AN ADDITION TO THE $\cos \pi p$-THEOREM FOR SUBHARMONIC AND ENTIRE FUNCTIONS OF ZERO LOWER ORDER

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ABSTRACT. We obtain a sharp asymptotic relation between the infimum and the maximum on a circle of a subharmonic function of zero lower order. An example is constructed, which shows the sharpness of the relation in the class of entire functions of zero order such that $\log M(r, f)/\log^2 r \to +\infty$, where $M(r, f) = \max(|f(z)| : |z| = r)$ as $r \to +\infty$.

1. INTRODUCTION

Let $u(z)$ be a subharmonic function in $\mathbb{C}$. We denote $A(r, u) = \inf\{u(z) : |z| = r\}$, $B(r, u) = \max\{u(z) : |z| = r\}$, $r > 0$. The classical $\cos \pi p$-theorem asserts that if $f(z)$ is an entire function of finite order $\rho \in [0; 1]$, then

$$A(r_n, \log |f|) > (C_1(\rho) + o(1))B(r_n, \log |f|),$$

where $C_1(\rho) = \cos \pi \rho$, $r_n \to +\infty$ as $n \to +\infty$. It is known ([11], Chapter 6) that one can replace $\log |f|$ by an arbitrary subharmonic in the plane function $u$ of lower order $\lambda \leq 1$ in (1.1). In this case, $C_1(\rho)$ must be replaced by $C_1(\lambda)$, and the constant is sharp. Hence, if $\lambda = 0$, then (1.1) implies $A(r_n, u) \sim B(r_n, u)$ as $r_n \to +\infty$. Naturally, the question on improvement of $\cos \pi p$-theorem in the case of subharmonic or entire functions of zero order arises.

In 1962 P. Barry [2] showed that for an arbitrary subharmonic function $u$ the inequality $B(r, u) \leq \psi(r)$ ($r \geq r_0$) implies

$$A(r_n, u) > B(r_n, u) - (1 + \varepsilon)\frac{\pi^2}{2}\psi_2(r_n), \quad r_n \to +\infty,$$

where $\psi \in C^2(\mathbb{R}_+)$ satisfies the following conditions: $\psi_2(r)$ is continuous and slowly varying on $[1, +\infty)$ (i.e. $\psi_2(2r) \sim \psi_2(r)$ as $r \to +\infty$), here $\psi_j(r) = \frac{d^j\psi(r)}{d(r \log r)^j}$, $j = 1, 2$. On the other hand, in 1980 P. Fenton [3] proved that for subharmonic function $u$ the condition $\liminf B(r, u)/\log^{p+1} r = \sigma < +\infty$ implies for $p > 0$ that $\forall \varepsilon > 0$ the inequality

$$A(r, u) > B(r, u) - (\sigma + \varepsilon)\Re\{\log^{p+1} r - (\log r + i\pi)^{p+1}\}$$

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holds for all \( r \) outside a set \( E \) satisfying
\[
\liminf_{r \to +\infty} \frac{1}{\log^p r} \int_{E(r)[1,r]} \frac{p \log^{p-1} t}{t} dt \leq \frac{\sigma}{\sigma + \varepsilon}.
\]

We remark that
\[
\text{Re}\{\log^{p+1} r - (\log r + i\pi)^{p+1}\} = (1 + o(1)) \frac{\pi^2(p + 1)p}{2} \log^{p-1} r
\]
\[
= (1 + o(1)) \frac{\pi^2 d^2 \log^{p+1} r}{2 (d \log r)^2}, \quad r \to +\infty.
\]

In the case \( \psi(r) = \sigma(\log^+ r)^2 \) \((r \geq 0)\) A. A. Gol’dberg [3] and P. Fenton [3][4] showed that
\[
(1.3) \quad \limsup_{r \to +\infty} \frac{A(r, |f|)}{B(r, |f|)} \geq C_2(\sigma) = \left( \prod_{n=1}^{+\infty} \frac{1 - q^{2n-1}}{1 + q^{2n-1}} \right)^2,
\]
where \( q = e^{-\frac{1}{\sigma}} \) and \( C_2(0) = 1 \), for every entire function \([5]\) such that \( \liminf_{r \to +\infty} \frac{B(r, \log |f|)}{\log^* r} = \sigma < +\infty \) (for every meromorphic function \([\overline{4}]\) such that \( \liminf_{r \to +\infty} N(r, 0, \infty, f) \log^{-2} r \leq \sigma < +\infty \), here \( N(r, 0, \infty, f) = N(r, 0, f) + N(r, \infty, f) \) is Nevanlinna’s counting function of zeros and poles. The constant \( C_2(\sigma) \) in (1.3) is sharp \([2], [4]\).

In 1982 P. Barry [7] established a sharp lower estimate for the remainder \( A(r, u) - B(r, u) \) in the case \( u = \log |f| \), \( f \) is an entire function with \( B(r, \log |f|) = O(\log^{p+1} r) \) as \( r \to +\infty \), \( 0 < p < 1 \). He also remarked, that a sharp estimate for \( p > 1 \) in the class on entire function was unknown. In this connection the following questions arise:

1) Is inequality (1.2) valid for subharmonic functions of zero lower order such that \( \liminf_{r \to +\infty} \frac{B(r, u)}{\psi(r)} \leq 1 ? \)

2) What is a sharp inequality between \( A(r, \log |f|) \) and \( B(r, \log |f|) \) in the class of entire functions such that \( B(r, \log |f|)/\log^* r \to +\infty \) as \( r \to +\infty ? \)

In this paper we give answers to both questions.

