ON THE TOTAL CURVATURE OF CONVEX HYPERSURFACES IN HYPERBOLIC SPACES

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Abstract. Let $C_1 \subset C_2 \subset H^n$ be two convex compact subsets of the hyperbolic space $H^n$ with smooth boundary. It is shown that the total curvature of the hypersurface $\partial C_2$ is larger than the total curvature of $\partial C_1$.

0. Introduction

Let $M^n$ be an $n$-dimensional Riemannian manifold and let $F$ be an $(n-1)$-dimensional smooth immersed hypersurface. Denote by $A_q : T_qF \to T_qF$ the shape operator of $F$ at $q \in F$ with respect to a normal field defined in a neighborhood of $q$ and set $K(q) = \det A_q$. This is well defined up to sign. When $M$ is the Euclidean space, it is called the Gauss-Kronecker curvature. We adopt the same name for $K$ in general although it is no longer an intrinsic quantity of the hypersurface.

Let $H^n$ denote the hyperbolic space and let $C_0 \subset C_1$ be two convex compact subsets with smooth boundaries. The goal of the paper is to show:

Theorem 1. With the notations introduced above we have

$$\int_{\partial C_0} K \leq \int_{\partial C_1} K.$$

Here, $K$ is computed with respect to the outward normal field of $\partial C_i$, $i = 0, 1$, and $Vol(S^{n-1})$ denotes the Euclidean $(n-1)$-dimensional volume of the unit sphere $S^{n-1}$.

It is well known that if $C$ is a convex compact subset of the Euclidean space, then

$$\int_F K = Vol(S^{n-1}).$$

In a general Hadamard manifold $M^n$, as a result of the Gauss-Bonnet theorem, we have for $n = 2$

$$Vol(S^{n-1}) \leq \int_{\partial C_0} K \leq \int_{\partial C_1} K$$

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and for \( n = 3 \)
\[
Vol(S^{n-1}) \leq \int_{\partial \mathcal{C}_0} K.
\]

It seems natural to wonder to what extent the above statements will hold in higher dimensions. Some partial results with respect to (2) were obtained in [1].

Although we have precise results about certain integrals on hypersurfaces due to Chern (the curvature integra [2]), the generalized Gauss-Bonnet-Chern theorem does not seem to help in higher dimension (at least not in an obvious way).

There is another motivation for trying to show that (2) is satisfied for a general nonpositively curved manifold. This is the so-called isoperimetric conjecture (see [3], [4]).

**Isoperimetric Conjecture.** Let \( M^n \) be a Hadamard manifold and \( D \subset M^n \) be a compact domain with smooth boundary. Then it satisfies the Euclidean isoperimetric inequality:
\[
\text{area}(\partial D) \geq d_n (\text{vol}(D))^{\frac{n-1}{n}},
\]
where \( d_n = \text{area}(S^{n-1})/(\text{vol}(B^n))^{\frac{n-1}{n}}. \)

This is now settled in dimension 4 by [3] and in dimension 3 by [1]. In fact, the main part of the proof in [4] is to show how (2) implies the isoperimetric inequality. Although it was carried out in dimension 3 only, it is very likely (and is explicitly mentioned in [4]) that it generalizes to higher dimensions. This means that a possible way of proving the isoperimetric conjecture is to establish (2) for a general Hadamard manifold.

### 1. Construction of a differential form

This is a general construction due to Chern [1] which works on any Riemannian manifold \( M^n \). Our notation follows the notation of the original paper.

Let \( e_n \) be a unit normal field defined on some open subset of \( M^n \). At each point extend this to an orthonormal frame \( e_1, \ldots, e_n \) such that \( e_i \) is a smooth vector field for \( i = 1, \ldots, n \). At least locally it is certainly possible. We now define the connection forms as
\[
\omega^i_j(X) = \langle \nabla_X e_j, e_i \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) denotes the metric on \( M^n \) and \( X \) is a vector field. The curvature form is defined as
\[
\Omega^i_j(X, Y) = -\langle R(X, Y) e_j, e_i \rangle,
\]
where \( R(X, Y) \) denotes the curvature tensor defined as: \( R(X, Y) Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z \). Then Cartan’s second structural equation reads as
\[
\Omega^i_j = d\omega^i_j + \omega^i_k \omega^k_j,
\]
where we use the usual summation convention, summing over repeated indices.

