ON THE ABSOLUTE CONTINUITY OF A CLASS OF INVARIANT MEASURES

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Abstract. Let $X$ be a compact connected subset of $\mathbb{R}^d$, let $S_j : X \to X$, $j = 1, \ldots, N$, be contractive self-conformal maps on a neighborhood of $X$, and let $\{p_j(x)\}_{j=1}^N$ be a family of positive continuous functions on $X$. We consider the probability measure $\mu$ that satisfies the eigen-equation

$$\lambda \mu = \sum_{j=1}^N p_j \circ S_j^{-1},$$

for some $\lambda > 0$. We prove that if the attractor $K$ is an $s$-set and $\mu$ is absolutely continuous with respect to $\mathcal{H}^s|_K$, the Hausdorff $s$-dimensional measure restricted on the attractor $K$, then $\mathcal{H}^s|_K$ is absolutely continuous with respect to $\mu$ (i.e., they are equivalent). A special case of the result was considered by Mauldin and Simon (1998). In another direction, we also consider the $L^p$-property of the Radon-Nikodym derivative of $\mu$ and give a condition for which $D\mu$ is unbounded.

1. Introduction

Let $X$ be a compact connected subset of $\mathbb{R}^d$ and let $S_j : X \to X$, $j = 1, \ldots, N$, be contractive maps. We call $\{S_j\}_{j=1}^N$ an iterated function system (IFS) on $X$. It is well known that there exists a unique non-empty compact subset $K \subset X$ invariant under $\{S_j\}_{j=1}^N$ in the sense that $K = \bigcup_{j=1}^N S_j(K)$. If we associate with probability weights $\{p_j\}_{j=1}^N$ to the IFS, then there is a unique probability measure $\mu$ on $X$ with $\text{supp}\mu = K$ satisfying

$$\mu(A) = \sum_{j=1}^N p_j \circ S_j^{-1}(A)$$

for every Borel set $A \subset X$. As is well-known the invariant measure is either continuously singular or absolutely continuous with respect to the Lebesgue measure $m$ on $\mathbb{R}^d$. It is easy to see that if $S_i(K) \cap S_j(K) = \emptyset$, $i \neq j$, then $\mu$ must be singular. However, it remains to be a challenging question to determine which is the case if the $S_i(K)$'s have nonempty intersection ([LNR], [PSS]). One of the most basic
examples of such measures is the classical Bernoulli convolution defined by
\[ \mu_\rho = \frac{1}{2}(\mu_\rho \circ S_1^{-1} + \mu_\rho \circ S_2^{-1}), \]
where \( S_1(x) = \rho x \) and \( S_2(x) = \rho x + (1 - \rho) \) and \( \rho \in (0, 1) \). It is known that \( \mu_\rho \) is purely singular for \( \rho \in (0, 1/2) \) and \( \mu_\rho \) is absolutely continuous with respect to the Lebesgue measure for \( m \)-a.e. \( \rho \in (1/2, 1) \) (see [PSS] and the references therein). In [MS] Mauldin and Simon proved that if \( \mu_\rho \) is absolutely continuous with respect to \( \mu_\rho \), i.e., \( \mu_\rho \) and \( m \) are equivalent. In this paper we will show, among the other results, that the equivalence is actually valid in a more general setting.

Let \( \{p_j(\cdot)\}_{j=1}^N \) be a family of positive continuous functions on \( X \) associated with a contractive IFS \( \{S_j\}_{j=1}^N \). We consider the probability measure \( \mu \) that satisfies the eigen-equation
\[ \lambda \mu = \sum_{j=1}^N p_j(\cdot)\mu \circ S_j^{-1} \]
for some \( \lambda > 0 \). (The notation means \( \lambda \mu(A) = \sum_{j=1}^N \int_A p_j(x)d\mu \circ S_j^{-1}(x) \) for every Borel set \( A \).) The measure is associated with the Ruelle-Perron-Frobenius operator \( T : C(K) \to C(K) \) and its adjoint \( T^* : M(K) \to M(K) \)
\[ Tf(x) = \sum_{j=1}^N p_j(S_j(x))f(S_j(x)), \quad T^* \nu = \sum_{j=1}^N p_j(\cdot)\nu \circ S_j^{-1}, \]
where \( C(K) \) is the space of continuous functions on \( K \) and \( M(K) \) is the space of bounded regular Borel measures on \( K \). The operator was introduced by Ruelle in a more restricted form to model the Gibbs distribution in statistical mechanics, and was adopted to study the discrete time evolution of flows on the Riemannian manifolds [B]. There has been extensive study on the operator in dynamical system in regard to \( \hbar = Th \) and \( \lambda \nu = T^*\nu \). The theory has also been used to study the multifractal structure of measures generated by conformal IFS [MU].