2. IMPROVING THE \( \cos \pi p \)-THEOREM

Let us introduce the class \( \Psi \) of functions \( \psi : \{1; +\infty) \to \mathbb{R}_+ \) such that
\[
\psi_1(r) = \frac{d\psi(r)}{d\log r} = K_1 \exp\left\{ \int_1^r \frac{\beta(s)}{s} ds \right\},
\]
where \( K_1 > 0 \) is a constant, \( \beta \in C(\mathbb{R}_+) \), \( \beta(r) > 0 \), \( \beta(r) \to 0 \) as \( r \to +\infty \), \( \int_1^{+\infty} \beta(s) s^{-1} ds \) is divergent and
\[
(2.1) \quad \log \frac{1}{\beta(r)} = O\left( \frac{1}{\hat{\beta}(r)} \right), \quad r \to +\infty,
\]
where \( \hat{\beta}(r) = \max\{ \beta(s) : s \geq r \} \). We remark that if function \( \beta \) is not required to satisfy condition (2.1), then the class of all such functions \( \psi_1 \) coincides with the class of functions that are positive continuously differentiable on \([1; +\infty)\), increasing to \( +\infty \) and slowly varying. On the other hand, the condition \( \psi_2(2r) \sim \psi_2(2r) \) as \( r \to +\infty \) implies \( \psi_2(2r) \sim \psi_1(r) \), and the latter one implies \( \psi(2r) \sim \psi(r) \).
Theorem 1. Let \( u(z) \) be a subharmonic function in \( \mathbb{C} \). If there exists \( \psi \in \Psi \) such that \( \liminf_{r \to +\infty} B(r, u)/\psi(r) \leq 1 \), then \( \forall \varepsilon > 0 \) the inequality

\[
A(r, u) \geq B(r, u) - (1 + \varepsilon) \frac{\pi^2}{2} \psi_2(r)
\]

holds outside a set \( E_\varepsilon \) of values \( r \) satisfying

\[
\liminf_{r \to +\infty} \frac{1}{\psi_1(r)} \int_{E_\varepsilon \cap [1, r]} d\psi_1(t) \leq \frac{1}{1 + \varepsilon}.
\]

In order to prove Theorem 1 we partially use Barry’s method. Let \( \mu_u \) be a Riesz measure associated with a subharmonic function \( u \), \( \mu_u^* (t) = \mu_u (\{ \zeta : |\zeta| \leq t \}) \). For \( R > 0 \) we define the following functions:

\[
u_1(z, R) = \int_{|\zeta| < R} \log \left| 1 - \frac{z}{\zeta} \right| d\mu_u (\zeta),
\]

\[
u_2(z, R) = \int_{|\zeta| < R} \log \left| 1 + \frac{z}{|\zeta|} \right| d\mu_u (\zeta) = \int_{[0, R)} \log \left| 1 + \frac{z}{t} \right| d\mu_u^* (t),
\]

\[
u_3(z, R) = u(z) - u_1(z, R).
\]

One can write \( \nu_2(z, R) = \int_0^R \log \left| 1 + \frac{r}{s} \right| d\mu_u^* (t) \) for almost all \( R > 0 \), where the last integral is understood in the Stieltjes sense.

Put \( B_j (r, R) = \max \{ u_j (z, R) : |z| = r \}, A_j (r, R) = \inf \{ u_j (z, R) : |z| = r \} \). It follows from the definitions of \( u_1 \) and \( u_2 \) that

\[
A_2 (r, R) \leq A_1 (r, R) \leq B_1 (r, R) \leq B_2 (r, R).
\]

Without loss of generality, we may assume that \( u(0) = 0 \) and \( u(z) \) is harmonic in the unit disk with the center in the origin. We need the following lemma [3].

**Lemma.** Suppose \( R > 2r \). Then

\[
B_3 (r, R) \leq \frac{4r}{R} B(2R, u), \quad -A_3 (r, R) \leq \frac{4r}{R} B(2R, u).
\]

The proof of a similar lemma can be found, for example, in [9], Lemma 6.1.

**Proof of Theorem 1.** We deduce from the definition of \( u_2 (z, R) \) that

\[
B_2 (t, R) - A_2 (t, R) = \int_0^R \log \left| 1 + \frac{t}{s} \right| d\mu_u^* (s) - \int_0^R \log \left| 1 - \frac{t}{s} \right| d\mu_u^* (s)
\]

\[
= \int_0^R \log \frac{t + s}{t - s} d\mu_u^* (s).
\]
Let us estimate the following integral using Fubini’s theorem and the latter relation ($0 < x < R$):

$$I(x, R) = \int_0^x \frac{B_2(t, R) - A_2(t, R)}{t} dt$$

$$= \int_0^x \frac{dt}{t} \int_0^R \log \left| \frac{t + s}{t - s} \right| d\mu^*_u(s)$$

$$= \int_0^R d\mu^*_u(s) \int_0^{x/s} \log \left| \frac{1 + \tau}{1 - \tau} \right| d\tau$$

$$\leq \int_{x/s}^{+\infty} \log \left| \frac{1 + \tau}{1 - \tau} \right| d\tau d\mu^*_u(R) = \frac{\pi^2}{2} u^*(R).$$

