Sums of Numbers with Small Partial Quotients

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Abstract. In a paper of James Hlavka it is stated that $F(3)+F(2)+F(2) \neq \mathbb{R}$. In this manuscript we show that this is false by establishing that $F(3) \pm F(2) \pm F(2) = \mathbb{R}$. We also describe the corresponding products and quotients.

1. Introduction

For any positive integer $m$ let $F(m)$ be the set of numbers

$$F(m) = \{ [t, a_1, a_2, \ldots]; t \in \mathbb{Z}, 1 \leq a_i \leq m \text{ for } i \geq 1 \}$$

where by $[a_0, a_1, a_2, \ldots]$ we denote the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

with partial quotients $a_0, a_1, a_2$ and so on. In 1947 Marshall Hall, Jr. [7] proved that

$$F(4) + F(4) = \mathbb{R}$$

where for two sets $A$ and $B$ of real numbers we denote by $A + B$ the set

$$A + B = \{ a + b ; a \in A \text{ and } b \in B \}.$$ 

James Hlavka improved this result in 1975 [8] establishing that in fact

$$F(4) + F(3) = \mathbb{R}.$$ 

Hlavka also proved several other similar results, including

$$F(7) + F(2) = \mathbb{R} \text{ and } F(3) + F(3) + F(2) = \mathbb{R}.$$ 

However, there were several cases that Hlavka was unable to treat with his techniques. In [4] the author used a new approach to establish that $F(5) \pm F(2) = \mathbb{R}$, where for sets $A$ and $B$ we let $A - B$ denote the pointwise difference of the two sets. Hanno Schecker [9] and Gregory Freiman [6] independently showed that $F(3) + F(3)$ contains many large intervals (it is easy to show that $F(3) + F(3) \neq \mathbb{R}$), while in [1]...
the author used similar techniques to give a complete characterization of $F(3) + F(3)$ and prove that $F(3) - F(3) = \mathbb{R}$.

In this manuscript we deal with one of the few remaining cases. Hlavka claimed (without proof) that $F(3) + F(2) + F(2) \neq \mathbb{R}$, but in this paper we will show that this is not true. We shall establish the following result.

**Theorem 1.1.** We have

$$F(3) \pm F(2) \mp F(2) = \mathbb{R}.$$  

Several cases still remain open. For example, at present we do not know whether or not either of the sets $F(3) + F(2)$ or $F(2) + F(2) + F(2)$ contain intervals. In fact, the best result concerning the Hausdorff dimensions of these sets (see [2]) is

$$\dim_H(F(3) + F(2)) \geq 0.808 \quad \text{and} \quad \dim_H(F(2) + F(2) + F(2)) \geq 0.886.$$  

Using the ideas behind the proof of Theorem 1.1 in [5] the author proved the following theorem.

**Theorem 1.2.** There exists a constant $c$ such that

$$F(3)F(2)F(2) \supseteq (-\infty, -c] \cup [c, \infty).$$

Further,

$$F(3)F(2)/F(2) = F(2)F(2)/F(3) = \mathbb{R}\setminus\{0\}.$$  

2. Background

We define a *generalized Cantor set* (henceforth known as a *Cantor set*) to be any set $C$ of real numbers of the form

$$C = I \setminus \bigcup_{i \geq 1} O_i,$$

where $I$ is a finite closed interval and $\{O_i \mid i \geq 1\}$ is a countable collection of disjoint open intervals contained in $I$. Equivalently, we may describe a Cantor set $C$ by construction from the interval $I$.

$$I$$

$$I^0$$

$$O_I$$

$$I^1$$

$$I^{00} \quad O_{I^0} \quad I^{01} \quad O_I \quad I^{10} \quad O_{I^1} \quad I^{11}$$

$$\vdots \quad \vdots \quad \vdots$$

We have

$$C = I \setminus \bigcup_{w} O_{I^w} = \bigcap_{n \geq 0} \left( \bigcup_{|w| = n} I^w \right)$$

where the first union is over all binary words and the second is over all binary words of length $n$ (for a more detailed explanation the reader is directed to [3]).
This process is called a derivation of \( C \) from \( I \). The intervals \( I, I^0, \ldots \) are called bridges of the derivation, while the open intervals \( O_I, O_{I^0}, \ldots \) are called gaps of \( C \). We define the thickness of the derivation \( D \) to be

\[
\tau(D) = \inf_w \left( \frac{\min(|I^0_w|,|I^1_w|)}{|O_{I^w}|} \right).
\]

