ON THE TOTAL CURVATURE OF CONVEX HYPERSURFACES IN HYPERBOLIC SPACES

ALBERT BORBÉLY

(Communicated by Wolfgang Ziller)

ABSTRACT. Let $C_1 \subseteq C_2 \subseteq H^n$ be two convex compact subsets of the hyperbolic space $H^n$ with smooth boundary. It is shown that the total curvature of the hypersurface $\partial C_2$ is larger than the total curvature of $\partial C_1$.

0. Introduction

Let $M^n$ be an $n$-dimensional Riemannian manifold and let $F$ be an $(n-1)$-dimensional smooth immersed hypersurface. Denote by $A_q : T_q F \to T_q F$ the shape operator of $F$ at $q \in F$ with respect to a normal field defined in a neighborhood of $q$ and set $K(q) = det A_q$. This is well defined up to sign. When $M$ is the Euclidean space, it is called the Gauss-Kronecker curvature. We adopt the same name for $K$ in general although it is no longer an intrinsic quantity of the hypersurface.

Let $H^n$ denote the hyperbolic space and let $C_0 \subseteq C_1$ be two convex compact subsets with smooth boundaries. The goal of the paper is to show:

Theorem 1. With the notations introduced above we have

$$\int_{\partial C_0} K \leq \int_{\partial C_1} K.$$

Here, $K$ is computed with respect to the outward normal field of $\partial C_i$, $i = 0, 1$, and $Vol(S^{n-1})$ denotes the Euclidean $(n-1)$-dimensional volume of the unit sphere $S^{n-1}$.

It is well known that if $C$ is a convex compact subset of the Euclidean space, then

$$\int_F K = Vol(S^{n-1}).$$

In a general Hadamard manifold $M^n$, as a result of the Gauss-Bonnet theorem, we have for $n = 2$

$$Vol(S^{n-1}) \leq \int_{\partial C_0} K \leq \int_{\partial C_1} K.$$

Received by the editors February 15, 2000 and, in revised form, September 20, 2000.

1991 Mathematics Subject Classification. Primary 53C21.

Key words and phrases. Total curvature, Gauss-Kronecker curvature, isoperimetric inequality.

This research was supported by the Kuwait University Research Grant SM 03/99.

©2001 American Mathematical Society
and for $n = 3$

$$Vol(S^{n-1}) \leq \int_{\partial C_0} K.$$

It seems natural to wonder to what extent the above statements will hold in higher dimensions. Some partial results with respect to (2) were obtained in [1]. Although we have precise results about certain integrals on hypersurfaces due to Chern (the curvature integral [2]), the generalized Gauss-Bonnet-Chern theorem does not seem to help in higher dimension (at least not in an obvious way).

There is another motivation for trying to show that (2) is satisfied for a general nonpositively curved manifold. This is the so-called isoperimetric conjecture (see [3], [4]).

**Isoperimetric Conjecture.** Let $M^n$ be a Hadamard manifold and $D \subset M^n$ be a compact domain with smooth boundary. Then it satisfies the Euclidean isoperimetric inequality:

$$\text{area}(\partial D) \geq d_n(\text{vol}(D))^{\frac{n-1}{n}},$$

where $d_n = \text{area}(S^{n-1})/(\text{vol}(B^n))^{\frac{n-1}{n}}$.

This is now settled in dimension 4 by [3] and in dimension 3 by [1]. In fact, the main part of the proof in [3] is to show how (2) implies the isoperimetric inequality. Although it was carried out in dimension 3 only, it is very likely (and is explicitly mentioned in [3]) that it generalizes to higher dimensions. This means that a possible way of proving the isoperimetric conjecture is to establish (2) for a general Hadamard manifold.

### 1. Construction of a differential form

This is a general construction due to Chern [1] which works on any Riemannian manifold $M^n$. Our notation follows the notation of the original paper.

Let $e_n$ be a unit normal field defined on some open subset of $M^n$. At each point extend this to an orthonormal frame $e_1, \ldots, e_n$ such that $e_i$ is a smooth vector field for $i = 1, \ldots, n$. At least locally it is certainly possible. We now define the connection forms as

$$\omega^i_j(X) = \langle \nabla_X e_j, e_i \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the metric on $M^n$ and $X$ is a vector field. The curvature form is defined as

$$\Omega^i_j(X, Y) = -\langle R(X, Y)e_j, e_i \rangle,$$

where $R(X, Y)$ denotes the curvature tensor defined as: $R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z$. Then Cartan’s second structural equation reads as

$$\Omega^i_j = d\omega^i_j + \omega^i_k \omega^k_j,$$

where we use the usual summation convention, summing over repeated indices.