Let \( D \) be an open set in \( \mathbb{R}^d \). We use \( C^1 \) to denote the class of continuously differentiable maps on \( D \). A \( C^1 \)-map \( S : D \to \mathbb{R}^d \) is conformal if \( S'(x) \) is a similar matrix, i.e., \( S'(x) \) is a positive scalar multiple of an orthogonal matrix. In this case \( \|S'(x)\| \), the operator norm of \( S'(x) \), is the square root of the maximum eigenvalue of the product of \( S'(x) \) and its transpose and equals \( |\det S'(x)|^{1/d} \). We say that \( \{S_j\}_{j=1}^N \) is a self-conformal iterated function system on a compact connected set \( X \subset \mathbb{R}^d \) if each \( S_j \) extends to an injective map \( S_j : D \to D \) on an open neighborhood \( D \supset X \) and
\[ \sup\{\|S_j'(x)\| : x \in D, j = 1, 2, \ldots, N\} < 1. \]

For the IFS \( \{S_j\}_{j=1}^N \), let \( J = (j_1, \ldots, j_n) \in \{1, \ldots, N\}^n \) and let \( S_J = S_{j_1} \circ \cdots \circ S_{j_n} \). The conformal IFS is said to have the bounded distortion property (BDP) if there exists a constant \( C > 0 \) such that for any index \( J \)
\[ \frac{\|S_J'(x)\|}{\|S_J'(y)\|} \leq C \quad \text{for any} \quad x, y \in D. \]

It is easy to see that if \( \{S_j\}_{j=1}^N \) are affine maps, then it has the BDP. Moreover, by adopting the proof in [FL] Lemma 2.3, we can show that \( \{S_j\}_{j=1}^N \) also has the
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BDP if \( \log \| S_j^p \|, j = 1, \cdots, N, \) satisfy the Dini condition, i.e.,

\[
\int_0^a \frac{\Omega(\log \| S_j^p \|, t)}{t} \, dt < \infty
\]

for some \( a > 0, \) where \( \Omega(\psi, t) := \max\{\psi(x) - \psi(y) : |x - y| \leq t\}. \)

Let \( \mathcal{H}^s \) and \( \mathcal{H}^s|_K \) be the Hausdorff \( s \)-dimensional measure on \( \mathbb{R}^d \) and its restriction on the set \( K \) respectively. Recall that a set \( E \subset \mathbb{R}^d \) is called an \( s \)-set if \( 0 < \mathcal{H}^s(E) < \infty. \)

**Theorem 1.1.** Suppose that \( \{S_j\}_{j=1}^N \) is a self-conformal iterated function system defined on \( X \) and has the BDP. If the attractor \( K \) is an \( s \)-set and the measure \( \mu \) in (1.2) is absolutely continuous with respect to \( \mathcal{H}^s|_K \), then in reverse, \( \mathcal{H}^s|_K \) is also absolutely continuous with respect to \( \mu \) on \( K \).

