METRIC ENTROPY OF CONVEX HULLS
IN TYPE p SPACES—THE CRITICAL CASE

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Abstract. Given a precompact subset \( A \) of a type \( p \) Banach space \( E \), where \( p \in (1, 2] \), we prove that for every \( \beta \in [0, 1) \) and all \( n \in \mathbb{N} \)

\[
\sup_{k \leq n} k^{1/p'} \left( \log k \right)^{\beta - 1} e_k(\text{aco} A) \leq c \cdot \sup_{k \leq n} k^{1/p'} \left( \log k \right)^{\beta - 1} e_k(A)
\]

holds, where \( \text{aco} A \) is the absolutely convex hull of \( A \) and \( e_k(\cdot) \) denotes the \( k \)th dyadic entropy number. With this inequality we show in particular that for given \( A \) and \( \beta \in (0, 1) \) with \( e_n(\text{aco} A) \leq c \cdot n^{1/p'} \left( \log n \right)^{-\beta + 1} \) holds true for all \( n \in \mathbb{N} \). We also prove that this estimate is asymptotically optimal whenever \( E \) has no better type than \( p \). For \( \beta = 0 \) this answers a question raised by Carl, Kyrezi, and Pajor which has been solved up to now only for the Hilbert space case by F. Gao.

1. Introduction and Results

In the following, \( E \) shall always denote a Banach space and \( B_E \) its closed unit ball. For a bounded subset \( A \subseteq E \) we define the entropy numbers of \( A \) to be

\[
\varepsilon_n(A) := \inf \left\{ \varepsilon > 0 : \exists x_1, \ldots, x_n \in A \text{ such that } A \subseteq \bigcup_{i=1}^{n} (x_i + \varepsilon B_E) \right\}, \quad n \in \mathbb{N}.
\]

Alternatively, one can consider the covering numbers of \( A \), namely

\[
N(\varepsilon, A) := \min \left\{ n \in \mathbb{N} : \exists x_1, \ldots, x_n \in A \text{ such that } A \subseteq \bigcup_{i=1}^{n} (x_i + \varepsilon B_E) \right\}, \quad \varepsilon > 0.
\]

In the setting of our problems, we prefer to deal with \( e_n(A) := \varepsilon_{2^{n-1}}(A) \), resp. \( H(\varepsilon, A) := \log N(\varepsilon, A) \). We remember that if \( A \) is precompact, so is \( \text{aco} A \). Now, it is natural to ask for entropy estimates of \( \text{aco} A \) in terms of entropy numbers of \( A \). Among others, the articles [Du], [BP], [CKP], [LL], [St1], [St2] and [Ga] are dedicated mainly to the study of this question with respect to different settings. For instance, asymptotically optimal estimates in type \( p \) spaces for the case of polynomially decaying \( e_n(A) \) were proved in [CKP]. Let us recall that a Banach...
space $E$ is said to be of type $p$ with $1 \leq p \leq 2$ iff there is a constant $c > 0$ such that for all $x_1, \ldots, x_n \in E$ we have the estimate
\[
\frac{1}{n} \left\| \sum_{i=1}^{n} x_i r_i(t) \right\| dt \leq c \left( \sum_{i=1}^{n} \|x_i\|^p \right)^{1/p},
\]
where $(r_i)$ shall denote the Rademacher functions, i.e. $r_i(t) := \text{sign}(\sin(2^i \pi t))$. The type $p$ constant $\tau_p(E)$ is the smallest constant $c$ satisfying the above inequality. We say that $E$ is of optimal type $p$ if $E$ is of type $p$, but of no greater type than $p$.

For $A \subseteq E$ bounded let us set $\|A\| := \sup_{x \in A} \|x\|$ and define $c_A := \frac{\|A\|}{e_{1/2}(A)}$. Then one of the main results of [CKP] reads as:

**Theorem 1.1.** Let $E$ be a Banach space of type $p \in (1, 2]$ and $A \subseteq E$ be precompact. Further, assume that there is a positive $\alpha \neq 1/p'$ such that for all $n \in \mathbb{N}$ we have $e_n(A) \leq n^{-\alpha}$. Then for all $n \in \mathbb{N}$ it holds that
\[
e_n(aco A) \leq c_A c_{\alpha,p} \begin{cases} n^{-\alpha}, & \text{if } 0 < \alpha < 1/p', \\ n^{-1/p'} (\log(n + 1))^{1/p' - \alpha}, & \text{if } \alpha > 1/p', \end{cases}
\]
where $c_{\alpha,p}$ depends on $\alpha, p$ and $\tau_p(E)$ only. These estimates are asymptotically optimal for some subsets $A \subseteq \ell_p$.

