

SOLUTIONS TO A CLASS OF SCHRÖDINGER EQUATIONS

YANHENG DING

(Communicated by David S. Tartakoff)

ABSTRACT. We establish existence and multiplicity of solutions to a class of nonlinear Schrödinger equations with, e.g., “atomic” Hamiltonians, via critical point theory.

1. INTRODUCTION AND RESULTS

Consider the Schrödinger equation

$$(1.1) \quad -i \frac{d\varphi}{dt} = A\varphi + G_\varphi(x, \varphi),$$

where A stands for a Schrödinger operator in $L^2(\mathbb{R}^3)$, and $G : \mathbb{R}^3 \times \mathbb{C} \rightarrow \mathbb{R}$ a nonlinear coupling with $G(x, 0) = 0$ and $G(x, e^{i\theta t}u) = G(x, u)$. We are interested in existence and multiplicity of solutions of the form

$$\varphi(t, x) = e^{i\lambda t}u(x) \quad \text{with} \quad u \in H^1(\mathbb{R}^3).$$

Substituting it in (1.1) one sees that u satisfies

$$(S) \quad \begin{cases} Au - \lambda u + g(x, u) = 0, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

where $g(x, u) := G_u(x, u)$. In this paper we deal with the following Schrödinger operators:

- The Schrödinger operator with Rollnik potentials

$$A_1 := -\Delta + V(x),$$

that is, for any $\epsilon > 0$, there is an expression $V = V_1 + V_2 \in \mathcal{R} + (L^\infty)_\epsilon$ where \mathcal{R} is the class of real functions W satisfying

$$\|W\|_{\mathcal{R}}^2 := \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|W(x)||W(y)|}{|x-y|^2} dx dy < \infty,$$

and $(L^\infty)_\epsilon$ indicates the class of real functions W satisfying $\|W\|_\infty < \epsilon$. Suppose that $V(x) \leq -ar^{-2+\epsilon}$ for $r := |x| > R_0$ some R_0 and $a > 0, \epsilon > 0$.

Received by the editors August 15, 2000.

1991 *Mathematics Subject Classification*. Primary 35Q55; Secondary 58E55.

Key words and phrases. Schrödinger equations, multiple solutions, critical point theory.

This research was supported by the Special Funds for Major State Basic Research Projects of China.

- The Hamiltonian of the helium atom

$$A_2 := -\Delta_1 - \Delta_2 - \frac{2}{|\mathbf{r}_1|} - \frac{2}{|\mathbf{r}_2|} + \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

where $\mathbf{r}_i = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\Delta_i = \sum_{k=1}^3 \frac{\partial^2}{\partial x_k^2}$.

- The general “atomic” Hamiltonian of the form

$$A_3 := \sum_{i=1}^m \left(-\frac{\Delta_i}{2\mu_i} - \frac{m}{|\mathbf{r}_i|} \right) + \sum_{i < j} \left(\frac{\nabla_i \cdot \nabla_j}{M} + \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} \right),$$

a system consisting of the nucleus of mass M and m electrons of masses μ_1, \dots, μ_m after the center of mass motion is removed.

Recall that the operator $A = A_i$ ($i = 1, 2, 3$) is selfadjoint with $\mathcal{D}(A_i) \subseteq H^2(\mathbb{R}^3)$. Moreover, the bound state energies $\sigma(A_i) \cap [\lambda_0, \lambda_e]$ is an infinite set (cf. [13]), where $\lambda_0 := \inf \sigma(A)$, $\lambda_e := \inf \sigma_{ess}(A)$ ($\sigma(A)$ and $\sigma_{ess}(A)$ denote the spectrum and essential spectrum). Assume first that g satisfies the following superlinear hypotheses:

- (g1) $g(x, u) = o(|u|)$ uniformly with respect to x as $u \rightarrow 0$;
- (g2) there exist $\gamma > 2$, $\delta > 0$ such that $0 < \gamma G(x, u) \leq ug(x, u)$ whenever $u \neq 0$ and $G(x, u) \geq \delta$ whenever $|u| = 1$;
- (g3) there are $p > 2$ and $c > 0$ such that

$$|g(x, u + v) - g(x, u)| \leq c(1 + |u|^{p-2} + |v|^{p-2})|v|.$$

For obtaining multiple solutions we assume g is odd in u , that is,

- (g4) $g(x, -u) = -g(x, u)$.