This generalizes the results in \([MS], [PSS\text{, Proposition 3.1}]\) and \([HP\text{, Proposition 1.2}]\) where \( \mu \) is a self-similar measure and \( \mathcal{H}^s \) is the Lebesgue measure. It is known that if the conformal \( \{S_j\}_{j=1}^N \) has Hölder continuous differential, and \( s \) is the unique solution of the Bowen equation \( P(s) = 0, \) where

\[
P(t) = \lim_{n \to \infty} \frac{1}{n} \log \sup_{x \in K} \sum_{|J|=n} \|S_j^p(x)\|^t \quad \text{for} \ t > 0,
\]

then \( K \) is an \( s \)-set if and only if \( \{S_j\}_{j=1}^N \) satisfies the open set condition \([PRSS].\)

(If each \( S_j \) is a similitude with a contraction ratio \( \rho_j, \) then \( P(s) = 0 \) is equivalent to the well known formula \( \sum_{i=1}^N \rho_i^s = 1. \) If \( s = d, \) then the absolute continuity of \( \mu \) with respect to \( \mathcal{H}^s|_K \) implies that \( K \) is an \( s \)-set.

For the absolutely continuous \( \mu, \) we also have the following interesting results on the equivalence of the local and global \( L^p \)-property and on a sufficient condition for the unboundedness of the Radon Nikodym derivative \( D\mu. \) Let \( B_d(x) \) represent the open ball centered at \( x \) with radius \( \delta. \)

**Theorem 1.2.** Suppose that \( \{S_j\}_{j=1}^N \) are contractive one-to-one \( C^1 \)-maps defined on \( X \) and the measure \( \mu \) in (1.2) is absolutely continuous with respect to the Lebesgue measure \( m. \) If there exists \( x \in K \) and \( \delta > 0 \) such that \( D\mu \in L^p(B_d(x) \cap K) \) for some \( 1 \leq p \leq \infty, \) then \( D\mu \in L^p(K). \)

**Theorem 1.3.** Let \( \{S_j\}_{j=1}^N \) be contractive one-to-one \( C^1 \)-maps defined on \( X \) and let \( p_j(x) = p_{j}, j = 1, \cdots, N, \) be probability weights. If the invariant measure \( \mu \) in (1.1) is absolutely continuous with respect to the Lebesgue measure \( m \) and there is at least one \( p_j > \beta_j, \) where \( \beta_j = \max_{x \in \mathcal{K}} \{ |\det S_j^p(x)| \}, \) then \( D\mu \) is unbounded on the attractor \( K. \)

We will prove these theorems in Section 2 and make some remarks in Section 3.

2. PROOF OF THEOREMS

Let \( K = \bigcup_{j=1}^N S_j(K) \) be the attractor of the IFS and let \( K_j = S_j(K), J = (j_1, \cdots, j_n) \in \{1, 2, \cdots, N\}^n. \) It is easy to see that for each \( n, K = \bigcup_{|J|=n} K_j. \) We let \( \mathcal{K} = \{K_j : |J| = n, n \in \mathbb{N}\}. \) Then \( \mathcal{K} \) is a countable family of compact subsets with the following properties:

(P1) For any \( \delta > 0, \) there are only finitely many members of \( \mathcal{K} \) whose diameters are \( > \delta. \)

(P2) For any \( \epsilon > 0, x \in K, \) there exists \( K_j \in \mathcal{K}, \) such that \( x \in K_j \subset B_\epsilon(x) \cap K. \)
We first show that for any open set $U$, $K$ has a finite or countable disjoint subfamily which covers $U \cap K$ except for an $\mathcal{H}^{s}$-zero set.

**Lemma 2.1.** Suppose that the IFS defined on $X$ is self-conformal and has the BDP. If the attractor $K$ is an $s$-set, then for any open set $U \subset \mathbb{R}^d$, there exists a finite or countable disjoint subfamily $\mathcal{G}$ of $K$ contained in $U$ and $\mathcal{H}^s((U \cap K) \setminus G) = 0$, where $G = \bigcup \mathcal{G}$.

**Proof.** Let $K_U = \{ A \in K : A \subset U \}$. Using the Vitali covering theorem [F, Theorem 1.10], we can select a finite or countable disjoint subfamily $\mathcal{G}$ of $K_U$ such that either $\sum_{V \in \mathcal{G}} (\text{diam}(V))^s = \infty$ or $\mathcal{H}^s((U \cap K) \setminus G) = 0$. In the following, we will exclude the first case to complete the proof.