It is a strange phenomenon that the asymptotic behaviour of the absolutely convex hull drastically changes when the limit $1/p'$ is crossed. In particular, the asymptotic behaviour in the limit case $\alpha = 1/p'$ is an interesting problem raised in [CKP]. The following result due to Gao (cf. [Ga]) gives an answer to this question for Hilbert spaces.

**Theorem 1.2.** Let $H$ be a Hilbert space and $A \subseteq H$ be precompact. Then $e_n(A) \leq n^{-1/2}$ for all $n \in \mathbb{N}$ implies
\[
e_n(aco A) \leq c n^{-1/2} \log(n + 1)
\]
for all $n \in \mathbb{N}$ and a suitable constant $c > 0$ independent of $n$. This estimate is asymptotically optimal.

It turns out that the techniques used by Gao also work in the more general setting of type $p$ spaces. With the help of the ideas from [ST], Gao’s estimate then becomes an inequality. Our first result is

**Theorem 1.3.** Let $E$ be a Banach space of type $p \in (1, 2]$ and $\beta \in [0, 1)$. Then there is a constant $c_{\beta,p} > 0$ such that for every precompact $A \subseteq E$ and all $n \geq 1$ we have
\[
\sup_{k \leq n} k^{1/p'} (\log(k + 1))^{\beta - 1} e_k(aco A) \leq c_{\beta,p} c_A \sup_{k \leq (\log(n + 1))^{\beta + 1}} k^{1/p'} (\log(k + 1))^{\beta} e_k(A),
\]
with $c_{\beta,p}$ only depending on $p, \beta$ and $\tau_p(E)$.

As a trivial consequence we can establish the following estimate.
Corollary 1.4. Let $E$ be a Banach space of type $p \in (1, 2]$, $A \subset E$ be precompact, $\beta \in (-\infty, 1]$ and $f : [1, \infty) \to (0, \infty)$ be a positive and increasing function. Then
\[
e_n(A) \leq n^{-1/p'} (\log(n+1))^{-\beta} f(n)
\]
for all $n \in \mathbb{N}$ implies
\[
e_n(aco A) \leq c n^{-1/p'} (\log(n+1))^{-\beta+1} f(n)
\]
for every $n \in \mathbb{N}$ and a suitable constant $c > 0$ independent of $n$.

This estimate is asymptotically optimal under natural conditions on $E$ and $f$ as the next theorem shows.

Theorem 1.5. Let $E$ be an infinite dimensional Banach space of optimal type $p \in (1, 2]$ and $\beta \in (-\infty, 1)$. Further, let $g : [1, \infty) \to (0, \infty)$ be a positive and increasing function with $g(2x) \leq c g(x)$ for all $x \geq 1$ and a constant $c > 0$ independent of $x$. We define $f(x) := g(\log(x + 3))$. Then there exist a precompact subset $A \subseteq E$ and constants $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$ we have
\[
e_n(A) \leq c_1 n^{-1/p'} (\log(n+1))^{-\beta} f(n)
\]
and
\[
e_n(aco A) \geq c_2 n^{-1/p'} (\log(n+1))^{-\beta+1} f(n).
\]

For the construction of these subsets with asymptotically optimal behaviour, we need the following two results. The first one is due to Schütt (cf. [Sch]).

Theorem 1.6. For all $p \in (1, 2]$ there exists a constant $c_p > 0$ such that for all integers $n$ and $N$ with $\log N \leq n \leq N$ we have
\[
e_n(\text{id} : \ell^N_p \to \ell^N_p) \geq c_p \left( \frac{\log(N/n + 1)}{n} \right)^{1/p'}.
\]

The second result is a deep theorem due to Maurey and Pisier (see [MP, Théorème 2.1], or [MS, Theorem 13.1]).

Theorem 1.7. Let $E$ be an infinite dimensional Banach space of optimal type $p \in (1, 2]$. Then for all $n \in \mathbb{N}$ there are subspaces $E_n \subseteq E$ and isomorphisms $T_n : \ell^N_p \to E_n$ with $\|T_n\| \|T_n^{-1}\| \leq 2$.