It follows from the definitions of $u_1, u_2, u_3$ and the lemma that

$$A(t, u) \geq A_1(t, R) + A_3(t, R) \geq A_2(t, R) + A_3(t, R),$$

$$B(t, u) \leq B_2(t, R) + B_3(t, R).$$

Therefore, for $R > 2r$, in accordance with Lemma and (2.5) we have

$$\int_0^x \frac{B(t, u) - A(t, u)}{t} dt \leq \int_0^x \frac{B_2(t, R) - A_2(t, R)}{t} dt$$

$$+ \int_0^x \frac{B_3(t, R) - A_3(t, R)}{t} dt \leq I(x, R) + \frac{8 \pi}{R} B(2R, u)$$

$$\leq \frac{\pi^2}{2} \mu^*_u(R) + \frac{8 \pi}{R} B(2R, u).$$

The conditions of the theorem imply that for arbitrary $\eta > 0$ an arbitrary large $r$ exists such that, $B(r, u) < (1 + \eta) \psi(r)$, and, due to Jensen’s formula for subharmonic functions [10, Chap. 4]

$$\int_0^r \frac{\mu^*_u(t)}{t} dt \leq B(r, u) < (1 + \eta) \psi(r).$$

Define $r^* = \max\{t \leq r : \mu^*_u(t) \leq (1 + 2\eta) \psi_1(t)\}$. The value $r^*$ is well defined, because, if we assume that there is no $t$ satisfying $\mu^*_u(t) \leq (1 + 2\eta) \psi_1(t)$, then integrating the inverse inequality divided on $t$ leads us to a contradiction with the choice of $r$.

Thus

$$(1 + 2\eta)(\psi(r) - \psi(r^*)) = (1 + 2\eta) \int_{r^*}^r \frac{\psi_1(t)}{t} dt \leq \int_{r^*}^r \frac{\mu^*_u(t)}{t} dt \leq (1 + \eta) \psi(r).$$

From the last inequality we find that $\psi(r^*) \geq \frac{1}{1 + 2\eta} \psi(r)$. Therefore,

$$B(r^*, u) \leq B(r, u) \leq (1 + \eta) \psi(r) \leq \frac{(1 + \eta)(1 + 2\eta)}{\eta} \psi(r^*).$$

In particular, there exists an arbitrary large $r^*$ which satisfies

$$\mu^*_u(r^*) \leq (1 + 2\eta) \psi_1(r^*), \quad B(r^*, u) \leq \frac{(1 + \eta)(1 + 2\eta)}{\eta} \psi(r^*).$$
Choose \( R \) such that inequalities (2.7) hold with \( r^* = 2R \). Then, from (2.6), using the slow variation of \( \psi_1 \) and \( \psi \), we obtain for sufficiently large \( R \)

\[
\int_0^x \frac{B(t,u) - A(t,u)}{t} \, dt \leq \frac{\pi^2}{2} \left( 1 + 2\eta \right) \psi_1(2R) + \frac{8(1 + 2\eta)(1 + \eta)}{\eta} \frac{x}{R} \psi(2R)
\]

(2.8)

\[
\leq \frac{\pi^2}{2} \left( 1 + 3\eta \right) \psi_1(1) + \frac{8(1 + 2\eta)^2}{\eta} \frac{x}{R} \psi(R).
\]

In accordance with (2.1) there exists a constant \( K_2 \in (0; +\infty) \) such that \( \beta(t) \leq (\beta(t))^{2/3} \) as \( t \geq t_0 \). Put \( \gamma(r) = \exp\left\{ (\beta(r))^{-\frac{3}{2}} \right\} \). Since \( \beta(r) \) is nonincreasing function and \( \beta(r) \to 0 \) as \( r \to +\infty \), \( \gamma(r) \) is a nondecreasing function, and \( \gamma(r) \to +\infty \) as \( r \to +\infty \). Let \( x \) be defined by the equality \( x\gamma(x) = R \). Then we find that

\[
\frac{\psi_1(R)}{\psi_1(x)} = \exp\left\{ \int_x^x \frac{\beta(s)}{s} \, ds \right\} \leq \exp\left\{ \int_x^x \frac{\beta(s)}{s} \, ds \right\} \leq \exp\left\{ \beta(x) \log \gamma(x) \right\} = \exp\left\{ \sqrt[3]{\beta(x)} \right\} \to 1, \quad x \to +\infty.
\]

(2.9)

Using the slow variation of \( \psi(r) \), it is easy to check that \( \psi(r)r^{-1/2} \downarrow 0 \) as \( r \to +\infty \). Therefore

\[
\frac{x}{R} \psi(R) = \frac{1}{\gamma(x)} \psi(x\gamma(x)) \leq \psi(x)(\gamma(x))^{-\frac{1}{2}}.
\]

(2.10)