The differential form which is of interest to us is defined as:

$$\Phi = \omega^1_n \omega^2_n \cdots \omega^{n-1}_n.$$

This is an $(n-1)$-form on $M^n$ which is essentially the same as Chern’s form denoted by $\Phi_0$ in [2]. More precisely, $\Phi_0$ is an $(n-1)$-form on the unit tangent bundle and
\(\Phi\) is its pull back via the map \(E_n : M^n \to TM^n\) defined as \(E_n(p) = e_n(p)\). As a consequence we have the following important observation:

**Fact.** The differential form \(\Phi\) depends only on the vector field \(e_n\). It does not depend on how \(e_n\) is extended to an orthonormal frame \(e_1, ..., e_n\).

From the second structural equation (4) one can derive that

\[
\rho = \sum \epsilon_{i_1...i_{n-1}} \Omega_n^{i_1} \omega_N^{i_2} ... \omega_N^{i_{n-1}},
\]

where \(\epsilon_{i_1...i_{n-1}}\) is the Kronecker index which is equal to +1 or -1 according to whether the permutation \(i_1...i_{n-1}\) of the numbers 1, 2, ..., \(n-1\) is even or odd and the summation is extended over all the indices \(i_1...i_{n-1}\) subject to the condition \(i_2 < i_3 < ... < i_{n-1}\). It is essentially the same as the \(n\)-form \(\Psi_0\) in [2].

2. Convex exhaustion

The other important ingredient is a lemma about convex exhaustion. Although we only need this in the hyperbolic space we state it for Hadamard manifolds. We say that a convex set \(C\) with smooth boundary is strictly convex if the second fundamental form of the boundary (with respect to the outward normal) is positive definite everywhere.

**Lemma 1.** Let \(M^n\) be a Hadamard manifold and \(C_0 \subset int(C_1)\) be two compact strictly convex subsets with smooth boundary. Then there exists a continuous function \(F : C_1 - int(C_0) \to [0,1]\) which is smooth in the interior such that: \(\nabla F \neq 0, \partial C_i = \{p \in C_2 - int(C_1) : F(p) = i\} \) for \(i = 0, 1\) and the sublevel sets \(C_b = \{p \in C_2 - int(C_1) : F(p) \leq b\}\) for \(0 \leq b \leq 1\) are convex.

**Proof.** The statement is clear intuitively. Denote by \(t\) the distance function from the set \(\partial C_i\) for \(i = 0, 1\).

For \(\delta > 0\) we set \(N_{2\delta} = \{p \in C_1 : t(p) < 2\delta\}\). Since \(\partial C_1\) has a positive definite second fundamental form, we can choose \(\delta > 0\) small enough such that: \(3\delta < dist(C_0, \partial C_1)\), the function \(t\) is smooth on \(N_{2\delta}\) (there are no focal points of \(\partial C_1\) inside \(N_{2\delta}\)) and

\[
-D^2 t > c_1 > 0,
\]

when restricted to \(\nabla t^+\) for some positive constant \(c_1\). Here, \(D^2 t\) denotes the Hessian and we adapted the notation that \(-D^2 t > c_1\) on \(\nabla t^+\) if \(-D^2 t(X, X) > c_1\) for every unit tangent vector \(X \in \nabla t^+\). We observe also that on \(N_{2\delta}\) the following inequality holds for the angle between the gradients:

\[
\angle(\nabla t, -\nabla t) < \pi/2 - \alpha,
\]

for some \(\alpha > 0\) depending on \(\delta\).

We are going to construct \(F\) in the form

\[
F = 1 - e^{-a t} f^\epsilon.
\]

Here \(f = h(t)\) is the reparametrized distance function from \(\partial C_1\) and \(h\) is a fixed smooth increasing real function \(h : [0, \infty) \to [0, 1]\) such that \(h(t) = \frac{1}{2\delta} t\) for \(0 \leq t \leq \delta\) and \(h(t) = 1\) on \([2\delta, \infty)\). For the derivative and the Hessian of \(f\) we have

\[
d^f = h \, d^t t \quad \text{and} \quad D^2 f = h'' d^t t \otimes t + h' D^2 t.
\]

The choice of \(a, \epsilon\) will be discussed later.
First, we show that the derivative of $F$ is never zero. From the definition we obtain
\begin{equation}
    dF = e^{-a_\theta_0} f^\epsilon (ad_\theta_0 - \epsilon \frac{d}{df}).
\end{equation}

Then, taking into consideration that $\nabla f \parallel \nabla \theta_1$, the statement follows from (7).