In the following, for any $\lambda \leq \lambda_e$, let $\mathcal{N}(\lambda)$ denote the number of all eigenvalues of A which are less than λ and counted in multiplicity.

Theorem 1. (i) Assume (g1) – (g3). Then (S) has no nontrivial solutions if $\lambda \leq \lambda_0$, and has at least two nontrivial solutions if $\lambda \in (\lambda_0, \lambda_e)$.

- (ii) Assume (g1) – (g3) and (g4). Then (S) has at least $\mathcal{N}(\lambda)$ pairs of solutions.

Next consider the case where $g(x, u)$ is sublinear with respect to $u \in \mathbb{R}$. Assume

- (g5) $g(x, u)u \geq 0$, and there are $1 < \alpha \leq \beta < 2$, $c_1, c_2 > 0$ such that $|g(x, u)| \leq c_1(|u|^{\alpha-1} + |u|^{\beta-1})$ and $2G(x, u) - g(x, u)u \geq c_2|u|^\alpha$.

Theorem 2. (i) Assume (g5) holds. Then (S) has no nontrivial solutions if $\lambda \leq \lambda_0$, and has at least one nontrivial solution if $\lambda \in (\lambda_0, \lambda_e)$.

- (ii) Assume (g4) – (g5) hold. Then (S) has $\mathcal{N}(\lambda)$ pairs of solutions.

A typical example satisfying the above hypotheses is $g(x, u) = |u|^{p-2}u$. With this nonlinearity, the above results conclude that the problem (S) has $\mathcal{N}(\lambda)$ pairs of solutions provided $p \in (1, 2) \cup (2, \infty)$ and $\lambda \in (\lambda_0, \lambda_e)$. Clearly, if $p = 2$, (S) has solutions if and only if $\lambda - 1$ is an eigenvalue of A .

The problems looking for stationary solutions to Schrödinger equations have been studied extensively via variational methods. We refer, e.g., to [10] for Hartree-Fock equations, [5] for Schrödinger operators of atoms and molecules, and [1], [3], [4], [6], [8], [12], [14] for the Schrödinger operators with continuous (periodic etc.) potentials. We also mention that a similar problem of semilinear elliptic equations on bounded domains was studied in [2], [7]. The arguments used below extend to higher dimensions and more operators.

2. THE SUPERLINEAR CASE

Fix $\lambda < \lambda_e$ and set $A_\lambda := A - \lambda$. Let E^+ be the subspace of $L^2(\mathbb{R}^3)$ spanned by all eigenfunctions corresponding to negative eigenvalues of A_λ , and let $E^0 := \ker A_\lambda$. Then $\dim E^+ = \mathcal{N}(\lambda)$ and $L^2 = E^+ \oplus E^0 \oplus L^-$, $u = u^+ + u^0 + u^-$, where L^- denotes the orthogonal complement of $E^+ \oplus E^0$ in L^2 . Let $E := \mathcal{D}(|A_\lambda|^{1/2})$ be the Hilbert space with the inner product

$$(u, v) := (|A_\lambda|^{1/2}u, |A_\lambda|^{1/2}v)_{L^2} + (u^0, v^0)_{L^2}$$

and norm $\|u\| = (u, u)^{1/2}$, where $(\cdot, \cdot)_{L^2}$ denotes the usual L^2 -inner product. There is a decomposition

$$E = E^+ \oplus E^0 \oplus E^-, \quad u = u^+ + u^0 + u^-$$

with $E^- = L^- \cap E$, orthogonal with respect to both $(\cdot, \cdot)_{L^2}$ and (\cdot, \cdot) . Since $\mathcal{D}(A) \subseteq H^2(\mathbb{R}^3)$, E is embedded continuously in $H^1(\mathbb{R}^3)$, hence continuously in $L^t(\mathbb{R}^3)$ for all $t \in [2, 6]$ and compactly in $L^t_{loc}(\mathbb{R}^3)$ for all $t \in [1, 6]$.