2. Proof of the results

First, let us fix some notations and simple facts used in the following. For $A, B \subseteq E$ we write $A + B$ for the Minkowski sum. If $A, B$ are symmetric, it holds that $(aco A) + (aco B) = aco (A + B)$, while this is in general wrong for nonsymmetric sets.

For $x > 0$ we set $\lfloor x \rfloor$ to be the integer part of $x$. If $E$ is a Banach space of type $p$ and $Y_1, \ldots, Y_n$ are independent $E$-valued random variables with finite $p$th moment, the inequality
\[
E \left\| \sum_{i=1}^{n} (Y_i - EY_i) \right\| \leq 4 \tau_p(E) \left( \sum_{i=1}^{n} E \|Y_i\|^p \right)^{1/p}
\]
holds (cf. [MP] and [Ho]).
Due to technical reasons, we prefer to prove a statement similar to Theorem 1.3 only formulated for covering numbers first. For the proof we will combine techniques from [Ga] and [St1].

**Proposition 2.1.** Let $E$ be a Banach space of type $p \in (1, 2]$ and $\alpha \in [0, p')$. Then there is a constant $c_{\alpha, p} > 0$ depending only on $\alpha, p$ and $\tau_p(E)$ such that for all precompact and symmetric $A \subseteq B_E$ and every $\varepsilon_0 \in [0, 1]$ we have

$$\sup_{\varepsilon \in [2\varepsilon_0, 2]} H(\varepsilon, aco A) \varepsilon^{-\alpha} (\log(3/\varepsilon))^{\alpha - p'} \leq c_{\alpha, p} \sup_{\varepsilon \in [\varepsilon_0, 1]} H(\varepsilon, A) \varepsilon^{-\alpha} (\log(3/\varepsilon))^{\alpha - p'}.$$

**Proof.** For brevity’s sake we first define

$$f(\varepsilon) := \varepsilon^{-\alpha} (\log(3/\varepsilon))^{-\alpha} \quad \text{and} \quad g(\varepsilon) := \varepsilon^{-\alpha} (\log(3/\varepsilon))^{p' - \alpha}.$$

We fix $\varepsilon_0 \in (0, 1]$ and set

$$K(\varepsilon_0) := \sup_{\varepsilon \in [\varepsilon_0, 1]} \frac{H(\varepsilon, A)}{f(\varepsilon)}.$$

Since the latter is decreasing in $\varepsilon_0$ it suffices to show that

$$H(2\varepsilon_0, aco A) \leq c K(\varepsilon_0) g(2\varepsilon_0)$$

for some constant $c > 0$ depending only on $\alpha, p$ and $\tau_p(E)$. We can restrict ourselves to $e_1(A) = 1$ by a rescaling argument. Moreover, $A \subseteq B_E$ implies $H(1, aco A) = 0$, and hence we have to show assertion [4] only for $\varepsilon_0 \in (0, 1/2]$.

Now, let us fix $n \in \mathbb{N}$ and $\gamma \in (1/2, 1]$ such that $\varepsilon_0 = \gamma 2^{-n}$. The definition of $K(\varepsilon_0)$ yields

$$H(\varepsilon, A) \leq K(\varepsilon_0) f(\varepsilon) \quad \text{for all} \quad \varepsilon \in [\varepsilon_0, 1/2].$$

In particular there exist $\gamma 2^{-k}$-nets $N_k$ of $A$ with cardinality

$$|N_k| \leq \exp(K(\varepsilon_0) f(\gamma 2^{-k})), \quad k = 1, \ldots, n.$$

We define $D_1 := N_1$ and

$$D_k := \{z \in N_k - N_{k-1} : \|z\| \leq \gamma 2^{-k+1}\}, \quad k = 2, \ldots, n.$$

