FINITE DIMENSIONAL REPRESENTATIONS OF THE SOFT TORUS

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Abstract. The soft tori constitute a continuous deformation, in a very precise sense, from the commutative C*-algebra $C(T^2)$ to the highly non-commutative C*-algebra $C^*(F_2)$. Since both of these C*-algebras are known to have a separating family of finite dimensional representations, it is natural to ask whether that is also the case for the soft tori. We show that this is in fact the case.

1. Introduction

Knowing that a given C*-algebra has many representations on finite dimensional Hilbert spaces is of great importance to understanding structural properties of it. Among C*-algebras, those who possess a separating family of finite dimensional representations are called residually finite dimensional or just RFD. This class was studied in [12, 11] and [1], and more recent insight about it has lead to important advances in classification theory and the theory of quasidiagonal C*-algebras (see, e.g., [2, 5] and [6]).

For any $\varepsilon \geq 0$ we define a C*-algebra $A_\varepsilon$ as the universal (unital) C*-algebra defined by the generators $u, v$ subject to the relations

$$uu^* = u^*u = 1, \quad vv^* = v^*v = 1, \quad \|uv - vu\| \leq \varepsilon.$$ 

As recorded in [8], $A_0$ is the commutative C*-algebra of functions over the torus $T^2$, and $A_2$ is the full C*-algebra of the free group of two generators $F_2$ whenever $\varepsilon \geq 2$. For $\varepsilon$ between 0 and 2 we get a class of C*-algebras which are commonly referred to as soft tori. These C*-algebras are of relevance to several problems in operator algebra theory (see [10]) and have been extensively studied in [3, 8, 9, 7].

The starting point of the investigation reported on in the present paper is a result from [9], stating that the soft tori form a continuous field interpolating between the (hard) torus and the group C*-algebra of the free group. Since $C(T^2)$ is obviously RFD, and since $C^*(F_2)$ was proved to be RFD in [4] — a surprise at the time — we are naturally led to the question of whether the same is true for the interpolating family $A_\varepsilon$. We are going to prove that this is the case.

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2. Methods

We prove that $A_\varepsilon$ is RFD using an adaptation of the compression argument developed by Choi in [4] to prove that $C^*({\mathbb F}_2)$ is RFD. However, Choi’s argument does not apply directly to $A_\varepsilon$ since one cannot arrange for the compressions to satisfy the commutation relation. We instead argue via an auxiliary $C^*$-algebra, thus employing a method from [3] and [8] which lies behind many results about the structural properties of $A_\varepsilon$.

We define $B_\varepsilon$ as the universal $C^*$-algebra given by the generators $\{u_n\}_{n \in \mathbb Z}$ and the relations

$$u_n u_n^* = u_n^* u_n = 1, \quad \|u_{n+1} - u_n\| \leq \varepsilon.$$  

(1)

Clearly one can define an automorphism $\alpha$ on $B_\varepsilon$ by

$$u_n \mapsto u_{n+1},$$

and as seen in [8] one has $A_\varepsilon = B_\varepsilon \rtimes_\alpha \mathbb Z$.

There is a faithful conditional expectation $E_\alpha : A_\varepsilon \to B_\varepsilon$.

Our strategy will be to prove that $A_\varepsilon$ is RFD by proving that $B_\varepsilon$ is RFD in a way which is covariant with $\alpha$.

3. Finite dimensional representations of $B_\varepsilon$

We start out by finding a new picture of $B_\varepsilon$ by generators and relations.

Lemma 3.1. For any $\varepsilon < 2$, $B_\varepsilon$ is isomorphic to the universal $C^*$-algebra generated by $v_0, \{h_n\}_{n \in \mathbb Z}$ subject to the relations

$$v_0 v_0^* = v_0^* v_0 = 1, \quad h_n = h_n^*, \quad \|h_n\| \leq \frac{2}{\pi} \arcsin(\varepsilon/2).$$  

(2)

Proof. Let us denote the $C^*$-algebra generated by $v_0$ and $h_n$ subject to (2) by $B'_\varepsilon$. We can define a map $\varphi : B_\varepsilon \to B'_\varepsilon$ by

$$u_n \mapsto \begin{cases} e^{i\pi h_n} \ldots e^{i\pi h_1} v_0, & n > 0, \\ v_0, & n = 0, \\ e^{-i\pi h_n} \ldots e^{-i\pi h_{-1}} v_0, & n < 0, \end{cases}$$

since the elements to the right of the arrow above satisfy the relations (1). Similarly, the universal property of $B'_\varepsilon$ allows for a map $\psi : B'_\varepsilon \to B_\varepsilon$ defined by

$$v_0 \mapsto u_0, \quad h_n \mapsto \frac{1}{i\pi} \Log(u_n u_{n-1}^*).$$

Clearly $\varphi$ and $\psi$ are each others’ inverse.