Applying l’Hospital’s rule we obtain

\[
\lim_{x \to +\infty} \frac{\psi(x)}{\psi_1(x)\gamma(x)^{1/2}} = \lim_{x \to +\infty} \frac{\psi_1(x)}{\psi_1(x)\beta(x)^{1/2} + \psi_1(x)(\gamma(x)^{1/2})'} 
\leq \lim_{x \to +\infty} \frac{1}{\beta(x)(\gamma(x))^{1/2}} \leq \lim_{x \to +\infty} \frac{1}{\beta(x)} \exp\left\{ \frac{1}{2\sqrt[3]{\beta(x)}} \right\} = 0.
\]

(2.11)

Relations (2.10) and (2.11) imply

\[
\frac{x}{R} \psi(R) \leq \frac{\psi(x)}{\gamma(x)^{1/2}} = o(\psi_1(x)),
\]

(2.12)

on a sequence of values \( x \to +\infty \). Combining (2.8), (2.9) and (2.12), we deduce

\[
\int_0^x \frac{B(t,u) - A(t,u)}{t} \, dt \leq \frac{\pi^2}{2} (1 + 4\eta) \psi_1(x),
\]

(2.13)

for an arbitrary large \( x \).

Using the generalized l’Hospital rule, we get

\[
\frac{\pi^2}{2} (1 + 4\eta) \geq \liminf_{x \to +\infty} \frac{1}{\psi_1(x)} \int_0^x \frac{B(t,u) - A(t,u)}{t} \, dt 
\geq \liminf_{x \to +\infty} \frac{B(x,u) - A(x,u)}{\psi_2(x)}.
\]

(2.14)

This means that on a sequence \( r_n \to +\infty \),

\[
A(r_n, u) > B(r_n, u) - \frac{\pi^2}{2} (1 + 5\eta) \psi_2(r_n).
\]
Let \( E_\varepsilon \) be the set of those \( r \), for which (2.2) is not fulfilled. Then taking into account (2.13) we obtain
\[
\frac{\pi^2}{2}(1+4\eta)\psi_1(x) \geq \int_0^x \frac{B(t, u) - A(t, u)}{t} dt \geq \int_{[0, x] \cap E_\varepsilon} \frac{B(t, u) - A(t, u)}{t} dt
\]
\[
\geq \int_{[0, x] \cap E_\varepsilon} \frac{\pi^2}{2}(1+\varepsilon)\psi_2(t) dt = \frac{\pi^2}{2}(1+\varepsilon) \int_{[0, x] \cap E_\varepsilon} d\psi_1(t).
\]
Now, estimate (2.3) follows from the arbitrariness of \( \eta > 0 \). Theorem 1 is proved. \( \square \)

**Corollary.** Let \( f(z) \) be an entire function. If there exists \( \psi \in \Psi \) such that
\[
\liminf_{r \to +\infty} B(r, \log |f|)/\psi(r) \leq 1, \quad \forall \varepsilon > 0
\]
outside a set \( E_\varepsilon \) of values \( r \) satisfying (2.3).

3. An example

Sharpness of inequality (2.2) in the class of subharmonic functions of zero order follows from Barry’s results [2]. However, inequality (2.15) can be improved when
\[
B(r, \log |f|) = O(\log^2 r) \quad (r \to +\infty)
\]
(see (1.3) and the following remark).

**Remark 1.** For entire functions \( f \) satisfying \( B(r, \log |f|) \leq \psi(r) \) \( (r \geq r_0) \), \( \psi(r) = (\log^+ r)^{p+1} \), \( 0 < p < 1 \), Barry [7] proved that
\[
\forall \varepsilon \in (0, 1) \quad \frac{A(r_n, |f|)}{B(r_n, |f|)} > 1 - \exp\left\{ \frac{\varepsilon - 1}{2\psi_2(r_n)} \right\}, \quad r_n \to +\infty.
\]
Using the Wiman-Valiron method, Barry also showed [7] that \( \varepsilon - 1 \) cannot be replaced by \(-1\) in the last inequality, and estimated the set through which \( r_n \) may approach \(+\infty\).

We construct an entire function of zero order, for which in (2.15) the asymptotic equality holds with \( \varepsilon = 0 \) on some sequence of \( r = r_n \to +\infty \) as \( n \to +\infty \).

**Theorem 2.** Let \( \psi(r) = \frac{1}{p\pi^2} (\log^+ r)^{p+1} \), \( (r \geq 0), \quad p > 0, \quad 0 < \sigma < +\infty \). There exists an entire function \( f \) such that
\[
\lim_{r \to +\infty} B(r, \log |f|)/\psi(r) = 1,
\]
and on a sequence \( r_n \to +\infty \)
\[
A(r_n, \log |f|) = B(r_n, \log |f|) - \left( \frac{\pi^2}{2} + o(1) \right)\psi_2(r_n), \quad p > 1,
\]
\[
\frac{A(r_n, |f|)}{B(r_n, |f|)} = 1 - \exp\left\{ -\frac{1 + o(1)}{2\psi_2(r_n)} \right\}, \quad 0 < p < 1,
\]
\[
\frac{A(r_n, |f|)}{B(r_n, |f|)} = C_2(\sigma/2) + o(1), \quad p = 1.
\]