Now, let E_p be the Banach space of the completion of $C_0^\infty(\mathbb{R}^3)$ in the norm $\|u\|_p = (\|u\|^2 + |u|_p^2)^{1/2}$. Here and below by $|\cdot|_p$ we denote the usual L^p -norm. By the Sobolev embedding theorem $E_p = E$ if $p \in [2, 6]$. In general, E_p embeds continuously in L^s for all $s \in [p, 6]$ (resp. $s \in [2, p]$) if $p \in (1, 6)$ (resp. $p \geq 6$). It is clear that E_p possesses the direct sum decomposition

$$E_p = E^+ \oplus E^0 \oplus E_p^-, \quad u = u^+ + u^0 + u^-$$

where E_p^- is the completion of $C_0^\infty(\mathbb{R}^3) \cap L^-$ under the norm $\|\cdot\|_p$. Throughout this section we assume (g1) – (g3) and fix E_p with $p > 2$ from (g3). Define

$$I(u) = I_\lambda(u) := \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) - \int_{\mathbb{R}^3} G(x, u).$$

Under the assumptions, $I \in C^1(E_p, \mathbb{R})$ and solutions of (S) can be found as critical points of I . It follows from the assumptions that, for any $\varepsilon > 0$, there are $C_\varepsilon, C'_\varepsilon > 0$ such that

$$(2.1) \quad C_\varepsilon |u|^\gamma - \varepsilon |u|^2 \leq G(x, u) \leq \varepsilon |u|^2 + C'_\varepsilon |u|^p \quad \text{for all } (x, u).$$

Since $\dim(E^+ \oplus E^0) < \infty$ and $\gamma > 2$, the following lemma clearly holds.

Lemma 2.1. *I is bounded from above: $\sup I(E) < \infty$. In addition, there is $R > 0$ such that $\sup I(E_p \setminus B_R) \leq 0$.*

Using (2.1) one gets easily the following

Lemma 2.2. *If $\lambda > \lambda_0$, there is $\rho > 0$ such that $\mu := \inf I(S_\rho^+) > 0$ where $S_\rho^+ := \partial B_\rho \cap E^+$.*

Recall that a sequence $(u_n) \subset E_p$ is called a $(PS)_c$ sequence if $I(u_n) \rightarrow c$ and $I'(u_n) \rightarrow 0$. I is said to satisfy the $(PS)_c$ condition if any $(PS)_c$ sequence has a convergent subsequence.

Lemma 2.3. *I satisfies the $(PS)_c$ condition.*

Proof. Let $I(u_n) \rightarrow c$ and $I'(u_n) \rightarrow 0$. A standard argument shows that (u_n) is bounded (cf. [8] or [4], [15]). We can suppose, without loss of generality, that $u_n \rightharpoonup u$. Since $\dim(E^+ \oplus E^0) < \infty$, $u_n^0 + u_n^+ \rightarrow u^0 + u^+$. It remains to show that $u_n^- \rightarrow u^-$.