Notice that each $S_f$ extends to an injective map on an open bounded set $D$ which is also connected. Let $\delta_0 = \inf \{|x-y| : x \in X, y \notin D\}$. Then the Mean Value Theorem and the property of conformal map imply that for all $J \in \bigcup_{n=1}^{\infty} \{1, 2, \cdots, N\}^n$ and $x, y \in X$ with $|x-y| < \delta_0$,

$$\min_{u \in D} \|S'_f(u)\| \cdot |x-y| \leq |S_J(x) - S_J(y)| \leq \max_{u \in D} \|S'_f(u)\| \cdot |x-y|.$$  

Using this bi-Lipschitz property we obtain

$$\left( \min_{u \in D} \|S'_f(u)\| \right)^s \mathcal{H}^s(K) \leq \mathcal{H}^s(S_J(K)) \leq \left( \max_{u \in D} \|S'_f(u)\| \right)^s \mathcal{H}^s(K).$$

We will show that the second inequality in (2.1) holds even if $|x-y| \geq \delta_0$. In fact, since $D$ is connected and $X$ is bounded, we can find $M$ balls of radius $\delta_0$ contained in $D$ such that their union is connected and covers $X$. We then select $B_i = B_{\delta_0}(x_i), i = 1, 2, \cdots, m, (m \leq M)$ from the covering such that $x \in B_i$, $y \in B_m, B_i \cap B_{i+1} \neq \emptyset$, $i = 1, 2, \cdots, m-1$. Using the Mean Value Theorem we obtain

$$|S_J(x) - S_J(y)| \leq |S_J(x) - S_J(x_1)| + \sum_{i=1}^{m-1} |S_J(x_i) - S_J(x_{i+1})| + |S_J(x_m) - S_J(y)| \leq 2(m+1)\max_{u \in D} \|S'_f(u)\| \leq 2(M+1)\max_{u \in D} \|S'_f(u)\||x-y|.$$  

It follows that

$$\text{diam}(S_J(K)) \leq 2(M+1)\max_{u \in D} \|S'_f(u)\| \text{diam}(K).$$

Applying the BDP, (2.2) and the above inequality, we can find a constant $C > 0$ such that

$$\text{diam}(S_J(K))^s \leq C \mathcal{H}^s(S_J(K)).$$

This implies that

$$\sum_{V \in \mathcal{G}} (\text{diam}(V))^s \leq C \sum_{V \in \mathcal{G}} \mathcal{H}^s(V) \leq C \mathcal{H}^s(U \cap K) < \infty.$$  

Proof of Theorem 1.1. Suppose otherwise there exists a Borel subset \( E \subset K \) such that \( \mu(E) = 0 \) but \( \mathcal{H}^r|_K(E) > 0 \). Using (1.2) we have
\[
0 = \lambda \mu(E) = \sum_{j=1}^{N} \int_{E} p_j(x) d\mu \circ S_j^{-1}(x) = \sum_{j=1}^{N} \int_{S_j^{-1}(E)} p_j \circ S_j(x) d\mu(x).
\]
Since the \( p_j \)'s are positive functions, \( \mu(S_j^{-1}(E)) = 0 \) for all \( j \). Let
\[
Z = \bigcup_{k=0}^{\infty} \bigcup_{|J|=k} S_j^{-1}(E \cap K_j).
\]
It follows that \( \mu(Z) = 0 \). Note that \( Z \subset K \), so \( \mu(K \setminus Z) = \mu(K) \). Let us denote \( \mathcal{H}^r|_K \) by \( \nu \) for short. We claim that \( \nu(K \setminus Z) = 0 \). This will imply that \( \mu \) is concentrated on a \( \nu \)-zero subset of \( K \), so \( \nu \) and \( \mu \) are mutually singular. It contradicts the hypothesis that \( \mu \) is absolutely continuous with respect to \( \nu \) on \( K \), and completes the proof of the theorem.