The differential form which is of interest to us is defined as:
\[
\Phi = \omega^1_n \omega^2_n \cdots \omega^{n-1}_n.
\]
This is an \((n-1)\)-form on \( M^n \) which is essentially the same as Chern’s form denoted by \( \Phi_0 \) in [2]. More precisely, \( \Phi_0 \) is an \((n-1)\)-form on the unit tangent bundle and
Φ is its pull back via the map $E_n : M^n \to TM^n$ defined as $E_n(p) = e_n(p)$. As a consequence we have the following important observation:

**Fact.** The differential form $\Phi$ depends only on the vector field $e_n$. It does not depend on how $e_n$ is extended to an orthonormal frame $e_1, \ldots, e_n$.

From the second structural equation (4) one can derive that

\[
\frac{d\Phi}{dt} = \sum_{i_1 \ldots i_n} c_{i_1 \ldots i_n} \omega^{i_1}_{\mu} \cdots \omega^{i_n}_{\mu},
\]

where $c_{i_1 \ldots i_{n-1}}$ is the Kronecker index which is equal to $+1$ or $-1$ according to whether the permutation $i_1 \ldots i_{n-1}$ of the numbers $1, 2, \ldots, n-1$ is even or odd and the summation is extended over all the indices $i_1 \ldots i_{n-1}$ subject to the condition $i_2 < i_3 < \ldots < i_{n-1}$. It is essentially the same as the $n$-form $\Psi_0$ in [2].

## 2. Convex exhaustion

The other important ingredient is a lemma about convex exhaustion. Although we only need this in the hyperbolic space we state it for Hadamard manifolds. We say that a convex set $C$ with smooth boundary is strictly convex if the second fundamental form of the boundary (with respect to the outward normal) is positive definite everywhere.

**Lemma 1.** Let $M^n$ be a Hadamard manifold and $C_0 \subset \text{int}(C_1)$ be two compact strictly convex subsets with smooth boundary. Then there exists a continuous function $F : C_1 - \text{int}(C_0) \to [0, 1]$ which is smooth in the interior such that: $\nabla F \neq 0$, $\partial C_i = \{p \in C_2 - \text{int}(C_1) : F(p) = i\}$ for $i = 0, 1$ and the sublevel sets $C_b = \{p \in C_2 - \text{int}(C_1) : F(p) \leq b\}$ for $0 \leq b \leq 1$ are convex.

**Proof.** The statement is clear intuitively. Denote by $\varrho_i$ the distance function from the set $\partial C_i$ for $i = 0, 1$.

For $\delta > 0$ we set $N_{2\delta} = \{p \in C_1 : \varrho_1(p) < 2\delta\}$. Since $\partial C_1$ has a positive definite second fundamental form, we can choose $\delta > 0$ small enough such that: $3\delta < \text{dist}(C_0, \partial C_1)$, the function $\varrho_1$ is smooth on $N_{2\delta}$ (there are no focal points of $\partial C_1$ inside $N_{2\delta}$) and

\[
-D^2 \varrho_1 > c_1 > 0,
\]

when restricted to $\nabla \varrho_1^+$ for some positive constant $c_1$. Here, $D^2 \varrho_1$ denotes the Hessian and we adapted the notation that $-D^2 \varrho_1 > c_1$ on $\nabla \varrho_1^+$ if $-D^2 \varrho_1(X, X) > c_1$ for every unit tangent vector $X \in \nabla \varrho_1^+$. We observe also that on $N_{2\delta}$ the following inequality holds for the angle between the gradients:

\[
\angle(\nabla \varrho_0, -\nabla \varrho_1) < \pi/2 - \alpha,
\]

for some $\alpha > 0$ depending on $\delta$.

We are going to construct $F$ in the form

\[
F = 1 - e^{-a\varrho} f^\epsilon.
\]

Here $f = h(\varrho_1)$ is the reparametrized distance function from $\partial C_1$ and $h$ is a fixed smooth increasing real function $h : [0, \infty) \to [0, 1]$ such that $h(t) = \frac{1}{1+t}$ for $0 \leq t \leq \delta$ and $h(t) = 1$ on $[2\delta, \infty)$. For the derivative and the Hessian of $f$ we have

\[
df = h' d \varrho_1 \quad \text{and} \quad D^2 f = h'' d \varrho_1 \otimes \varrho_1 + h' D^2 \varrho_1.
\]

The choice of $a, \epsilon$ will be discussed later.
First, we show that the derivative of $F$ is never zero. From the definition we obtain
\begin{equation}
\label{eq:dF}
dF = e^{-a\theta_0} f^\epsilon (ad\theta_0 - \epsilon f).\end{equation}

Then, taking into consideration that $\nabla f \parallel \nabla \theta_1$, the statement follows from (7).