**Proof of Theorem 2.** In the notations of Theorem 2 we have
\[
\psi_1(t) = \begin{cases} \sigma \log^n t, & t \geq 1, \\ 0, & 0 < t < 1. \end{cases}
\]
Let \( \varphi_1(r) = \psi_1^{-1}(r) = \exp\{\left(\frac{x}{\sigma}\right)^{\frac{1}{p}}\} \) be the inverse to \( \psi_1 \) function, \( x_n = \varphi(n) \). We shall denote by \( [x] \) an integral part of \( x \) and by \( K \) with a subscript a positive constant depending only on \( p \) and \( \sigma \). Define the entire function \( f(z) \) by the canonical product of zero genus

\[
f(z) = \prod_{n=1}^{+\infty} \left(1 + \frac{z}{x_n}\right).
\]

This representation becomes, for \( r \in (x_n, x_{n+1}) \),

\[
B(r, \log |f|) - A(r, \log |f|) = \sum_{k=1}^{+\infty} \left( \log \left|1 + \frac{r}{x_k}\right| - \log \left|1 - \frac{r}{x_k}\right| \right)
\]

(3.1)

\[= \left(\sum_{k=1}^{n} + \sum_{k=n+1}^{+\infty}\right) \log \left|\frac{r}{x_k} + 1\right|.
\]

We choose \( r = r_n = \sqrt{x_n x_{n+1}} \). Then

\[
r_n = \exp\left\{\frac{1}{2} \left(\left(\frac{n+1}{\sigma}\right)^{\frac{1}{p}} + \left(\frac{n}{\sigma}\right)^{\frac{1}{p}}\right)\right\}
\]

(3.2)

\[= \exp\left\{\left(\frac{n}{\sigma}\right)^{\frac{1}{p}} + \frac{1}{2\sigma^p \pi n^{\frac{1}{p}}} + O\left(\frac{1}{n^{1+1/q}}\right)\right\}, \quad n \to +\infty,
\]

where \( q \) is defined by the equality \( p^{-1} + q^{-1} = 1 \). Let \( t_{-k} = \frac{r}{x_{n-k}} \) \((k = 1, \ldots, n)\), \( t_k = \frac{r_n}{x_n} \) \((k = 1, 2, \ldots)\). We need asymptotics for \( t_{\pm k} \) as \( n \to +\infty \). For arbitrary \( \alpha \in (0, 1) \) and \( 1 \leq k \leq n^\alpha \) using (3.2) we find that

\[
\log t_k = \left(\frac{n}{\sigma}\right)^{\frac{1}{p}} \left(\left(1 + \frac{k}{n}\right)^{\frac{1}{p}} - \left(1 + \frac{1}{2p} + O\left(\frac{1}{n^2}\right)\right)\right)
\]

(3.3)

\[= \frac{2k - 1}{2p\sigma^\frac{1}{p} n^{\frac{1}{p}}} \left(1 + O\left(\frac{k}{n}\right)\right), \quad n \to +\infty.
\]

It is easy to see that relation (3.3) also holds with \( t_{-k} \) instead of \( t_k \) for the same \( k \).

For \( k > n^\alpha \) we obtain

\[
\log t_k = \left(\frac{n + k}{\sigma}\right)^{\frac{1}{p}} - \left(\frac{n}{\sigma}\right)^{\frac{1}{p}} \left(1 + o(1)\right)
\]

\[
= \left(\frac{n + k}{\sigma}\right)^{\frac{1}{p}} \left(1 - \left(\frac{n + k}{n}\right)^{\frac{1}{p}} \left(1 + o(1)\right)\right)
\]

(3.4)

\[\geq \left(\frac{n + k}{\sigma}\right)^{\frac{1}{p}} \left(1 - \frac{1 + o(1)}{1 + n^\alpha} \right)^{\frac{1}{p}}
\]

\[= \left(1 + o(1)\right) \left(\frac{n + k}{\sigma}\right)^{\frac{1}{p}} \frac{n^{\alpha-1}}{p} > \frac{(n + k)^{\alpha-\frac{1}{p}}}{2p\sigma^\frac{1}{p}}, \quad n \to +\infty,
\]

and for \( n^\alpha < k \leq n \)

\[
\log t_{-k} = \left(\frac{n}{\sigma}\right)^{\frac{1}{p}} \left(1 + o(1)\right) - \left(\frac{n + k - 1}{n}\right)^{\frac{1}{p}}
\]

(3.4')

\[\geq \left(\frac{n}{\sigma}\right)^{\frac{1}{p}} \left(1 + o(1)\right) - (1 - n^\alpha)^{-\frac{1}{p}}
\]

\[= \left(1 + o(1)\right) \left(\frac{n}{\sigma}\right)^{\frac{1}{p}} \frac{n^{\alpha-1}}{p} > \frac{(n + k)^{\alpha-\frac{1}{p}}}{2(\frac{1}{1+\alpha}+1)\sigma^\frac{1}{p}}, \quad n \to +\infty.
\]
Put $\tau_k = \exp\left\{ \frac{2k-1}{2p\sigma n^2} \right\}$ and fix an arbitrary positive $\alpha$ from the interval $\left(\frac{1}{q}, 1\right)$. By a similar manner for $\log \tau_k$ we have

$$\log \tau_k = \frac{(1 + o(1))k}{p \sigma n^2} > \frac{n^{a - \frac{1}{q}}}{2p \sigma} > \frac{(n + k)n^{a - \frac{1}{q} - 1}}{4p \sigma}, \quad n^{\alpha} < k \leq n,$$

$$\log \tau_k = \frac{(1 + o(1))k}{p \sigma n^2} > \frac{(n + k)n^{1 - \frac{1}{q}}}{4p \sigma} \geq \frac{(n + k)n^{a - \frac{1}{q} - 1}}{4p \sigma}, \quad k > n.$$