Especially, we have $\|D_k\| \leq 2^{-k+1}$ for every $k = 1, \ldots, n$. Moreover, $|D_k| \leq |N_k| |N_{k-1}| \leq \exp(2K(\varepsilon_0) f(\gamma 2^{-k}))$ holds and hence for $D'_k := D_k \cup (-D_k) \cup \{0\}$ we obtain

$$|D'_k| \leq 3 |D_k| \leq 3 \exp(2K(\varepsilon_0) f(\gamma 2^{-k})).$$

Since $e_1(A) = 1$ implies $N(1/2, A) \geq 2$ we get

$$K(\varepsilon_0) \geq \frac{H(1/2, A)}{f(1/2)} \geq 2^{-p'} (\log 6)^{\alpha} \log 2 \geq 2^{-2p'} \log 3,$$

which allows us to estimate

$$\log 3 + 2 K(\varepsilon_0) f(\varepsilon) \leq \log 3 + 2 \left(2^{2p'} + 2 f(\varepsilon)\right) \leq 2^{2p'} K(\varepsilon_0) f(\varepsilon)$$

for all $\varepsilon \in [\varepsilon_0, 1/2]$. Thus, [3] can be continued to

$$|D'_k| \leq \exp(2^{2p'} K(\varepsilon_0) f(\gamma 2^{-k})), \quad k = 1, \ldots, n.$$

Let us now define $C_k := co D'_k = aco D_k$ and $E_n := \sum_{k=1}^n C_k$. For $k \geq 2$ and $t_k \in N_k \subseteq A$ there is always a $t_{k-1} \in N_{k-1}$ such that $t_k - t_{k-1} \in D_k$. From
this one easily deduces $N_n \subseteq E_n$, so $E_n$ is an $\varepsilon_0$-net of $A$. Since $E_n$ is absolutely convex, it is even an $\varepsilon_0$-net of $aco A$. Hence we have

\begin{equation}
H(2\varepsilon_0, aco A) \leq H(\varepsilon_0, E_n)
\end{equation}

by the triangle inequality. We will prove in the following that, for suitable numbers $m_1, \ldots, m_n$ which we specify later, the set

\begin{equation}
X := \left\{ \sum_{k=1}^{n} \frac{1}{m_k} \sum_{i=1}^{m_k} d_{k,i} : d_{k,i} \in D'_k \right\}
\end{equation}

forms an $\varepsilon_0$-net of $E_n$. For this we will use an argument which originally goes back to Maurey (cf. [Pis]). We denote by $x_1^k, \ldots, x_{d_k}^k$ the elements of $D'_k \setminus \{0\}$. Now we fix $z \in E_n$ and write $z = \sum_{k=1}^{n} z_k$ with $z_k \in C_k$. Then every $z_k$ can be represented by

\[ z_k = \sum_{i=1}^{d_k} a_{k,i} x_i^k, \text{ where } a_{k,i} \geq 0 \quad \text{and} \quad \sum_{i=1}^{d_k} a_{k,i} \leq 1. \]

Let us define $Z_k$ to be a random vector with values in $D'_k$, namely

\[ P(Z_k = x_i^k) = a_{k,i} \quad \text{for } i = 1, \ldots, d_k, \quad \text{and} \quad P(Z_k = 0) = 1 - \sum_{i=1}^{d_k} a_{k,i}. \]

Trivially, we obtain $EZ_k = z_k$. Moreover, we take independent random vectors $Z_{1,1}, \ldots, Z_{1,m_1}, \ldots, Z_{n,1}, \ldots, Z_{n,m_n}$ such that $Z_{k,i}$ is distributed as $Z_k$ for all $k = 1, \ldots, n$ and $i = 1, \ldots, m_k$. With $Y_{k,i} := \frac{1}{m_k} Z_{k,i}$ and inequality (8) it then yields

\begin{align*}
\mathbb{E} \left\| \sum_{k=1}^{n} z_k - \sum_{k=1}^{n} \frac{1}{m_k} \sum_{i=1}^{m_k} Z_{k,i} \right\| &\leq \mathbb{E} \left\| \sum_{k=1}^{n} \sum_{i=1}^{m_k} (\mathbb{E} Y_{k,i} - Y_{k,i}) \right\| \\
&\leq 4 \tau_p(E) \left( \sum_{k=1}^{n} \sum_{i=1}^{m_k} \mathbb{E} \| Y_{k,i} \|^{p} \right)^{1/p} \\
&\leq 4 \tau_p(E) \left( \sum_{k=1}^{n} \frac{1}{m_k p^{k-1}} 2^{-p(k-1)} \right)^{1/p}.
\end{align*}