This characterization can be used to shorten the proof of [8, 2.2], stating that $B_\varepsilon$ is homotopic to $C(T)$. To see this, define maps $\varphi : B'_\varepsilon \to C(T)$ and $\psi : C(T) \to B'_\varepsilon$ by the correspondence $v_0 \leftrightarrow [z \mapsto z], h_n \leftrightarrow 0$. Clearly $\varphi \psi = \text{id}_{C(T)}$, and $\chi_t : B'_\varepsilon \to B'_\varepsilon$ given by

$$\chi_t(v_0) = v_0, \quad \chi_t(h_n) = t h_n$$

provides a homotopy from $\text{id}_{B'_\varepsilon}$ to $\psi \varphi$.

In the following proof, we denote by $\text{Alg}(X)$ the smallest $*$-algebra, not necessarily closed, generated by the set $X$ inside some $C^*$-algebra.
Proposition 3.2. For any $\varepsilon < 2$, $B_\varepsilon$ is RFD. In fact, for any $0 \neq b \in B_\varepsilon$ there exists $n \in \mathbb{N}$, an automorphism $\beta$ of $M_n$ and a representation $\rho : B_\varepsilon \to M_n$ with the properties

$$\rho(b) \neq 0, \quad \beta \rho = \rho \alpha.$$  

Proof. For the first claim we use the characterization of $B_\varepsilon$ given by the relations in (2) and proceed as in (3). Fix a faithful non-degenerate representation $\pi : B_\varepsilon \to \mathbb{B}(\mathcal{H})$, where we may assume that $\mathcal{H}$ is a separable Hilbert space.

Let $P_m$ be a sequence of projections, with $\text{rank}(P_m) = m$, converging strongly to the unit of $\mathbb{B}(\mathcal{H})$, and abbreviate $T_{0,m} = P_m \pi(v_0) P_m$, $K_{n,m} = P_m \pi(h_n) P_m$.

Now note that for each $m$ the collection of elements $\{V_{0,m}, H_{n,m} : n \in \mathbb{Z}\}$ defined by

$$V_{0,m} = \begin{pmatrix} T_{0,m} & \sqrt{T_{0,m} - T_{0,m}^* T_{0,m}^*} \\ \sqrt{T_{0,m}^* - T_{0,m} T_{0,m}^*} & -T_{0,m}^* \end{pmatrix},$$

$$H_{n,m} = \begin{bmatrix} K_{n,m} & 0 \\ 0 & K_{n,m} \end{bmatrix}$$

satisfies (2) in $M_2(\mathbb{B}(\mathcal{H})P_m) \simeq M_{2m}$. Consequently we get representations $\pi_m : B_\varepsilon \to M_{2m}$. We are going to check, following (3), that

$$\pi : B_\varepsilon \to \prod_{m=1}^{\infty} M_{2m}, \quad \pi(b) = (\pi_m(b))_{m=1}^{\infty}$$

is an isometry. It suffices to check that $\|\pi(x)\| \geq \|x\| - \eta$ for any $\eta > 0$ and any $x \in \text{Alg}(\{v_0, h_{-N}, \ldots, h_N\})$. Fix $\eta$, $N$ and $x$ and write

$$x = F(v_0, h_{-N}, \ldots, h_N)$$

where $F$ is some finite linear combination of finite words in $2N + 2$ variables and their adjoints. Since, when $m$ goes to infinity,

$$V_{0,m} \to [\pi(v_0) 0 \quad 0 -\pi(v_0)^*], \quad H_{n,m} \to [\pi(h_n) 0 \quad 0 \pi(h_n)^*],$$

strongly in the unit ball of $M_2(\mathbb{B}(\mathcal{H}))$ we conclude that

$$\limsup_m \|F(V_{0,m}, H_{-N,m}, \ldots, H_{N,m})\| \geq \lim_{m \to \infty} \|F(V_{0,m}, H_{-N,m}, \ldots, H_{N,m})\| \geq \|\pi(x)\| \geq \|x\| - \eta.$$  

