\Big)^2 = C_2(\sigma^*)$, ($h=\exp\{-\frac{1}{4\sigma^*}\}$, $\sigma = 2\sigma^*$). It is known ([11], Chap. IV) that

$$\prod_{k=1}^{+\infty}\Big(\frac{h^{2k-1}-1}{h^{2k-1}+1}\Big)^2 = \frac{\theta_0(0,h)}{\theta_3(0,h)},$$

where θ_0, θ_3 are theta-functions. If we associate with h the value $\tilde{k}$ of the modulus of a corresponding elliptic integral (see [11], p. 95), then ($\theta_j(0|\tau)\equiv\theta_j(0,h)$).

$$\tilde{k}^2 = \frac{\theta_2^4(0|\tau)}{\theta_3^4(0|\tau)} = 1-\frac{\theta_0^4(0|\tau)}{\theta_3^4(0|\tau)} = \Big\{\frac{2h^{\frac{1}{4}}+2h^{\frac{9}{4}}+\dots}{1+2h+2h^4+\dots}\Big\}^4, \tag{3.12}$$

where $h = e^{\pi i \tau}$. It follows from the latter formula (see also [11], p. 118) that $\tilde{k}^2 \sim 16h$ as $h \downarrow 0$, which is equivalent to

$$\tilde{k}'^2 \sim 16h', \quad h' \downarrow 0, \tag{3.13}$$

where $\tilde{k}'$ is a complementative modulus ($\tilde{k}'^2 + \tilde{k}^2 = 1$, $h' = e^{\pi i \tau'}$, $\tau' = -\frac{1}{\tau}$). The connections between h, h' and $\tilde{k}$, $\tilde{k}'$ yield that $\tilde{k}' \to 1$ as $h \uparrow 1$. Indeed, $h' = \exp\Big\{-\frac{i\pi}{\tau}\Big\} = \exp\Big\{\frac{\pi^2}{\log h}\Big\}$, so $k' \to 0$ as $h \uparrow 1$. Hence, using (3.12) and (3.13) we obtain

$$\frac{\theta_0(0,h)}{\theta_3(0,h)} = \sqrt[4]{1-\tilde{k}^2} \sim \sqrt[4]{16h'} = 2\exp\Big\{\frac{\pi^2}{4\log h}\Big\}, \quad h \uparrow 1.$$

Then if $q > 1$,

$$\begin{aligned}\log \prod_{k=1}^{+\infty}\Big(\frac{h^{2k-1}+1}{h^{2k-1}-1}\Big)^2 &= -\log\Big((2+o(1))\exp\Big\{\frac{\pi^2}{4\log h}\Big\}\Big)\\ &= -\log 2 + o(1) - \frac{\pi^2}{4\log h} = -\log 2 + o(1) + \frac{\pi^2}{2} p\sigma^{\frac{1}{p}} n^{\frac{1}{q}}, \quad n \to +\infty.\end{aligned}$$

But $p\sigma^{\frac{1}{p}} n^{\frac{1}{q}} = \sigma p\big(\frac{n}{\sigma}\big)^{\frac{p-1}{p}} \sim \sigma p \log^{p-1} r_n = \psi_2(r_n) \to +\infty$ as $n \to +\infty$. Therefore, according to (3.11) for $q > 1$ we have

$$B(r_n, \log|f|) - A(r_n, \log|f|) = \Big(\frac{\pi^2}{2} + o(1)\Big)\psi_2(r_n), \quad n \to +\infty,$$
$$B(r, \log|f|) \sim N(r,0,f) \sim \frac{\sigma}{p+1}\log^{p+1} r, \quad r \to +\infty.$$

If $0 < p < 1$ ($q < 0$), then $\psi_2(r_n) \to 0$ and $h = \exp\Big\{-\frac{1+o(1)}{2\psi_2(r_n)}\Big\} \to 0$ as $n \to +\infty$. In this case

$$\gamma(n) = o\Big(\exp\Big\{\frac{-3}{4\sigma^{\frac{1}{p}} p n^{\frac{1}{q}}}\Big\}\Big) = o\Big(\exp\Big\{-\frac{1}{2\psi_2(r_n)}\Big\}\Big), \quad n \to +\infty.$$

Consequently, taking into account (3.11) as in the previous case,

$$\begin{aligned}\frac{A(r_n,|f|)}{B(r_n,|f|)} &= \sqrt[4]{1-\tilde{k}^2}e^{-\gamma(n)} = \sqrt[4]{1-16h(1+o(1))}(1-\gamma(n)(1+o(1)))\\ &= (1-4h(1+o(1)))(1-\gamma(n)(1+o(1)))\\ &= 1-\exp\Big\{-\frac{1+o(1)}{2\psi_2(r_n)}\Big\}, \quad n \to +\infty.\end{aligned}$$

Theorem 2 is proved. □

Remark 2. In [9] (Chapter 6) constructing an example which showed sharpness of inequality (2.15) the author approximated a subharmonic function by the logarithm of modulus of an entire function. But sharpness of (2.15) was shown only under the restriction $B(r, \log|f|)/\log^{5/2} r \to +\infty$.

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Department of Mechanics and Mathematics, Lviv National University, Universytetska 1, Lviv, 79000, Ukraine

E-mail address: matstud@franko.lviv.ua