Now we want to specify the integers $m_k$. For this, we first set $\delta := \frac{1}{2} (p' - \alpha)$ and then let $\epsilon_{\alpha,p}^{(1)} := (4 \tau_p(E))^{p'} (\delta (p - 1))^{-p'/p}$. Since $\varepsilon_0 \leq 2^{-(n-1)}$ we know that

\[ \frac{k(p' - 1 - \delta)n^{\delta}}{\varepsilon_0^{p'} 2^{p'(k-1)}} \geq 1, \quad k = 1, \ldots, n, \]

and hence there is an integer $m_k$ with

\[ \frac{k(p' - 1 - \delta)n^{\delta}}{\varepsilon_0^{p'} 2^{p'(k-1)}} \leq m_k \leq 2 \epsilon_{\alpha,p}^{(1)} \frac{k(p' - 1 - \delta)n^{\delta}}{\varepsilon_0^{p'} 2^{p'(k-1)}}. \]

For these $m_k$’s we apply estimate (8) and obtain

\[ \mathbb{E} \left\| z - \sum_{k=1}^{n} \frac{1}{m_k} \sum_{i=1}^{m_k} Z_{k,i} \right\| \leq (\delta (p - 1))^{1/p} \varepsilon_0^{-(p-1)/p'} \left( \sum_{k=1}^{n} k^{-1+\delta(p-1)} \right)^{1/p} \leq \varepsilon_0. \]
Hence we get and define inequality (6) this yields (4) and therefore the assertion.

We let $x$ for all $x$ such that $\|z-x\| \leq \varepsilon_0$. Hence $X$ is an $\varepsilon_0$-net of $E_n$ and moreover, we have

$$\log |X| \leq \log \left( \prod_{k=1}^{n} |D_k^{\left( m_k \right)}| \right) \leq 2^{2p^2} K(\varepsilon_0) \sum_{k=1}^{n} m_k \left( \gamma 2^{-k} \right)$$

$$\leq c_{\alpha, p}^{(2)}(\varepsilon_0) \varepsilon_0^{-p} n^\delta \sum_{k=1}^{n} k^{p^2 - 1 - \delta - \alpha}$$

$$\leq c_{\alpha, p}^{(3)}(\varepsilon_0) \varepsilon_0^{-p} n^{p^2 - \alpha}$$

$$\leq c_{\alpha, p}^{(4)}(\varepsilon_0) \varepsilon_0^{-p} (\log(3/\varepsilon_0))^{p^2 - \alpha}$$

for suitable constants $c_{\alpha, p}^{(2)}, c_{\alpha, p}^{(3)}$ and $c_{\alpha, p}^{(4)}$ depending only on $\alpha, p$ and $\tau_p(E)$. With inequality (6) this yields (4) and therefore the assertion.

Although it seems self-evident that Proposition 2.1 is easily translated into Theorem 1.3, one needs to be careful with arising constants.

**Proof of Theorem 1.3.** Let us first assume that $A \subseteq E$ is symmetric and without loss of generality we also suppose $e_1(A) = 1$. We use $a_n := \frac{n}{(\log(n + 1))^{\beta}} + 1$ for short and define

$$C_n := 2 \sup_{k \leq a_n} k^{1/p} (\log(k + 1))^{\beta} e_k(A).$$

With the help of standard arguments, it suffices to show that

$$n^{1/p} (\log(n + 1))^{\beta - 1} e_{cn}(aco A) \leq 8 C_n$$

for all $n \geq (4p)^{4p}$ and some $c \in \mathbb{N}$ only depending on $\beta, p$ and $\tau_p(E)$. Therefore, we fix an arbitrary integer $n \geq (4p)^{4p}$. Since we have $e_k(A) < C_n k^{-1/p} (\log(k + 1))^{-\beta}$ for all $1 \leq k \leq a_n$, we obtain

$$N(C_n k^{-1/p} (\log(k + 1))^{-\beta}, A) \leq 2^{k - 1}, \quad 1 \leq k \leq a_n.$$  

We let

$$\varepsilon_0 := C_n [a_n]^{-1/p} (\log([a_n] + 1))^{-\beta}$$

and first assume $\varepsilon_0 < 1$. Then for all $\varepsilon \in [\varepsilon_0, 1]$ there is an integer $k_{\varepsilon}$ with $2 \leq k_{\varepsilon} \leq a_n$ such that

$$C_n k_{\varepsilon}^{-1/p} (\log(k_{\varepsilon} + 1))^{-\beta} \leq \varepsilon \leq C_n (k_{\varepsilon} - 1)^{-1/p} (\log(k_{\varepsilon}))^{-\beta}.$$  

