PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 3, Pages 673–674
S 0002-9939(01)06108-1
Article electronically published on September 28, 2001

RADICALS AND PLOTKIN'S PROBLEM CONCERNING GEOMETRICALLY EQUIVALENT GROUPS

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(Communicated by Stephen D. Smith)

ABSTRACT. If G and X are groups and N is a normal subgroup of X, then the G-closure of N in X is the normal subgroup $\overline{X}^G = \bigcap\{\ker\varphi|\varphi : X \to G$, with $N \subseteq \ker\varphi\}$ of X. In particular, $\overline{1}^G = R_G X$ is the G-radical of X. Plotkin calls two groups G and H geometrically equivalent, written $G \sim H$, if for any free group F of finite rank and any normal subgroup N of F the G-closure and the H-closure of N in F are the same. Quasi-identities are formulas of the form $(\bigwedge_{i\le n} w_i = 1 \to w = 1)$ for any words w, w_i $(i \le n)$ in a free group. Generally geometrically equivalent groups satisfy the same quasi-identities. Plotkin showed that nilpotent groups G and H satisfy the same quasi-identities if and only if G and H are geometrically equivalent. Hence he conjectured that this might hold for any pair of groups. We provide a counterexample.

In a series of paper, B. I. Plotkin and his collaborators [6, 3, 4, 5] investigated radicals of groups and their relation to quasi-identities. If G is a group, then the *G-radical $R_G X$ of a group X* is defined by

$$R_G X = \bigcap\{\ker\varphi|\varphi : X \to G \text{ any homomorphism}\}.$$

Clearly, $R_G X$ is a characteristic, hence a normal subgroup of X. The radical R_G can also be used to define the *G-closure $\overline{U}^G = \overline{U}$ of a normal subgroup U of X*, by saying that $\overline{U}/U = R_G(X/U)$. This immediately leads to Plotkin's definition of geometrically equivalent groups (see [6, 3, 4, 5] and [2, p. 113]).

Definition 0.1. Let G and H be two groups. Then G and H are geometrically equivalent, written $G \sim H$, if for any free group F of finite rank and any normal subgroup U of F the G- and H-closures of U in F are the same; i.e., for any normal subgroup U we have $\overline{U}^G = \overline{U}^H$.

It is easy to see that $G \sim H$ if and only if $R_G K = R_H K$ for all finitely generated groups K. Plotkin notes that geometrically equivalent groups satisfy the same quasi-identities. The well-known notion of quasi-identities relates to quasivarieties of groups. A *quasi-identity* is an expression of the form

$$w_1 = 1 \wedge \cdots \wedge w_n = 1 \to w = 1 \text{ where } w_i, w \in F \ (i \le n) \text{ are words.}$$

Received by the editors September 6, 2000 and, in revised form, September 21, 2000.
2000 *Mathematics Subject Classification.* Primary 20E06, 20E10, 20E32; Secondary 20F06.
The authors were supported by project No. G 0545-173, 06/97 of the German-Israeli Foundation for Scientific Research & Development. This paper is #GbSh 741 in Shelah's list of publications.

Moreover the following was shown in [6] (see [2, p.113]).

Theorem 0.2. (a) *If $G \sim H$, and G is torsion-free, then H is torsion-free.*

(b) *If G, H are nilpotent, then $G \sim H$ if and only if G and H satisfy the same quasi-identities.*

This led Plotkin to conjecture that two groups might be geometrically equivalent if and only if they satisfy the same quasi-identities (see the Kourovka Notebook [2, p.113, problem 14.71]). In this note we refute this conjecture. Clearly there are only countably many finitely presented groups which we enumerate as the set $\mathfrak{K} = \{K_n : n \in w\}$ and let $G = \prod_{n \in w} K_n$ be the restricted direct product. Then G satisfies only those quasi-identities satisfied by all groups and so if H is any group with $G \leq H$, G satisfies the same quasi-identities as H.

R. Camm [1, p. 68, p. 75 Corollary] proved there are $2^{\aleph_0}$ non-isomorphic, two-generator, simple groups (see also Lyndon, Schupp [7, p. 188, Theorem 3.2]). So there exists a 2-generated simple group L which cannot be mapped nontrivially into G. We consider the pair G, $H = L \times G$ and show the following:

Theorem 0.3. *If G, H and L are as above, $R_G L = L$ and $R_H L = 1$. In particular G and H are not geometrically equivalent. Since $G \leq H$ satisfy the same quasi-identities, this is the required counterexample.*

Proof. Since L is a two-generated simple group, L is an epimorphic image of a free group of rank 2. So it is enough to prove that $R_G L = L$ and $R_H L = 1$. The first equality follows since there is no nontrivial homomorphism of L into G. On the other hand, there is a canonical embedding $L \to H = L \times G$, so $R_H L = 1$. $\square$

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