We define the thickness of the Cantor set \( C \) to be

\[
\tau(C) = \sup_D \tau(D)
\]

where the supremum is over all derivations \( D \) of \( C \). It is not difficult to show that the supremum is attained if the sequence \( \{|O_i|\} \) is non-decreasing (see Lemma 3.1 of [3]). We also define the normalized thickness of \( C \), \( \gamma(C) \), to be

\[
\gamma(C) = \frac{\tau(C)}{\tau(C) + 1}.
\]

Let \( k \) be an integer which is at least 2, and assume that for \( 1 \leq j \leq k \), \( C_j \) is a Cantor set derived from \( I_j \). We would like to determine when

\[
C_1 + \cdots + C_k = I_1 + \cdots + I_k.
\]

If \( I_1, \ldots, I_{k-1} \) are all much smaller than one of the gaps in \( C_k \), then (1) cannot hold. Hence in our approach to finding sums of Cantor sets we will only consider sets that are approximately the same size, as follows. Let \( k \) be an integer which is at least 2, and assume that for \( 1 \leq j \leq k \), \( A_j \) is a bridge of the Cantor set \( C_j \), with \( O_j \) a gap of \( C_j \) of maximal size contained in \( A_j \). We say that the sequence of bridges \((A_1, \ldots, A_k)\) is compatible if

\[
|A_{r+1}| \geq |O_j| \quad \text{and} \quad |A_1| + \cdots + |A_r| \geq |O_{r+1}|
\]

for \( r = 1, \ldots, k-1 \) and \( j = 1, \ldots, r \). Note that if \( k = 2 \), then this is equivalent to the condition

\[
|A_1| \geq |O_2| \quad \text{and} \quad |A_2| \geq |O_1|.
\]

In [3] the author derived a result concerning the sum of a finite number of Cantor sets.

**Theorem 2.1.** Let \( k \) be a positive integer and for \( j = 1, \ldots, k \) let \( C_j \) be a Cantor set derived from \( I_j \). Put \( S_\gamma = \gamma(C_1) + \cdots + \gamma(C_k) \), and assume that \((I_1, \ldots, I_k)\) is compatible. If \( S_\gamma \geq 1 \), then

\[
C_1 + \cdots + C_k = I_1 + \cdots + I_k.
\]

Otherwise

\[
\gamma(C_1 + \cdots + C_k) \geq S_\gamma \quad \text{and} \quad \dim_H(C_1 + \cdots + C_k) \geq \frac{\log 2}{\log (1 + 1/S_\gamma)}.
\]

For any positive integer \( m \) we put

\[
g(m) = \frac{-m + \sqrt{m^2 + 4m}}{2}, \quad C(m) = [0,1] \cap F(m)
\]
and let $I(m)$ be the closed interval
\[
I(m) = \left[ \left[ 0, \overline{m, 1}, 0, \overline{1, m} \right] \right] = \left[ \frac{g(m)}{m}, g(m) \right]
\]
\[
= \left[ \frac{-1 + \sqrt{1 + 4/m}}{2}, \frac{-m + \sqrt{m^2 + 4}}{2} \right].
\]

We may characterize $C(m)$ as a Cantor set derived from $I(m)$ in the following manner. For any real $a$ and $b$, we denote by $[[a, b]]$ and $((a, b))$ the intervals
\[
[[a, b]] = \{ \min\{a, b\}, \max\{a, b\} \}
\]
and
\[
((a, b)) = (\min\{a, b\}, \max\{a, b\}).
\]

Assume that
\[
A = [[0, a_1, \ldots, a_r, b, m, 1], [0, a_1, \ldots, a_r, m, 1, m]]
\]
is a bridge of $C(m)$ of level $n$ with $b < m$. We define $A^0$, $A^1$, and $O_A$ by setting
\[
A^0 = [[0, a_1, \ldots, a_r, b, m, 1], [0, a_1, \ldots, a_r, m, 1, m]],
\]
\[
O_A = ((0, a_1, \ldots, a_r, b, 1, m], [0, a_1, \ldots, a_r, b + 1, m, 1])
\]
and
\[
A^1 = [[0, a_1, \ldots, a_r, b + 1, m, 1], [0, a_1, \ldots, a_r, m, 1, m]].
\]

Note that $A^0$ is of the form \((2)\) with $a_{r+1} = b$ and $b$ replaced by 1. Similarly $A^1$ is also of the form \((2)\). Since $I(m)$ is of the form \((2)\) with $r = 0$ and $b = 1$, by induction we obtain a derivation of $C(m)$ from $I(m)$.