Hence we get

$$H(\varepsilon, A) \leq H(C_n k_{\varepsilon}^{-1/p} (\log(k_{\varepsilon} + 1))^{-\beta}, A)$$

$$\leq k_{\varepsilon} - 1$$

$$\leq C_n^{\varepsilon} k_{\varepsilon}^{-1/p} (\log(k_{\varepsilon}))^{-\beta p^{2}}$$

$$\leq 5^{p^{2}} C_n^{\varepsilon} \varepsilon^{-p} (\log(3/\varepsilon))^{-\beta p^{2}}$$

for all $\varepsilon \in [\varepsilon_0, 1]$. Now Proposition 2.1 provides a constant $c_{\beta, p} > 1$ such that

$$H(2\varepsilon_0, aco A) \leq c_{\beta, p} C_n^{p^{2}} \varepsilon_0^{-p} (\log(3/2\varepsilon_0))^{p^{2}(1-\beta)} \leq (cn - 1) \log 2$$
with \( c := \left| 4^{2+2p'} e_{\beta,p} \right| + 2 \). On the other hand, if \( \varepsilon_0 \geq 1 \), we have \( H(2\varepsilon_0, aco A) = 0 \), hence estimate (10) holds in this case, too. This leads to

\[
e_{cn}(aco A) \leq 2\varepsilon_0 \leq 8 C_n n^{-1/p'} \left( \log(n + 1) \right)^{1-\beta},
\]

where the last estimate uses \( n \geq (4p')^{4p'} \). Thus we have shown the assertion for symmetric \( A \).

For arbitrary precompact \( A \subseteq E \), the set \( A' := A \cup (-A) \) is precompact and symmetric. Moreover, we have \( aco A' = aco A, e_1(A') \leq 2 \| A \| \) and \( e_{2k}(A') \leq e_k(A) \) for all \( k \geq 1 \). Therefore, we obtain

\[
\sup_{k \leq a_n} k^{1/p'} \left( \log(k + 1) \right)^{\beta} e_k(A') \leq 2 \max \left\{ \| A \|, \sup_{2 \leq k \leq a_n} k^{1/p'} \left( \log(k + 1) \right)^{\beta} e_k(A') \right\} \\
\leq 18 c_A \sup_{k \leq a_n} k^{1/p'} \left( \log(k + 1) \right)^{\beta} e_k(A),
\]

which completes the proof.

\( \square \)

Before we turn to the construction of \( A \) in Theorem 1.5, let us prove the following lemma which was essentially obtained in [Ga, proof of Theorem 1]:

**Lemma 2.2.** Let \( 1 \leq p < \infty \) and \( E_1, \ldots, E_N \) be Banach spaces. We equip the product space \( E_1 \times \cdots \times E_N \) with the \( p \)-product norm

\[
\|(x_1, \ldots, x_N)\|_p := \left( \sum_{i=1}^N \| x_i \|^p \right)^{1/p}.
\]

Then for all subsets \( A_i \subseteq E_i \) and every \( n \geq 6 \) we have

\[
N^{1/p} \min_{i \leq N} e_{n+1}(A_i) \leq 4 e_{\lfloor n/2 \rfloor} \left( A_1 \times \cdots \times A_N \right).
\]

**Proof.** Let \( \varrho < \min_{i \leq N} e_{n+1}(A_i) \) be arbitrary. Then there exist sets \( S_i \subseteq A_i \) of cardinality \( L := 2^n \) such that for all \( x_i, y_i \in S_i \) with \( x_i \neq y_i \) we have \( \| x_i - y_i \| \geq \varrho \).