We can hence find $m$ such that

$$\|\pi(x)\| \geq \|\pi_m(x)\| = \|F(V_{0,m}, H_{-N,m}, \ldots, H_{N,m})\| \geq \|x\| - \eta.$$  

For the second claim, we go back to the original presentation (1) of $B_\varepsilon$ by unitary generators only. For a given $b \in B_\varepsilon$ with $\|b\| = 1$ we fix, using the first part of the proposition, a finite dimensional representation $\pi : B_\varepsilon \to M_m$ with $\|\pi(b)\| > \frac{1}{4}$. We also fix $c$ and $N \in \mathbb{N}$ such that

$$\|b - c\| < \frac{1}{4}, \quad c \in \text{Alg}(u_{-N}, \ldots, u_N).$$
Choose $M > 0$ and unitaries $v_0^+, \ldots, v_M^+ \in M_m$ with the properties
\[ \|v_{n+1}^+ - v_n^+\| \leq \varepsilon, \quad v_0^+ = \pi(u_{nN}), \quad v_M^+ = 1. \]
There is then exactly one representation $\pi' : B_\varepsilon \rightarrow M_m$ which is $2(N + M)$-periodic in the sense that $\pi'(u_n) = \pi'(u_{n+2(N+M)})$ and satisfies
\[ \pi'(u_n) = \begin{cases} v_{-N-n}^-, & -M - N \leq n < -N, \\ \pi(u_n), & -N \leq n \leq N, \\ v_n^-N^-, & N < n \leq N + M. \end{cases} \]
Note that $\pi'(c) = \pi(c)$; in particular $\|\pi'(c)\| \geq \frac{1}{2}.$

Now let $n = 2(N + M)m$. With $\beta$ defined as the backward cyclic shift in block form (with period $2(N + M)$) we may define a covariant representation $\rho$ of $B_\varepsilon$ on $M_n$ by
\[ u_i \mapsto \begin{bmatrix} \pi'(u_i) & & \\ & \pi'(u_{i+1}) & \\ & & \ddots \\ & & & \pi'(u_{i+2(N+M)-1}) \end{bmatrix}. \]
We have
\[ \|\rho(b)\| \geq \|\pi'(b)\| \geq \|\pi'(c)\| - \frac{1}{4} > 0. \]

4. Finite dimensional representations of $A_\varepsilon$

**Theorem 4.1.** For any $\varepsilon > 0$, let $A_\varepsilon$ be the universal $C^*$-algebra generated by a pair of unitaries subject to the relation $\|uv - vu\| \leq \varepsilon$. Then $A_\varepsilon$ is residually finite dimensional in the sense that it admits a separating family of finite dimensional representations.

**Proof.** We may assume that $0 < \varepsilon < 2$. Let $0 \neq a \in A_\varepsilon$. Then also $b = E_\alpha(a^*a)$ is nonzero, for the conditional expectation is faithful. Choose $n$, $\rho$ and $\beta$ as in Proposition 3.2 and define
\[ \pi : A_\varepsilon = B_\varepsilon \rtimes_\alpha \mathbb{Z} \rightarrow M_n \rtimes_\beta \mathbb{Z} \]
as the extension to the crossed product of the covariant $*$-homomorphism $\rho$. We then have, with $E_\beta$ the conditional expectation from $M_n \rtimes_\beta \mathbb{Z}$ to $M_n$,
\[ E_\beta(\pi(a^*a)) = \pi(E_\alpha(a^*a)) = \rho(b) \neq 0, \]
so $\pi(a) \neq 0$.

Note finally that since $\beta$ is inner,
\[ M_n \rtimes_\beta \mathbb{Z} = M_n \rtimes_{1d} \mathbb{Z} \cong C(T) \otimes M_n. \]
Therefore we may compose $\pi$ with an evaluation map of $C(T) \otimes M_n$ to exhibit an $n$-dimensional representation which does not vanish on $a$. \qed

Linear algebra tells us that $M_n$ has a faithful tracial state and has the property that every matrix $x$ which is *hyponormal* is the sense that
\[ x^*x \geq xx^* \]
is in fact normal. As in [4], we may conclude:
Corollary 4.2. For any ε, $A_ε$ has a faithful tracial state, and any hyponormal operator in $A_ε$ is normal.

Proof. Such properties clearly pass from matrices to sums of the form $\prod_{n\in\mathbb{N}} M_{m_n}$, and from these sums to any of their subalgebras. By the theorem, $A_ε$ is one such.

References


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