To show that the sublevel sets are convex we need to show that the Hessian $D^2 F$ is positive definite on $\nabla F^\perp$. We have
\begin{equation}
\label{eq:D2F}
D^2 F = e^{-a\theta_0} (-a^2 d\theta_0 \otimes d\theta_0 + aD^2 \theta_0) f^\epsilon
+ \epsilon e^{-a\theta_0} f^{\epsilon-1} \left( \frac{1 - \epsilon}{f} df \otimes df - D^2 f \right)
+ ae^{-a\theta_0} f^{\epsilon-1} (d\theta_0 \otimes df + df \otimes d\theta_0).
\end{equation}

The argument depends on certain estimates of the Hessians $D^2 \theta_i$ for $i = 0, 1$, on various subspaces. It will be useful to keep in mind that $\theta_i$ (for $i = 0, 1$) are distance functions; therefore the gradients $\nabla \theta_i$ are eigenvectors of $D^2 \theta_i$ with eigenvalue zero.

First, we consider the region where $f \equiv 1$. On this region $df, D^2 f = 0$ and $\nabla F \parallel \nabla \theta_0$. Therefore $D^2 F = a f e^{-a\theta_0} D^2 \theta_0$, when it is restricted to vectors in $\nabla F^\perp$. But on this subspace $D^2 \theta_0$ is positive definite since it is the distance function from a convex set in a Hadamard manifold.

Next, we consider the region where $1/2 < f < 1$. Since $C_0$ is strictly convex and compact, we know that $D^2 \theta_0 > c_2 > 0$ for some $c_2 > 0$, when restricted to $(\nabla \theta_0)^\perp$. Therefore, taking (6) and (9) into account, we conclude that $D^2 \theta_0 > c_3 > 0$ for some positive constant $c_3$, when restricted to $\nabla F^\perp$. So, for a small enough $a$, where the choice of $a$ depends only on $c_3$, the term $aD^2 \theta_0$ will dominate $a^2 d\theta_0 \otimes d\theta_0$. This will remain true throughout the whole region $C_1 - C_0$. The derivatives of $h$ are bounded and so are the terms $df$ and $D^2 f$. Therefore, for a small enough $\epsilon$, the term $aD^2 \theta_0$ will dominate all the other terms as well. The choice of $\epsilon$ depends on $a$ and on the bounds for $df$ and $D^2 f$.

At last, we consider the region where $0 < f < 1/2$. It is clear from the definition of the function $f$ that this region is a subset of $N_{2\delta}$. As before, $aD^2 \theta_0$ will dominate $a^2 d\theta_0 \otimes d\theta_0$. The term involving $df \otimes df$ is positive semi-definite. As for the rest of the terms, we will show that the term involving $-D^2 f$ will dominate, for a small enough $a$, the term involving $d\theta_0 \otimes df + df \otimes d\theta_0$. Since $h$ is linear on this region, from (6) we obtain $-D^2 f = -h' D^2 \theta_1 > c_1 / 2\delta > 0$, when restricted to $(\nabla \theta_1)^\perp$. Taking into consideration (7) and (9) we conclude that $-D^2 f > c_4 > 0$ for some sufficiently small $c_4 > 0$, when restricted to $\nabla F^\perp$. The constant $c_4$ depends on $c_1, \delta$ and the angle $\alpha$. So, if the constant $a > 0$ was chosen small enough, where the choice of $a$ depended only on $c_2$ and $c_4$, then $-D^2 f$ dominates the term involving $d\theta_0 \otimes df + df \otimes d\theta_0$. This concludes the proof of the lemma. \hfill \Box

3. Proof of Theorem 1

With the preparation done in the previous sections, the proof of the theorem is simple. Let us return to the hyperbolic space $H^n$.

First, we prove the inequality between the two integrals. Assume that $C_0 \subset \text{int}(C_1)$ and both sets are strictly convex. We are going to show that
\begin{equation}
\label{eq:integrals}
\int_{\partial C_0} K \leq \int_{\partial C_1} K.
\end{equation}
The general case will follow by a trivial limiting procedure.