Since $\log\left|1 + \frac{1}{t}\right|$ is a decreasing function of $t$ on $(1, +\infty)$, using (3.4), (3.4') and the latter inequalities from (3.1) we have

$$B(r, \log |f|) - A(r, \log |f|) = \sum_{k=1}^{n} \log \left| \frac{t-k+1}{t-k-1} \right| + \sum_{k=1}^{+\infty} \log \left| \frac{t_k+1}{t_k-1} \right|$$

$$= 2 \sum_{k=1}^{+\infty} \log \left| \frac{t_k+1}{t_k-1} \right| + \sum_{k=1}^{n} \left( \log \left| \frac{t-k+1}{t-k-1} \right| - \log \left| \frac{t_k+1}{t_k-1} \right| \right)$$

$$+ \sum_{k=1}^{+\infty} \left( \log \left| \frac{t_k+1}{t_k-1} \right| - \log \left| \frac{t_k+1}{t_k-1} \right| \right) - \sum_{k=n+1}^{+\infty} \log \left| \frac{t_k+1}{t_k-1} \right|$$

$$= 2 \sum_{k=1}^{+\infty} \log \left| \frac{t_k+1}{t_k-1} \right| + \sum_{k=1}^{[n^a]} \left( \int_{t_k}^{t_k+1} \frac{2}{x^2 - 1} \right)$$

$$+ O \left( \sum_{k=1}^{[n^a]} \left( \int_{t_k}^{+\infty} + \int_{t_k}^{-\infty} \right) + \sum_{k=1}^{+\infty} \int_{t_k}^{+\infty} \frac{2}{x^2 - 1} \right)$$

$$= \sum_1 + \sum_2 + O \left( \sum_3 + \sum_4 \right).$$

Let us estimate the last sum in (3.5). Using l’Hospital’s rule one can obtain

$$\sum_3 \leq \sum_{k=[n^a]+1}^{+\infty} \exp\{-K_3(n+k)^{a-\frac{1}{q}}\} \leq \int_{n+n^a}^{+\infty} \exp\{-K_3t^{a-\frac{1}{q}}\} \, dt$$

$$= O\left( \exp\{-K_3(n+n^a)^{a-\frac{1}{q}}(n+n^a)^{1+\frac{1}{q}-a}\} \right)$$

$$= O\left( \exp\{-K_4n^{a-\frac{2}{q}}\}, \quad n \to +\infty. \right.$$
Similarly,
\[
\sum_{k=1}^{\lfloor n^{\frac{1}{2}} \rfloor} \left| \int_{t_k}^{t_k} \frac{2 \, dt}{t^2 - 1} \right| = \sum_{k=\lfloor n^{\frac{1}{2}} \rfloor + 1}^{\lfloor n^{\frac{1}{2}} \rfloor} \left| \int_{\psi_1(t_k)}^{\psi_1(t_k)} \frac{2 \phi'_{\chi}(s)}{\phi'_{\chi}(s) - 1} \, ds \right|
= \sum_{k=\lfloor n^{\frac{1}{2}} \rfloor + 1}^{\lfloor n^{\frac{1}{2}} \rfloor} \frac{2 \phi'_{\chi}(t_k)}{\phi'_{\chi}(t_k) - 1} |\psi_1(t_k) - \psi_1(t_k)|
= O \left( \sum_{k=\lfloor n^{\frac{1}{2}} \rfloor + 1}^{\lfloor n^{\frac{1}{2}} \rfloor} \frac{|\log^p t_k - \log^p \tau_k|}{\xi_k \phi_1(\xi_k)} \right)
= O \left( \sum_{k=\lfloor n^{\frac{1}{2}} \rfloor + 1}^{\lfloor n^{\frac{1}{2}} \rfloor} \frac{|\log^p t_k - \log^p \tau_k|}{\psi_1(\tau_k) \phi_1(\xi_k)} \right)
= O \left( \sum_{k=\lfloor n^{\frac{1}{2}} \rfloor + 1}^{\lfloor n^{\frac{1}{2}} \rfloor} \exp \left\{ - \left( 1 + o(1) \right) k \right\} \frac{k^2}{\psi_1(\tau_k) \phi_1(\xi_k)} \right)
= O \left( \frac{1}{n^{\frac{1}{2}}} \sum_{k=\lfloor n^{\frac{1}{2}} \rfloor + 1}^{\lfloor n^{\frac{1}{2}} \rfloor} \exp \left\{ - K_6 \frac{k}{n^{\frac{1}{2}}} \right\} \right) = O \left( n^{\frac{1}{2} - \frac{1}{q}} \right), \quad n \to +\infty.
\]
In a similar manner, \( \sum_{k=\lfloor n^{\frac{1}{2}} \rfloor + 1}^{\lfloor n^{\frac{1}{2}} \rfloor} |f_{\tau_k} \cdot k| \) can be estimated. Hence, (3.8) and (3.9) imply
\( \Sigma_2 = O(n^{\frac{1}{2} - \frac{1}{q}}) \) as \( n \to +\infty \) for \( q > 1 \).