Set $w_n = u_n - u$. Then $w_n \rightharpoonup 0$ and $w_n(x) \rightarrow 0$ a.e. in x since E_p embeds compactly in L^2_{loc} . Observe that

$$(I'(u_n) - I'(u)) w_n = \|w_n^+\|^2 - \|w_n^-\|^2 - \int_{\mathbb{R}^3} (g(x, u_n) - g(x, u)) w_n,$$

or

$$(2.2) \quad \|w_n^-\|^2 + \int_{\mathbb{R}^3} (g(x, u_n) - g(x, u)) w_n = o(1).$$

Write

$$\begin{aligned} \int_{\mathbb{R}^3} (g(x, u_n) - g(x, u)) w_n &= \int_{\mathbb{R}^3} g(x, w_n) w_n \\ &\quad + \int_{\mathbb{R}^3} (g(x, w_n + u) - g(x, w_n) - g(x, u)) w_n. \end{aligned}$$

Fix $\varepsilon > 0$. By assumptions it is easy to verify that there is $c_\varepsilon > 0$ satisfying

$$|g(x, w_n + u) - g(x, w_n) - g(x, u)| \leq \varepsilon |w_n|^{p-1} + c_\varepsilon (|u| + |u|^{p-1}).$$

Set $\Omega := \{x \in \mathbb{R}^3 : |u(x)| < 1\}$, $\Omega^c := \mathbb{R}^3 \setminus \Omega$, and

$$f_n^\varepsilon := (|g(x, w_n + u) - g(x, w_n) - g(x, u)| - \varepsilon |w_n|^{p-1})^+,$$

where $(a)^+ := \max\{a, 0\}$. Then $f_n^\varepsilon \leq 2c_\varepsilon |u|$ on Ω and $f_n^\varepsilon \leq 2c_\varepsilon |u|^{p-1}$ on Ω^c . The Lebesgue theorem then implies $\int_\Omega (f_n^\varepsilon)^2 \rightarrow 0$ and $\int_{\Omega^c} (f_n^\varepsilon)^{p'} \rightarrow 0$ where $p' = p/(p-1)$. Therefore,

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \int_\Omega |g(x, w_n + u) - g(x, w_n) - g(x, u)| |w_n| \\ &\leq \limsup_{n \rightarrow \infty} \left(\left(\int_\Omega (f_n^\varepsilon)^2 \right)^{1/2} |w_n|_2 + \varepsilon |w_n|_p^p \right) \leq c\varepsilon \end{aligned}$$

and

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \int_{\Omega^c} |g(x, w_n + u) - g(x, w_n) - g(x, u)| |w_n| \\ &\leq \limsup_{n \rightarrow \infty} \left(\left(\int_{\Omega^c} (f_n^\varepsilon)^{p'} \right)^{1/p'} |w_n|_p + \varepsilon |w_n|_p^p \right) \leq c\varepsilon, \end{aligned}$$

where $c := \sup_n |w_n|_p^p$. We obtain

$$\int_{\mathbb{R}^3} (g(x, u_n) - g(x, u)) w_n = \int_{\mathbb{R}^3} g(x, w_n) w_n + o(1).$$

Substituting in (2.2) yields

$$\|w_n^-\|^2 + \int_{\mathbb{R}^3} g(x, w_n) w_n = o(1).$$

Since $g(x, w_n) w_n \geq 0$, it follows that $w_n^- \rightarrow u^-$. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. (i) Assume (g1), (g2) and (g3) hold. Remark that if u is a nontrivial critical point of I , then

$$(2.3) \quad I(u) = I(u) - \frac{1}{2} I'(u)u \geq \frac{\gamma - 2}{2} \int_{\mathbb{R}^N} G(x, u) > 0.$$

If $\lambda \leq \lambda_0$, then $\sup I(E) \leq 0$, so I has no nontrivial critical points by (2.3).

Let $\lambda \in (\lambda_0, \lambda_e)$. By Lemma 2.1, I is bounded from above. In virtue of Lemma 2.3 I satisfies the $(PS)_c$. In addition, it follows from (2.1) that there exist $r, \varepsilon > 0$ such that

$$\sup I(\partial B_r \cap (E^- \oplus E^0)) \leq -\varepsilon < \sup I(E^- \oplus E^0) = 0$$

and

$$0 = \inf I(B_r \cap E^+) < \varepsilon \leq \inf I(\partial B_r \cap E^+).$$

Therefore I has at least two nontrivial critical points via the well-known “three solution theorem” (cf. [9]).