Let $F$ be the smooth function of Lemma 1 and define the unit vector field $e_n$ by $e_n = \nabla F/|\nabla F|$. This is defined on $\text{int}(C_1) - C_0$ but it extends continuously to the boundary. To the vector field $e_n$ we construct the form $\Phi$ as in the previous section and by Stokes’s theorem we have

$$
(12) \quad \int_{\partial C_0 \cup \partial C_1} \Phi = \int_{C_1 - C_0} d\Phi.
$$

To evaluate the integrals in (12) we are going to compute the forms $\Phi$ and $d\Phi$. Let $q \in C_1 - \text{int}(C_0)$ be an arbitrarily chosen point.

Since $\Phi$ depends only on the vector field $e_n$, we can express $\Phi$ in a special frame. Let us choose the frame $e_1, ..., e_n$ such that at the point $q \in C_1 - \text{int}(C_0)$ the vectors $e_1, ..., e_{n-1}$ are the principal directions for the hypersurface $\{F = F(q)\}$. For the other points of the hypersurface the vectors $e_1, ..., e_{n-1}$ may no longer be principal directions. Then, from the definition of the $\omega_i^n$’s we have

$$
(13) \quad \omega_i^n(e_j) = \delta^j_i \lambda_j, \quad \text{for} \quad 1 \leq i, j \leq n - 1
$$

at $q \in C_1 - \text{int}(C_0)$, where $\lambda_j$ denotes the principal curvature at $q$ of the hypersurface $\{F = F(q)\}$ in the direction of $e_j$ and $\delta^j_i$ is the Kronecker symbol. Therefore

$$
\Phi(e_1, ..., e_{n-1}) = \lambda_1 \cdot \cdot \cdot \lambda_{n-1} = K
$$

at $q \in C_1 - \text{int}(C_0)$, where $K$ denotes the Gauss-Kronecker curvature of the hypersurface $\{F = F(q)\}$ with respect to the normal field $e_n$. Since $q \in C_1 - \text{int}(C_0)$ was chosen arbitrarily, the left-hand side of (12) reads as follows:

$$
\int_{\partial C_0 \cup \partial C_1} \Phi = \epsilon_{n2...n-1} \left( \int_{\partial C_1} K - \int_{\partial C_0} K \right) = (-1)^{n-1} \left( \int_{\partial C_1} K - \int_{\partial C_0} K \right).
$$

The curvature tensor has the form $R(X, Y)Z = (Y, Z)X - (X, Z)Y$; therefore

$$
\Omega_i^n(e_j, e_k) = 0
$$

if $\{j, k\} \neq \{i, n\}$ as sets. From this and (13) we have

$$
\Omega_i^n \omega_i^n \omega_{i2}^{n-1} (e_1, ..., e_n) = \epsilon_{i1i2...i_{n-1}} K_{i1}^{-1} \lambda_{i2} \cdot \cdot \cdot \lambda_{i_{n-1}}
$$

at $q \in C_1 - \text{int}(C_0)$, where the indices satisfy the condition $i_2 < i_3 < ... < i_{n-1}$ and $K_{in}$ is the sectional curvature of the two-plane determined by $e_i, e_n$ and $\lambda_i$ is the principal curvature at the point $q \in C_1 - \text{int}(C_0)$ in the direction of $e_i$. Then (12) reads as follows:

$$
\int_{\partial C_1} K - \int_{\partial C_0} K = \int_{C_1 - C_0} \sum_{i=1}^{n-1} -K_{i1} \lambda_{i2} \cdot \cdot \cdot \lambda_{i_{n-1}},
$$

where the summation is extended over all the indices $i_1...i_{n-1}$ subject to the condition $i_2 < i_3 < ... < i_{n-1}$.

Since all sublevel sets are convex, all the principal curvatures are positive. Therefore, the integral on the right-hand side is positive. This completes the proof of the theorem, when $C_0 \subset \text{int}(C_1)$ and both sets are strictly convex. The general case follows by slightly "blowing up" the sets; that is, instead of $C_0$ we consider an $\eta$-neighborhood $C_0$ and instead of $C_1$ we take a $2\eta$-neighborhood $C_1 + 2\eta$. These are now strictly convex sets satisfying the conditions set forth at the beginning of the proof. Then letting $\eta$ go to 0 will yield the general case.
All that remains is to prove the inequality
\[ \text{Vol}(S^{n-1}) \leq \int_{\partial C_0} K. \]

This is a simple consequence of (11). Choose a ball \( B_\epsilon \) inside \( C_0 \). Applying (11), we obtain
\[ \int_{\partial B_\epsilon} K < \int_{\partial C_0} K. \]
Letting \( \epsilon \) go to 0 will yield the desired inequality. This completes the proof of the theorem.

REFERENCES


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