To prove the claim we note that \( \nu(E) > 0 \), hence we can apply the density theorem to find a point \( x \in E \) such that for any \( \varepsilon > 0 \), there exists an open ball \( B_r(x) \) with
\[
\frac{\nu(B_r(x) \cap E)}{\nu(B_r(x))} \geq 1 - \varepsilon.
\]
Replacing the \( U \) in Lemma 2.1 by \( B_r(x) \), we can find a finite or countable disjoint subfamily \( \mathcal{G} \) of \( K \) such that each member of \( \mathcal{G} \) is a subset of \( B_r(x) \cap K \) and
\[
\nu((B_r(x) \cap K) \setminus G) = 0,
\]
where \( G = \bigcup \mathcal{G} \). Note that since \( G \subset B_r(x) \), we have
\[
\frac{\nu(G \cap E)}{\nu(G)} = \frac{\nu(B_r(x) \cap G \cap E)}{\nu(B_r(x) \cap G)} \geq 1 - \varepsilon.
\]
Since members of \( \mathcal{G} \) are disjoint, there exists \( K_j \in \mathcal{G} \) with
\[
\frac{\nu(K_j \cap E)}{\nu(K_j)} \geq 1 - \varepsilon.
\]
Observe that since \( S_j^{-1}(K_j \cap E) \subset Z \), we have \( \nu(S_j Z) \geq \nu(K_j \cap E) \geq (1 - \varepsilon) \nu(K_j) \).
Since \( Z \subset K \), it gives
\[
\nu(S_j(K \setminus Z)) = \nu(K_j) - \nu(S_j Z) \leq \varepsilon \nu(K_j).
\]
Inequality (2.2) implies that
\[
\left( \min_{u \in D} ||S_j^*(u)|| \right)^* \nu(K \setminus Z) \leq \nu(S_j(K \setminus Z))
\]
and
\[
\nu(K_j) \leq \max_{u \in D} (||S_j^*(u)||)^* \nu(K).
\]
By the BDP it follows that \( \nu(K \setminus Z) \leq \varepsilon C \nu(K) \). Since \( \varepsilon \) is arbitrary, \( \nu(K \setminus Z) = 0 \) and the claim is proved. \( \square \)
Proof of Theorem 1.2. Let \( f = D\mu \). For any \( x \in K, \varepsilon > 0 \) and \( M > 0 \), denote
\[
A(x, \varepsilon, M) = \{ t \in K \cap B_\varepsilon(x) : f(t) > M \}.
\]
We first consider the case \( p = \infty \). It suffices to show the claim: If \( f \notin L^\infty(K) \), then for any given \( M_0 > 0, \varepsilon_0 > 0 \) and \( x_0 \in K \), we have \( m(A(x_0, \varepsilon_0, M_0)) > 0 \).

For this we first differentiate (1.2) with respect to the Lebesgue measure and get
\[
\lambda f(x) = \sum_{j=1}^N p_j(x) |\det((S_j^{-1})'(x))| f(S_j^{-1}(x)).
\]
Since \( S_j \) is contractive and one-to-one, hence \( |\det((S_j^{-1})'(x))| > 1 \). This implies that for every \( j \)
\[
(2.3) \quad \lambda f(x) \geq p_j(x) f(S_j^{-1}(x)).
\]
Given any \( M > 0 \), by assumption \( f \notin L^\infty(K) \), \( m\{t \in K : f(t) > M\} > 0 \). Using the Lebesgue density theorem, there exists \( x^* \in K \) such that for any \( \varepsilon > 0 \),
\[
m(A(x^*, \varepsilon, M)) > 0.
\]
Let \( x \in S_j(A(x^*, \varepsilon, M)) \). Then \( x = S_j(t) \) for some \( t \in K \cap B_\varepsilon(x^*) \) and \( f(t) > M \). Note that \( S_j \) is contractive, so \( x \in K \cap B_\varepsilon(S_j(x^*)) \). Let \( 0 < \alpha_j = : \min_{x \in K} p_j(x) \).