(ii) Assume (g4) also holds. Recall that I is even and bounded from above, satisfies the $(PS)_c$, $I(0) = 0$, and $\inf I(\partial B_\rho \cap E^+) \geq \mu$. Now an application of a Clark’s theorem (see [11, Theorem 9.1]) to $-I$ yields that I has at least $\mathcal{N}(\lambda)$ distinct pairs of critical points. \square

3. THE SUBLINEAR CASE

In this section we always assume that (g5) holds. It follows that

$$(3.1) \quad c_1|u|^\alpha \leq G(x, u) \leq c_2(|u|^\alpha + |u|^\beta) \quad \text{for all } (x, u).$$

We consider the functional defined on E_α

$$J(u) = J_\lambda(u) := \int_{\mathbb{R}^3} G(x, u) + \frac{1}{2}\|u^-\|^2 - \frac{1}{2}\|u^+\|^2.$$

Then $J \in \mathcal{C}^1(E_\alpha, \mathbb{R})$ and critical points of J are solutions of (S).

If $\lambda \leq \lambda_0$, $E^+ = \{0\}$ and clearly J has no nontrivial critical points. Therefore we fix $\lambda \in (\lambda_0, \lambda_e)$ from now on. Note that J is unbounded from either above or below.

Since $\dim(E^+ \oplus E^0) < \infty$ and $\beta < 2$, it follows from (3.1) that

(J₁) There is $R > 0$ such that $\sup J(E^+ \setminus B_R) \leq 0$;

(J₂) There is $\rho > 0$ such that $\mu := \inf J(S_\rho) > 0$.

Let $\{e_n : n \in \mathbb{N}\}$ be a normal orthogonal base of E_α^- , $Y_n := \text{span}\{e_1, \dots, e_n\}$ and $E_n := E^+ \oplus E^0 \oplus Y_n$. Note that $\dim E_n < \infty$. Set $J_n := J|_{E_n}$, the restriction of J on E_n . A sequence $(u_n) \subset E_\alpha$, $u_n \in E_n$, is called a $(PS)_c^*$ sequence if $J(u_n) \rightarrow c$ and $J'_n(u_n) \rightarrow 0$. J is said to satisfy the $(PS)_c^*$ condition if any $(PS)_c^*$ sequence has a convergent subsequence.

Lemma 3.1. *J satisfies the $(PS)_c^*$ condition.*

Proof. Let $u_n \in E_n$ be such that $J(u_n) \rightarrow c$ and $\varepsilon_n = \|J'_n(u_n)\| \rightarrow 0$. It follows from (g5) that

$$J(u_n) + \varepsilon_n \|u_n\|_\alpha \geq \int_{\mathbb{R}^3} \left(G(x, u_n) - \frac{1}{2}g(x, u_n)u_n \right) \geq c|u_n|_\alpha^\alpha,$$

hence

$$(3.2) \quad |u_n|_\alpha \leq c \left(1 + \|u\|_\alpha^{1/\alpha} \right).$$

By the Hölder inequality

$$\begin{aligned} \int_{\mathbb{R}^3} |g(x, u_n)u_n^+| &\leq c \int_{\mathbb{R}^3} (|u_n|^{\alpha-1} + |u_n|^{\beta-1}) |u_n^+| \\ &\leq c \left(|u_n|_{\alpha}^{\alpha-1} |u_n^+|_{\alpha} + |u_n|_{\beta}^{\beta-1} |u_n^+|_{\beta} \right) \end{aligned}$$

and so, since $\dim E^+ < \infty$,

$$\int_{\mathbb{R}^3} |g(x, u_n)u_n^+| \leq c (\|u_n\|_{\alpha}^{\alpha-1} + \|u_n\|_{\alpha}^{\beta-1}) \|u_n^+\|.$$