To show that the sublevel sets are convex we need to show that the Hessian $D^2 F$ is positive definite on $\nabla F^\perp$. We have
\begin{equation}
    D^2 F = e^{-a_\theta_0} (-a^2 d\theta_0 \otimes d\theta_0 + aD^2 \theta_0) f^\epsilon \\
    + \epsilon e^{-a_\theta_0} f^{\epsilon - 1} \left( \frac{1}{f} \frac{d}{df} \otimes \frac{d}{df} - D^2 f \right) \\
    + \epsilon e^{-a_\theta_0} f^{\epsilon - 1} (d\theta_0 \otimes d\theta_0 + d\theta \otimes d\theta_0).
\end{equation}

The argument depends on certain estimates of the Hessians $D^2 \theta_i$ for $i = 0, 1$, on various subspaces. It will be useful to keep in mind that $\theta_i$ (for $i = 0, 1$) are distance functions; therefore the gradients $\nabla \theta_i$ are eigenvectors of $D^2 \theta_i$ with eigenvalue zero.

First, we consider the region where $f \equiv 1$. On this region $df, D^2 f = 0$ and $\nabla F \parallel \nabla \theta_0$. Therefore $D^2 F = a f e^{-a_\theta_0} D^2 \theta_0$, when it is restricted to vectors in $\nabla F^\perp$. But on this subspace $D^2 \theta_0$ is positive definite since it is the distance function from a convex set in a Hadamard manifold.

Next, we consider the region where $1/2 < f < 1$. Since $C_0$ is strictly convex and compact, we know that $D^2 \theta_0 > c_2 > 0$ for some $c_2 > 0$, when restricted to $(\nabla \theta_0)^\perp$. Therefore, taking (6) and (9) into account, we conclude that $D^2 \theta_0 > c_3 > 0$ for some positive constant $c_3$, when restricted to $\nabla F^\perp$. So, for a small enough $a$, where the choice of $a$ depends only on $c_3$, the term $aD^2 \theta_0$ will dominate $a^2 d\theta_0 \otimes d\theta_0$. This will remain true throughout the whole region $C_1 - C_0$. The derivatives of $h$ are bounded and so are the terms $df$ and $D^2 f$. Therefore, for a small enough $\epsilon$, the term $aD^2 \theta_0$ will dominate all the other terms as well. The choice of $\epsilon$ depends on $a$ and on the bounds for $df$ and $D^2 f$.

At last, we consider the region where $0 < f < 1/2$. It is clear from the definition of the function $f$ that this region is a subset of $N_{\epsilon}$. As before, $aD^2 \theta_0$ will dominate $a^2 d\theta_0 \otimes d\theta_0$. The term involving $df \otimes df$ is positive semi-definite. As for the rest of the terms, we will show that the term involving $-D^2 f$ will dominate, for a small enough $a$, the term involving $d\theta_0 \otimes df + df \otimes d\theta_0$. Since $h$ is linear on this region, from (6) we obtain $-D^2 f = -h' D^2 \theta_1 > c_3 / 2 > 0$, when restricted to $(\nabla \theta_1)^\perp$. Taking into consideration (7) and (9) we conclude that $-D^2 f > c_4 > 0$ for some sufficiently small $c_4 > 0$, when restricted to $\nabla F^\perp$. The constant $c_4$ depends on $c_1, \delta$ and the angle $\alpha$. So, if the constant $a > 0$ was chosen small enough, where the choice of $a$ depended only on $c_2$ and $c_4$, then $-D^2 f$ dominates the term involving $d\theta_0 \otimes df + df \otimes d\theta_0$. This concludes the proof of the lemma. \hfill \Box

3. Proof of Theorem 1

With the preparation done in the previous sections, the proof of the theorem is simple. Let us return to the hyperbolic space $H^n$.

First, we prove the inequality between the two integrals. Assume that $C_0 \subset \text{int}(C_1)$ and both sets are strictly convex. We are going to show that
\begin{equation}
    \int_{\partial C_0} K \leq \int_{\partial C_1} K.
\end{equation}
The general case will follow by a trivial limiting procedure.

Let $F$ be the smooth function of Lemma 1 and define the unit vector field $e_n$ by $e_n = \nabla F/|\nabla F|$. This is defined on $\text{int}(C_1) - C_0$ but it extends continuously to the boundary. To the vector field $e_n$ we construct the form $\Phi$ as in the previous section and by Stokes’s theorem we have

\begin{equation}
\Phi = \int_{C_1 - C_0} d\Phi.
\end{equation}

To evaluate the integrals in (12) we are going to compute the forms $\Phi$ and $d\Phi$. Let $q \in C_1 - \text{int}(C_0)$ be an arbitrarily chosen point.