By (2.3) we have
\[
f(x) \geq \lambda^{-1} p_j(x) f(S_j^{-1}(x)) \geq \lambda^{-1} \alpha_j M.
\]
It follows that
\[
S_j(A(x^*, \varepsilon, M)) \subset A(S_j(x^*), \varepsilon, \lambda^{-1} \alpha_j M).
\]
Hence
\[
m(A(S_j(x^*), \varepsilon, \lambda^{-1} \alpha_j M)) \geq m(A(S_j(x^*), \varepsilon, M))
\]
\[
= \int_{A(x^*, \varepsilon, M)} |\det S_j'(x)| \, dx
\]
\[
\geq \left( \min_{x \in K} |\det S_j'(x)| \right) m(A(x^*, \varepsilon, M)) > 0.
\]
By repeating this process, we can prove that for any \( J = j_1 \cdots j_n \in \{ 1, \cdots, N \}^n \) and for any \( \varepsilon > 0 \),
\[
m(A(S_J(x^*), \varepsilon, \lambda^{-n} \alpha_J M)) > 0,
\]
where \( \alpha_J = \alpha_{j_1} \cdots \alpha_{j_n} \).

Now for any fixed \( x_0 \in K, \varepsilon_0 > 0 \) and \( M_0 > 0 \), let \( J_0 \in \bigcup_{k \geq 1} \{ 1, \cdots, N \}^k \) be such that \( |S_{J_0}(x) - x_0| < \varepsilon_0/2 \) for all \( x \in K \). We choose \( \varepsilon = \varepsilon_0/2 \), \( M = \lambda^{1/\varepsilon_0} \alpha_{J_0}^{-1} M_0 \).

Then from the above, we have \( x_0 \in K \) such that
\[
m(A(S_{J_0}(x^*), \varepsilon_0/2, M_0)) > 0.
\]
By the definition of \( J_0 \), we have \( |S_{J_0}(x^*) - x_0| < \varepsilon_0/2 \). It follows that
\[
A(S_{J_0}(x^*), \varepsilon_0/2, M_0) \subset A(x_0, \varepsilon_0, M_0).
\]
Thus
\[
m(A(x_0, \varepsilon_0, M_0)) \geq m(A(S_{J_0}(x^*), \varepsilon_0/2, M_0)) > 0.
\]
The claim is proved.
For the case $1 \leq p < \infty$, we let $C > 0$ be such that $p_j(x) \geq C$ for all $x \in K, j = 1, \cdots, N$. By (2.3), $\lambda^p f^p(x) \geq C^p f^p(S_j^{-1}(x))$. Note that since $f(x)$ is supported on $K$, we have

$$\lambda^p > C \int_{B_\delta(x)} f^p(t)dt \geq C \int_{B_\delta(x)} f^p(S_j^{-1}(t))dt$$

$$= C \int_{S_j^{-1}(B_\delta(x))} f^p(u) | \det S_j'(u) | du$$

$$\geq C \min_{t \in K} | \det S_j'(t) | \int_{S_j^{-1}(B_\delta(x))} f^p(u)du.$$  

Since $\min_{t \in K} | \det S_j'(t) | > 0$, we conclude that

$$\int_{S_j^{-1}(B_\delta(x))} f^p(u)du < \infty.$$  

By replacing $B_\delta(x)$ with $S_i^{-1}(B_\delta(x))$, we can show inductively that for all $J \in \{1, \cdots, N\}^n$

$$\int_{S_j^{-1}(B_\delta(x))} f^p(t)dt < \infty.$$  

The theorem thus follows from the fact that there exists $J$ such that $S_J(K) \subset B_\delta(x)$, and hence $K \subset S_j^{-1}(B_\delta(x))$. \hfill \Box