Therefore,

$$\begin{aligned} \|u_n^+\|^2 &= \int_{\mathbb{R}^3} g(x, u_n)u_n^+ - I'_n(u_n)u_n^+ \\ &\leq c (1 + \|u_n\|_{\alpha}^{\beta-1}) \|u_n^+\| \end{aligned}$$

or

$$(3.3) \quad \|u_n^+\| \leq c (1 + \|u_n\|_{\alpha}^{\beta-1}).$$

Moreover, since

$$\begin{aligned} \|u_n^-\|^2 &= 2I(u_n) + \|u_n^+\|^2 - 2 \int_{\mathbb{R}^3} G(x, u_n) \\ &\leq c (1 + \|u_n\|_{\alpha}^{2(\beta-1)}), \end{aligned}$$

one has

$$(3.4) \quad \|u_n^-\| \leq c (1 + \|u_n\|_{\alpha}^{\beta-1}).$$

Hence, in virtue of (3.2)–(3.4), (u_n) is bounded. We can suppose $u_n \rightharpoonup u$ and $u_n^0 + u_n^+ \rightarrow u^0 + u^+$.

Let $P_n : E_{\alpha} \rightarrow E_n$ denote the projection. Then

$$\begin{aligned} J'_n(u_n)(u_n^- - P_n u^-) &= \int_{\mathbb{R}^3} g(x, u_n)(u_n^- - P_n u^-) + (u_n^-, u_n^- - P_n u^-) \\ &= \int_{\mathbb{R}^3} g(x, u_n)(u_n - u) + (u_n^-, u_n^- - u^-) + o(1) \end{aligned}$$

or

$$\int_{\mathbb{R}^3} g(x, u_n)u_n = \int_{\mathbb{R}^3} g(x, u_n)u - \|u_n^-\|^2 + (u_n^-, u^-) + o(1).$$

Thus

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} g(x, u_n)u_n \leq \int_{\mathbb{R}^3} g(x, u)u.$$

On the other hand, by a Fatou lemma,

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} g(x, u_n)u_n \geq \int_{\mathbb{R}^3} g(x, u)u.$$

Thus

$$\int_{\mathbb{R}^3} g(x, u_n)u_n \rightarrow \int_{\mathbb{R}^3} g(x, u)u.$$

We obtain $\|u_n^-\| \rightarrow \|u^-\|$, which implies $u_n^- \rightarrow u^-$. □

Similar to the above arguments one can prove the following lemma.

Lemma 3.2. *J and J_n satisfy the (PS)_c condition.*

Proof of Theorem 2. It follows from (J₁) and (J₂) that J has the Mountain-Pass geometry [11], [15]. Since J satisfies the (PS)_c condition for all c, J possesses a critical value c ≥ μ, hence conclusion (i) follows.

Next we prove conclusion (ii). We write gen(A) ∈ ℕ ∪ {0, ∞} for the Krasnoselski genus of a symmetric subset A of E_α, that is, gen(A) is the least integer k such that there is an odd continuous map A → S^{k-1}. If no such map exists, then gen(A) := ∞. Let n⁰ := dim E⁰ and set

$$\Gamma_k^n := \{A \subset E_n \setminus \{0\} : A \text{ closed, symmetric, } \text{gen}(A) \geq n + n^0 + k\},$$

$$c_k^n := \sup_{A \in \Gamma_k^n} \inf_{u \in A} J(u).$$

If A ∈ Γ_kⁿ, then A ∩ E⁺ ≠ ∅ (see [11]). Hence

$$c_k^n \leq J(E^+) < \infty.$$

Since gen(S_ρ ∩ E_n) = n + n⁰ + N(λ), one has S_ρ ∩ E_n ∈ Γ_kⁿ for k = 1, ⋯, N(λ), and consequently by (J₂)

$$c_k^n \geq \inf J(S_\rho \cap E_n) \geq \mu.$$

Since J_n satisfies the (PS)_c condition, c_kⁿ is a critical value of J_n for k = 1, ⋯, N(λ). Clearly Γ_{k+1}ⁿ ⊂ Γ_kⁿ and one has

$$c_k^n \geq c_{k+1}^n.$$

Therefore, letting c_kⁿ → c_k as n → ∞ (along a subsequence), we have

$$c_N \leq \dots \leq c_1,$$

and, by (PS)_c^{*}, c_k is a critical value of J.