Since $\Phi$ depends only on the vector field $e_n$, we can express $\Phi$ in a special frame. Let us choose the frame $e_1, \ldots, e_n$ such that at the point $q \in C_1 - \text{int}(C_0)$ the vectors $e_1, \ldots, e_{n-1}$ are the principal directions for the hypersurface $\{F = F(q)\}$. For the other points of the hypersurface the vectors $e_1, \ldots, e_{n-1}$ may no longer be principal directions. Then, from the definition of the $\omega^n_i$’s we have

\begin{equation}
\omega^n_i(e_j) = \delta^n_{ij} \lambda_j, \quad \text{for } 1 \leq i, j \leq n - 1
\end{equation}

at $q \in C_1 - \text{int}(C_0)$, where $\lambda_j$ denotes the principal curvature at $q$ of the hypersurface $\{F = F(q)\}$ in the direction of $e_j$ and $\delta^n_{ij}$ is the Kronecker symbol. Therefore

$\Phi(e_1, \ldots, e_{n-1}) = \lambda_1 \cdot \ldots \cdot \lambda_{n-1} = K$

at $q \in C_1 - \text{int}(C_0)$, where $K$ denotes the Gauss-Kronecker curvature of the hypersurface $\{F = F(q)\}$ with respect to the normal field $e_n$. Since $q \in C_1 - \text{int}(C_0)$ was chosen arbitrarily, the left-hand side of (12) reads as follows:

$$
\int_{\partial C_0 \cup \partial C_1} \Phi = e_n \iota_{\pi_{12\ldots n-1}}(\int_{\partial C_1} K - \int_{\partial C_0} K) = (-1)^{n-1}(\int_{\partial C_1} K - \int_{\partial C_0} K).
$$

The curvature tensor has the form $\Omega^n_i(e_j, e_k)$ from this and (13) we have

\begin{equation}
\Omega^n_i(e_j, e_k) = 0
\end{equation}

if $\{j, k\} \neq \{i, n\}$ as sets. From this and (13) we have

\begin{equation}
\Omega^n_i(e_{i_1}, e_{i_2}, \ldots, e_{i_{n-1}}) = e_{i_1} i_2 \ldots i_{n-1} K_{i_1 i_2 \ldots i_{n-1}} \lambda_{i_2} \cdot \ldots \cdot \lambda_{i_{n-1}}
\end{equation}

at $q \in C_1 - \text{int}(C_0)$, where the indices satisfy the condition $i_2 < i_3 < \ldots < i_{n-1}$ and $K_{i_1 i_2 \ldots i_{n-1}}$ denotes the sectional curvature of the two-plane determined by $e_{i_1}, e_{i_n}$ and $\lambda_i$ is the principal curvature at the point $q \in C_1 - \text{int}(C_0)$ in the direction of $e_{i_i}$. Then (12) reads as follows:

$$
\int_{\partial C_1} K - \int_{\partial C_0} K = \int_{C_1 - C_0} \sum_{i_2 < i_3 < \ldots < i_{n-1}} K_{i_1 i_2 \ldots i_{n-1}} \lambda_{i_2} \cdot \ldots \cdot \lambda_{i_{n-1}},
$$

where the summation is extended over all the indices $i_1 \ldots i_{n-1}$ subject to the condition $i_2 < i_3 < \ldots < i_{n-1}$.

Since all sublevel sets are convex, all the principal curvatures are positive. Therefore, the integral on the right-hand side is positive. This completes the proof of the theorem, when $C_0 \subset \text{int}(C_1)$ and both sets are strictly convex. The general case follows by slightly "blowing up" the sets; that is, instead of $C_0$ we consider an $\eta$-neighborhood $C_0$ and instead of $C_1$ we take a $2\eta$-neighborhood $C_1 + 2\eta$. These are now strictly convex sets satisfying the conditions set forth at the beginning of the proof. Then letting $\eta$ go to 0 will yield the general case.
All that remains is to prove the inequality
\[ \text{Vol}(S^{n-1}) \leq \int_{\partial C_0} K. \]
This is a simple consequence of (11). Choose a ball \( B_\epsilon \) inside \( C_0 \). Applying (11), we obtain
\[ \int_{\partial B_\epsilon} K < \int_{\partial C_0} K. \]
Letting \( \epsilon \) go to 0 will yield the desired inequality. This completes the proof of the theorem.

REFERENCES

Department of Mathematics and Computer Science, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait
E-mail address: borbely@mcs.sci.kuniv.edu.kw