**Proof of Theorem 1.3.** Let $f = D\mu$. It suffices to show that $m\{x \in K : f(x) > M\} > 0$ for any fixed $M > 0$.

For any $J = (j_1, \cdots, j_n)$, let $p_J = p_{j_1} \cdots p_{j_n}$. By using (1.1) repeatedly, we have for any $n \in \mathbb{N}$

$$\mu(A) = \sum_{j=1}^N \sum_{i=1}^N p_i p_j \mu \circ S_i^{-1} \circ S_j^{-1} (A)$$

$$= \sum_{|J|=n} p_J \mu \circ S_J^{-1} (A).$$

It follows that for any fixed $J = (j_1, \cdots, j_n)$,

(2.4)  

$$p_J = p_{j_1} \cdots p_{j_n} \leq \mu(K_J).$$

Let $j$ be the index such that $p_j > \beta_j$ as in the hypothesis. Then for any $M > 0$, there exists $n_0$ such that for all $n \geq n_0$,

$$M m(K) \beta_j^n < p_j^n.$$  

Let $J^* = (j, j, \cdots, j)$ have length $n \geq n_0$; then

$$m(K_{J^*}) = \int_K | \det S_j'(x) | dx \leq \beta_j^n m(K).$$
On the other hand, we have

\[ \mu(K_{j^*}) = \int_{K_{j^*}} f(x)dx \]

\[ = \int_{K_{j^*} \cap \{f(x) > M\}} f(x)dx + \int_{K_{j^*} \cap \{f(x) \leq M\}} f(x)dx \]

\[ \leq \int_{K_{j^*} \cap \{f(x) > M\}} f(x)dx + M \cdot m(K_{j^*}) \]

\[ < \int_{K_{j^*} \cap \{f(x) > M\}} f(x)dx + p_j^m. \]

By (2.4) we have

\[ \int_{K_{j^*} \cap \{f(x) > M\}} f(x)dx > \mu(K_{j^*}) - p_j^m \geq 0, \]

so that \( m\{x \in K_{j^*} : f(x) > M\} > 0 \). This proves the theorem. \( \square \)

3. Some remarks

Let \( \mu \) be the Bernoulli convolution defined by \( \mu = p\mu \circ S_1^{-1} + (1 - p)\mu \circ S_2^{-1} \), where \( S_1x = \rho x, S_2x = \rho x + (1 - \rho) \). In [PS] Peres and Solomyak proved that if \( p \in [\frac{1}{3}, \frac{2}{3}] \), then \( \mu \) is absolutely continuous for almost all \( \rho \in [p^p(1 - p)^{1-p}, 1] \). Note that \( (\frac{1}{3})^{1/3}(\frac{2}{3})^{2/3} = \frac{2^{2/3}}{3} = 0.5291 < \frac{2}{3} \). If we take \( \frac{2^{2/3}}{3} \leq \rho < \frac{2}{3} \), we see from Theorem 1.3 that there are invariant measures with unbounded density. We also note that there are \( \rho \in [\frac{2^{2/3}}{3}, \frac{2}{3}] \) such that \( \rho^{-1} \) is a P.V. number, for such \( \rho \) the invariant measure will be purely singular for any weight [LNR].

Theorem 1.2 implies that the \( \mu \) in (1.2) has a self-similar property that if \( D\mu \) is unbounded on \( K \), then it is unbounded in \( B_\delta(x) \) for every \( x \in \text{supp}\mu \) and for every \( \delta > 0 \). It is clear that the \( L^p \)-property in the theorem cannot be replaced by the \( C^k \)-property, namely, the fact that \( D\mu \) is \( C^k \)-differentiable on a ball does not imply that it is \( C^k \)-differentiable on \( K \). For example, let \( S_j(x) = x/2 + j, j = 0, 1, 2, \) with weights \( p_0 = p_2 = 1/4 \) and \( p_1 = 1/2 \). Then \( D\mu \) is a tent function on \([0,4] \), which has continuous derivatives of all order in \((0,4) \setminus \{2\} \), but it is not differentiable at 2.

In regard to the question of the eigen-measure in (1.2) being of pure type, we see that if \( \mu \) is the unique measure satisfying (1.2), then it is either discrete or singularly continuous or absolutely continuous. This can easily be proved by writing \( \mu \) as the three components and show that each component satisfies (1.2) and then applying the uniqueness of the eigen-measure to make the conclusion. Nevertheless the measure of (1.2) may not be unique if \( p_0(x) \) is merely assumed to be continuous [Q], and stronger continuity assumption has to be added [FL]. Therefore we do not have a complete answer for the pure type.

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