The definition \( \tau_k \) and (3.3) yield and that \( \min \{ \log \tau_k, \log t_k \} \geq K_5 > 0 \) for \( n^{\frac{1}{2}} < k \leq n^a \). Therefore, using the mean value theorem, we obtain (\( \xi_k \in [\psi_1(t_k), \psi_1(\tau_k)] \))
\[
\sum_{k=\lfloor n^{\frac{1}{2}} \rfloor + 1}^{\lfloor n^{\frac{1}{2}} \rfloor} \left| \int_{t_k}^{t_k} \frac{2 \, dt}{t^2 - 1} \right| = O \left( \sum_{k=\lfloor n^{\frac{1}{2}} \rfloor + 1}^{\lfloor n^{\frac{1}{2}} \rfloor} \frac{|\log^p t_k - \log^p \tau_k|}{\xi_k \phi_1(\xi_k)} \right)
= O \left( \sum_{k=\lfloor n^{\frac{1}{2}} \rfloor + 1}^{\lfloor n^{\frac{1}{2}} \rfloor} \frac{|\log^p t_k - \log^p \tau_k|}{\psi_1(\tau_k) \phi_1(\xi_k)} \right)
= O \left( \sum_{k=\lfloor n^{\frac{1}{2}} \rfloor + 1}^{\lfloor n^{\frac{1}{2}} \rfloor} \exp \left\{ - \left( 1 + o(1) \right) k \right\} \frac{k^2}{\psi_1(\tau_k) \phi_1(\xi_k)} \right)
= O \left( \frac{1}{n^{\frac{1}{2}}} \sum_{k=\lfloor n^{\frac{1}{2}} \rfloor + 1}^{\lfloor n^{\frac{1}{2}} \rfloor} \exp \left\{ - K_6 \frac{k}{n^{\frac{1}{2}}} \right\} \right) = O \left( n^{\frac{1}{2} - \frac{1}{q}} \right), \quad n \to +\infty.
\]
Now, suppose that $0 < p \leq 1$, i.e. $q < 0$ or $q = \infty$. Then there exists a positive constant $K_7$ such that $\tau_k, t_k, t_{-k}$ are greater than $K_7$ for all $k$. Thus, proceeding in a similar fashion as in (3.9) we can see that $(\zeta_k \in [\psi_1(\tau_k), \psi_1(t_k)])$

$$\sum_{k=1}^{[n^n]} \left| \int_{\tau_k}^{t_k} \frac{2 \ dx}{x^2 - 1} \right| = O\left( \sum_{k=1}^{[n^n]} \frac{\phi_k' (\zeta_k)}{\phi_2' (\zeta_k)} |\psi_1(t_k) - \psi_1(\tau_k)| \right)$$

$$= O\left( \sum_{k=1}^{[n^n]} \left| \log^p t_k - \log^p \tau_k \right| \right)$$

$$= O\left( \sum_{k=1}^{[n^n]} \exp \left\{ -(1 + o(1)) \frac{2k - 1}{2p^q \tau_n^q} \right\} \frac{k^2}{n^{1+1/q}} \right)$$

$$= O\left( \sum_{k=1}^{[n^n]} \exp \left\{ -\frac{3k}{4p^q \tau_n^q} \right\} \right)$$

$$= O\left( \frac{1}{n^q} \sum_{k=1}^{\infty} \exp \left\{ -\frac{3k}{4p^q \tau_n^q} \right\} \right)$$

$$= O\left( \frac{n^{-\frac{1}{q}}} {1 - \exp(-K_7(p) n^{-\frac{1}{q}})} \right)$$

$$= o\left( \exp\left\{ -\frac{3}{4p^q \tau_n^q} \right\} \right), \quad n \to +\infty.$$

Recall that due to (3.3) estimate (3.10) also holds for $\sum_{k=1}^{[n^n]} | f_{-t_k} |$, so $\sum_2 = o\left( \exp\left\{ -\frac{3n^{-\frac{1}{q}}} {4p^q \tau_n^q} \right\} \right)$ as $n \to +\infty$. Therefore, using (3.6)–(3.10) one can obtain from (3.5) that

$$B(r_n, \log |f|) - A(r_n, \log |f|) = \sum_{k=1}^{\infty} \log \left| \frac{1 + \tau_k^{-1}}{1 - \tau_k^{-1}} \right| + \gamma(n)$$

$$= \log \prod_{k=1}^{\infty} \left| \frac{h^{2k-1} + 1}{h^{2k-1} - 1} \right| + \gamma(n),$$

where

$$h = \exp\left\{ -\frac{1}{2p^q \tau_n^q} \right\}, \quad \gamma(n) = \begin{cases} O(n^{\frac{1}{q} - \frac{1}{2} + \varepsilon}), & q > 1, \\ o\left( \exp\left\{ -\frac{3}{4p^q \tau_n^q} \right\} \right), & q < 0 \text{ or } q = \infty. \end{cases}$$