We set

\[
S := S_1 \times \cdots \times S_N
\]

and \( M := \lfloor N/2 \rfloor \). For \( x, y \in S \) with \( x = (x_i) \) and \( y = (y_i) \) we define the Hamming distance

\[
h(x, y) := |\{ i : x_i \neq y_i \}| \quad \text{and set} \quad B_h(y, M) := \{ x \in S : h(x, y) \leq M \}.
\]

Then for every \( y \in S \) we obtain

\[
|B_h(y, M)| \leq \binom{N}{M} L^M \leq 2^N L^{N/2} \leq L^{N/6} L^{N/2} = L^{N/3}.
\]

Thus, for any \( k < L^{N/3} \) and \( x^1, \ldots, x^k \in S \) we have

\[
\left| \bigcup_{j=1}^k B_h(x^j, M) \right| \leq k \sum_{j=1}^k |B_h(x^j, M)| \leq k L^{N/3} < |S|.
\]

Therefore, one can find elements \( x^1, \ldots, x^m \in S \) with \( m \geq L^{N/3} \geq 2^{\lfloor n/3 \rfloor} \) such that for \( i \neq j \) we have \( h(x^i, x^j) \geq M+1 \geq N/2 \). Thus we get \( \| x^i - x^j \|_p \geq (N/2)^{1/p} \varrho \) for \( i \neq j \) and hence \( N^{1/p} \varrho \leq 4 e_{\lfloor n/3 \rfloor} (A_1 \times \cdots \times A_N) \). But this finally yields the assertion.

\( \square \)
For the proof of Theorem 1.5, we will combine ideas and techniques from [Ga] and [St2].

Proof of Theorem 1.5. Before we start the proof, we would like to point out that for \( p, \beta \) and \( f \) given as in Theorem 1.5, there is a constant \( c > 0 \) which we keep fixed throughout the proof such that

\[
\sum_{k=1}^{\infty} 2^{-k} k^{\max(0,-\beta)} f(2^p k) \leq c
\]

as well as \( f(xy) \leq cf(x)f(y) \) and \( f(x \log x) \leq cf(\sqrt{x}) \) for all \( x, y \geq 1 \).

For the construction of the subset \( A \subseteq E \) we set

\[
 C_j := \{ \pm 2^{j \cdot \beta} f(2^p) e_i : \alpha_{j} - \beta < i \leq \alpha_j \}
\]

and \( D_j := T_k(C_j^k) \). Moreover, we let \( A_k := \sum_{j=k+1}^{2k} D^k_j \). For further calculations we mention that we have

\[
\left| \sum_{j=k+1}^{N} D^k_j \right| \leq \prod_{j=k+1}^{N} |D^k_j| \leq \prod_{j=k+1}^{N} 2 \alpha_j \leq \alpha_{N+2}
\]

for \( k < N \leq 2k \). This implies \( |A_k| \leq \alpha_{2k+2} \) in particular. We finally set \( A \) to be

\[
A := \bigcup_{k=1}^{\infty} A_k,
\]

which completes the desired construction.

Now we begin with verifying estimate (\text{[1]}). We fix \( n > 1 + \log_2 \alpha_{11} \). Then there is an \( N \geq 8 \) with \( \alpha_{N+3} \leq 2^{n-1} \leq \alpha_{N+4} \). We divide \( A \) into

\[
B_1 := \bigcup_{k=1}^{\lfloor N/2 \rfloor} A_k, \quad B_2 := \bigcup_{k=\lfloor N/2 \rfloor + 1}^{N-2} A_k, \quad B_3 := \bigcup_{k=N-1}^{\infty} A_k
\]

and set \( \varepsilon := 2c2^{k} N^{-\beta} f(2^p N) \). Of course, \( B_1 \) is an \( \varepsilon \)-net for itself, and (\text{[1]}1) yields

\[
|B_1| \leq \sum_{k=1}^{\lfloor N/2 \rfloor} |A_k| \leq \sum_{k=1}^{\lfloor N/2 \rfloor} \alpha_{2k+2} \leq N \alpha_{N+2}.
\]
To see that \{0\} is an \(\varepsilon\)-net of \(B_3\) we estimate

\[
\|A_k\| \leq \sum_{j=k+1}^{2k} \|D_j\| \\
\leq \sum_{j=N}^{\infty} 2^{-j} j^{-\beta} f(2^j) \\
\leq c 2^{-N+1} N^{-\beta} f(2^N) \left( \sum_{j=1}^{\infty} 2^{-j} j^{\max\{0,-\beta\}} f(2^j) \right) \\
\leq \varepsilon
\]

for \(k \geq N - 1\). Finally, for considering \(B_2\) we first investigate \(A_k\) for \(N + 1 \leq k \leq N - 2\). Given an \(x = \sum_{j=k+1}^{2k} x_j \in A_k\) with \(x_j \in D_j\), we define \(y := \sum_{j=k+1}^{N} x_j \in \sum_{j=k+1}^{N} D_j\). Then