By calculation it can be shown (see Lemma 4.2 of \([3]\)) that
\[
\tau(C(m)) = \frac{g(m)(m - 1)}{m - g(m)(m - 1)}.
\]

Hence we may easily calculate $\tau(C(m))$ and $\gamma(C(m))$. For example,
\[
\gamma(C(2)) = 0.267 \ldots \quad \text{and} \quad \gamma(C(3)) = 0.451 \ldots .
\]

3. PROOF OF THE MAIN RESULT

Proof of Theorem 1.1 We first consider the set $C(3) + C(2) + C(2)$. Note that from \((3)\) we have
\[
\gamma(C(3)) + \gamma(C(2)) + \gamma(C(2)) = 0.98 \ldots < 1;
\]
hence we cannot use Theorem 2.1 with the Cantor sets $C(3)$, $C(2)$, and $C(2)$ to find intervals in $C(3) + C(2) + C(2)$. However, since thickness (and therefore normalized thickness) is an infimum, we might hope to increase it by looking at Cantor sets properly contained in $C(3)$ or $C(2)$. For $m = 2$ or $m = 3$ let $w = a_1 \ldots a_r$ be a word with digits $a_i$ between 1 and $m$ inclusive. We denote by $I(m; w)$ the bridge of $C(m)$
\[
I(m; w) = [[0, a_1, \ldots, a_r, \overline{m, 1}, [0, a_1, \ldots, a_r, \overline{m, 1}]]
\]
and put $C(m; w) = C(m) \cap I(m; w)$. By calculation we find that if $w, v \neq \emptyset$, then
\[
\gamma(C(2; v)) \geq 0.2800 \quad \text{and} \quad \gamma(C(3; w)) \geq 0.4565.
Note that this implies that
\[\gamma(C(3; w)) + \gamma(C(2)) + \gamma(C(2; v)) \geq 1.004\]  
and
\[\gamma(C(3)) + \gamma(C(2; v)) + \gamma(C(2; v)) \geq 1.011.\]

By calculation we have
\[|I(2)| = 0.3660 \ldots, \quad |O_{I(2)}| = 0.1547 \ldots,\]
\[|I(2; 1)| = 0.1547 \ldots, \quad |O_{I(2; 1)}| = 0.0689 \ldots,\]
\[|I(2; 2)| = 0.0566 \ldots, \quad |O_{I(2; 2)}| = 0.0247 \ldots,\]
\[|I(3)| = 0.5275 \ldots, \quad |O_{I(3)}| = 0.1165 \ldots,\]
\[|I(3; 1)| = 0.2330 \ldots, \quad |O_{I(3; 1)}| = 0.0518 \ldots,\]
\[|I(3; 3)| = 0.0426 \ldots, \quad |O_{I(3; 3)}| = 0.0095 \ldots.\]

Therefore
\[(I(2; 2), I(3; 3), I(2; 1)) \quad \text{and} \quad (I(2; 1), I(2; 1), I(3))\]
are compatible. By (4), (5), and Theorem 2.1 we have
\[C(2; 2) + C(3; 3) + C(2; 1) = I(2; 2) + I(3; 3) + I(2; 1) = \[1.20 \ldots, 1.46 \ldots]\]
and
\[C(2; 1) + C(2; 1) + C(3) = I(2; 1) + I(2; 1) + I(3) = \[1.41 \ldots, 2.25 \ldots].\]

Thus
\[C(3) + C(2) + C(2) \supseteq \[1.20 \ldots, 2.25 \ldots]\]
so
\[F(3) + F(2) + F(2) = \mathbb{Z} + C(3) + C(2) + C(2) = \mathbb{R}\]

as required.

The remaining results are proved in an analogous fashion. Note that if \(C\) is a Cantor set, then \(-C\) is a Cantor set, and \(\tau(-C) = \tau(C)\). Now,
\[(I(2; 1), -I(3; 1), I(2)) \quad \text{and} \quad (I(2; 1), I(2; 1), -I(3))\]
are compatible. Since
\[I(2; 1) - I(3; 1) + I(2) = [0.15 \ldots, 0.90 \ldots]\]
and
\[I(2; 1) + I(2; 1) - I(3) = [0.36 \ldots, 1.20 \ldots]\]
we have
\[F(2) + F(2) - F(3) = \mathbb{R}.\]

Similarly
\[(I(2; 1), -I(2; 1), I(3)) \quad \text{and} \quad (I(2; 1), I(3; 1), -I(2))\]
are compatible, with
\[I(2; 1) - I(2; 1) + I(3) = [0.10 \ldots, 0.94 \ldots]\]
and

\[ I(2;1) + I(3;1) - I(2) = [0.40\ldots, 1.15\ldots]. \]

Thus

\[ F(2) - F(2) + F(3) = \mathbb{R} \]

and the theorem follows. \qed

References


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