Finally, it remains to compute the infinite product in (3.11). If $p = 1$ ($q = \infty$) we have

$$\prod_{k=1}^{\infty} \left( \frac{h^{2k-1} - 1}{h^{2k-1} + 1} \right)^2 = C_2(\sigma^*), \quad (h = \exp\left\{ -\frac{1}{4\tau_n^q} \right\}, \quad \sigma = 2\sigma^*).$$

It is known ([11], Chap. IV) that

$$\prod_{k=1}^{\infty} \left( \frac{h^{2k-1} - 1}{h^{2k-1} + 1} \right)^2 = \frac{\theta_0(0,h)}{\theta_3(0,h)},$$

where $\theta_0, \theta_3$ are theta-functions. If we associate with $h$ the value $\tilde{k}$ of the modulus of a corresponding elliptic integral (see [11], p. 95), then $(\theta_j(0) = \theta_j(0, h))$.

$$\tilde{k}^2 = \frac{\theta_3' (0|\tau)}{\theta_3'^2 (0|\tau)} = 1 - \frac{\theta_3'' (0|\tau)}{\theta_3'^2 (0|\tau)} = \left( \frac{2h \tilde{k}^2 + 2h^2 \tilde{k}^4 + \ldots}{1 + 2h + 2h^4 + \ldots} \right)^4.$$
where \( h = e^{\pi i r} \). It follows from the latter formula (see also [11], p. 118) that 
\( \tilde{k}^2 \sim 16 h \), \( h \downarrow 0 \), which is equivalent to 
(3.13) 
\( \tilde{k}^2 \sim 16 h', \quad h' \downarrow 0, \)
where \( \tilde{k}' \) is a complementative modulus \( (\tilde{k}'^2 + \tilde{k}^2 = 1, \quad h' = e^{\pi i r}, \quad r' = -\frac{r}{2}) \).
The connections between \( h, h' \) and \( \tilde{k}, \tilde{k}' \) yield that \( \tilde{k}' \to 1 \) as \( h \uparrow 1 \). Indeed, 
\( h' = \exp \left\{ -\frac{r}{2h} \right\} = \exp \left\{ \frac{2}{1 \log h} \right\}, \) so \( \tilde{k}' \to 0 \) as \( h \uparrow 1 \). Hence, using (3.12) and (3.13) we obtain 
\( \frac{\theta_0(0, h)}{\theta_3(0, h)} = \sqrt{1 - \tilde{k}^2} \approx \sqrt{16 h'} = 2 \exp \left\{ \frac{\pi^2}{4 \log h} \right\}, \quad h \uparrow 1. \)

Then if \( q > 1, \)
\[
\log \prod_{k=1}^{\infty} \left( \frac{h^{2k-1} + 1}{h^{2k-1} - 1} \right)^2 = -\log \left( 2 + o(1) \right) \exp \left\{ \frac{\pi^2}{4 \log h} \right\}
= -\log 2 + o(1) - \frac{\pi^2}{4 \log h} = -\log 2 + o(1) + \frac{\pi^2}{2} p \sigma^{\frac{1}{n}} n^{\frac{1}{2}}, \quad n \to +\infty.
\]
But \( \sigma p^{\frac{1}{n}} n^{\frac{1}{2}} = \sigma p \left( \frac{n}{\sigma} \right)^{\frac{1}{n}} \sim \sigma p \log^{p-1} r_n = \psi_2(r_n) \to +\infty \) as \( n \to +\infty \). Therefore, according to (3.11) for \( q > 1 \) we have 
\[
B(r_n, \log |f|) - A(r_n, \log |f|) = \left( \frac{\pi^2}{2} + o(1) \right) \psi_2(r_n), \quad n \to +\infty, \quad B(r, \log |f|) \sim N(r, 0, f) \sim \frac{\sigma}{p + 1} \log^{p+1} r, \quad r \to +\infty.
\]
If \( 0 < p < 1 \) (\( q < 0 \)), then \( \psi_2(r_n) \to 0 \) and \( h = \exp \left\{ -\frac{1 + o(1)}{2 \psi_2(r_n)} \right\} \to 0 \) as \( n \to +\infty \). In this case 
\[
\gamma(n) = o \left( \exp \left\{ \frac{-3}{4 \sigma^{\frac{1}{2}} n^{\frac{1}{2}}} \right\} \right) = o \left( \exp \left\{ \frac{-1}{2 \psi_2(r_n)} \right\} \right), \quad n \to +\infty.
\]
Consequently, taking into account (3.11) as in the previous case, 
\[
\frac{A(r_n, |f|)}{B(r_n, |f|)} = \sqrt{1 - \tilde{k}^2 e^{-\gamma(n)}} = \sqrt{1 - 16 h(1 + o(1))(1 - \gamma(n)(1 + o(1)))}
= (1 - 4 h(1 + o(1))(1 - \gamma(n)(1 + o(1))))
= 1 - \exp \left\{ -\frac{1 + o(1)}{2 \psi_2(r_n)} \right\}, \quad n \to +\infty.
\]
Theorem 2 is proved.

Remark 2. In [9] (Chapter 6) constructing an example which showed sharpness of inequality (2.15) the author approximated a subharmonic function by the logarithm of modulus of an entire function. But sharpness of (2.15) was shown only under the restriction \( B(r, \log |f|) / \log^{3/2} r \to +\infty \).

References


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