\[
\|x - y\| = \left\| \sum_{j=N+1}^{2k} x_j \right\| \leq \sum_{j=N+1}^{2k} \|D_j\| \leq \varepsilon
\]

holds similar to estimate (12). Thus, \(\sum_{j=k+1}^{N} D_j\) is an \(\varepsilon\)-net of \(A_k\) and hence

\[
R := \bigcup_{k=\lceil \frac{N+1}{2} \rceil}^{N-2} \sum_{j=k+1}^{N} D_j
\]

is an \(\varepsilon\)-net of \(B_2\). Moreover, (11) ensures that

\[
|R| \leq \sum_{k=\lceil \frac{N+1}{2} \rceil}^{N-2} \alpha_N < N \alpha_{N+2}.
\]

Summing up our results for \(B_1, B_2\) and \(B_3\), we have shown that \(B_1 \cup R \cup \{0\}\) is an \(\varepsilon\)-net of \(A\) of at most \(2N\alpha_{N+2}\) points. Since \(2N\alpha_{N+2} \leq \alpha_{N+3} \leq 2^{n-1}\) we conclude

\[
e_n(A) \leq \varepsilon = 2 c^2 2^{-N} N^{-\beta} f(2^N) \leq c_{p,\beta,f} n^{-1/p'} (\log(n+1))^{-\beta} f(n),
\]

where \(c_{p,\beta,f} > 0\) is independent of \(n\).

Now we verify estimate (11). Given \(n \geq 8 \cdot 2^{16p'}\), we fix an \(N \geq 9\) with

\[
(N - 1) \cdot 2^{p(N-1)} \leq 3n \leq N \cdot 2^{pN}.
\]

We set \(m := 2^{[2pN]} + 1\) and observe that \(n \leq mN/3\). Thus, with \(T_n^{-1}(\aco A_N) = \sum_{j=N+1}^{N+1} \aco C_j\) and Gao’s Lemma we obtain

\[
e_n(\aco A) \geq \epsilon_{[\frac{mN}{3}]}(\aco A_N) \\
\geq \frac{1}{2} \epsilon_{[\frac{mN}{3}]}(\aco C_{N+1} \times \cdots \times \aco C_{2N}) \\
\geq \frac{1}{8} N^{-1/p} \min_{N \leq j \leq 2N} \epsilon_{m+1}(\aco C_j).
\]

(13)
To continue this estimate we first note that for \( j \) with \( N \leq j \leq 2N \) we have
\[
2^{p'j} \leq m + 1 \leq \exp(2^{p'(j-2)-1}),
\]
where the right inequality follows from \( N \geq 9 \). Moreover, for \( d_j := \dim \left( \text{span } C_j^N \right) \) we know \( \alpha_{j-2} \leq d_j \leq \alpha_j \) and with (13) this implies \( 2^{p'(j-2)} \leq \log d_j \leq m + 1 \leq d_j^{1/2} \).

Now applying Theorem 1.6 for \( j \) with \( N \leq j \leq 2N \) we get
\[
e_{m+1}(aco C_j^N) = 2^{-j} j^{-\beta} f(2^{p'j}) e_{m+1}(id: \ell_1^d \to \ell_1^d)
\geq c_p 2^{-j} j^{-\beta} f(2^{p'j}) \left( \frac{\log d_j^{1/2}}{m + 1} \right)^{1/p'}
\geq \frac{1}{8} c_p N^{-\beta} f(2^{p'N}) m^{-1/p'}.
\]
Therefore, we can continue (13) by
\[
e_n(aco A) \geq \frac{1}{64} c_p N^{1/p - \beta} f(2^{p'N}) m^{-1/p'}
= \frac{1}{64} c_p (mN)^{-1/p'} N^{1-\beta} f(2^{p'N})
\geq c_{p,\beta, f}^{(1)} (mN)^{-1/p'} (\log (mN))^{1-\beta} f(mN)
\geq c_{p,\beta, f}^{(2)} n^{-1/p'} (\log n)^{1-\beta} f(n),
\]
where \( c_{p,\beta, f}^{(1)} \) and \( c_{p,\beta, f}^{(2)} \) are suitable constants depending only on \( p, \beta \) and \( f \).

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\section*{References}

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