It remains to show that if c = c_k = ⋯ = c_{k+j}, then gen(K(c)) ≥ j + 1 where K(c) := {u ∈ E_α : J(u) = c, J'(u) = 0}. For this goal we note that, since J(0) = 0 while c ≥ μ > 0, 0 ∉ K(c). The continuity of genus implies that there is δ > 0 such that gen(N_{3δ}(K(c))) = gen(K(c)). Invoking (PS)_c we have ε̄, σ > 0 and n̄ ∈ ℕ satisfying

$$\|J'_n(u)\| \geq \sigma \quad \text{for all } u \in (J_n)_{c-\bar{\varepsilon}} \setminus N_\delta^n, \quad n \geq \bar{n},$$

where N_δⁿ = N_δ(K(c)) ∩ E_n. By a standard deformation argument we choose ε > 0 and construct a sequence of odd homeomorphisms h_n : E_n → E_n which maps (J_n)_{c-ε} \ N_δⁿ into (J_n)_{c+ε} (see [15]). Now choose a n ≥ n̄ large so that

$$c - \varepsilon/2 < c_{k+j}^n \leq \dots \leq c_k^n < c + \varepsilon/2.$$

Choose A ∈ Γ_{k+j}ⁿ with

$$\inf J(A) > c - \varepsilon/2.$$

If gen(K(c)) ≤ j, then

$$\text{gen}(\overline{A \setminus N_\delta^n}) \geq n + n^0 + k + j - \text{gen}(K(c)) \geq n + n^0 + k,$$

that is, $\overline{A \setminus N_\delta^n} \in \Gamma_k^n$. Therefore h_n($\overline{A \setminus N_\delta^n}$) ∈ Γ_kⁿ and, consequently,

$$c + \varepsilon \leq \inf J(h_n(\overline{A \setminus N_\delta^n})) \leq c_k^n \leq c + \varepsilon/2,$$

a contradiction. □

Remark 3.3. There is a three-body system with two-body potentials, $V = V_{01}(x) + V_{02}(y) + V_{12}(x-y)$, so that the number of bound states $E(\nu)$ of $A_\nu := -\Delta_x - \Delta_y + \nu V$ has the following property: There is an increasing sequence $0 < \nu_1 < \nu_2 < \dots$ so that $E(\nu) = \infty$ if $\nu \in [\nu_1, \nu_2] \cup [\nu_3, \nu_4] \cup \dots$ and $E(\nu) < \infty$ if $\nu \in \mathcal{T} := [0, \nu_1) \cup (\nu_2, \nu_3) \cup \dots$; see [13]. All the previous existence and multiplicity results can be established for the equation (S_ν) obtained from (S) with A replaced by A_ν in parallel.

Theorem 3.4. *Let $\nu \in \mathcal{T}$. (a) Assume (g1) – (g3). Then (S_ν) has no nontrivial solutions if $\lambda \leq \lambda_0$, has at least two nontrivial solutions if $\lambda \in (\lambda_0, \lambda_e)$, and has $\mathcal{N}(\lambda)$ pairs solutions if (g4) also holds. (b) Assume (g5) holds. Then (S_ν) has no nontrivial solutions if $\lambda \leq \lambda_0$, has at least one nontrivial solution if $\lambda \in (\lambda_0, \lambda_e)$, and has $\mathcal{N}(\lambda)$ pairs solutions if (g4) also holds.*

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MORNINGSIDE CENTER OF MATHEMATICS AND INSTITUTE OF MATHEMATICS, AMSS, CHINESE ACADEMY OF SCIENCES, 100080 BEIJING, PEOPLE’S REPUBLIC OF CHINA
E-mail address: dingyh@math03.math.ac.cn