PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 3, Pages 613–618
S 0002-9939(01)06393-6
Article electronically published on October 23, 2001

ON THE NATANZON-TURAEV COMPACTIFICATION OF THE HURWITZ SPACE

STEVEN P. DIAZ

(Communicated by Michael Stillman)

Abstract. Natanzon and Turaev have constructed by topological methods a compactification of the Hurwitz space, that is, the space of simple branched covers of the two-sphere. Here we show that this compactification is homeomorphic to a compactification mentioned by Diaz and Edidin (in 1996) that was constructed by algebraic methods. Using this we are able to show by example that the Natanzon-Turaev compactification can be singular, that is, not a manifold.

1. Introduction

The Hurwitz space $H^{n,w}$ is the set of all n sheeted connected coverings of the sphere S^2 simply branched over exactly w distinct points and otherwise unbranched. Hurwitz [Hu] showed a natural way to make $H^{n,w}$ into a complex manifold of complex dimension w. The space $H^{n,w}$ is not compact. When two or more distinct branch points approach each other the limit of the corresponding covers will not be a cover of the same type. Various compactifications of $H^{n,w}$ and closely related spaces have been constructed and studied by methods of algebraic geometry; see for instance [HM], [DE], and [M]. In [NT] Natanzon and Turaev construct a compactification of $H^{n,w}$ using topological methods. In this paper we show that the compactification constructed in [NT] is homeomorphic to a compactification in [DE].

This allows us to answer some questions brought up in [NT]. Natanzon and Turaev point out that there are no known topological descriptions of the compactifications in algebraic geometry. In view of the homeomorphism we construct, the Natanzon-Turaev compactification is a topological description of one of the compactifications from algebraic geometry. Finally, Natanzon and Turaev asked about the local structure of their compactification, in particular, is it a complex manifold? Methods of algebraic geometry allow one to analyse the local structure of the compactification from [DE] to which the Natanzon-Turaev compactification is homeomorphic. Using this we construct an example to show that the Natanzon-Turaev compactification can be singular, that is, not a complex manifold.

Received by the editors August 17, 1999 and, in revised form, March 8, 2000.
2000 *Mathematics Subject Classification.* Primary 14H10; Secondary 57M12.
Key words and phrases. Hurwitz space.

2. The homeomorphism

We start by describing the Natanzon-Turaev compactification and the compactification from [DE] to which it is homeomorphic.

First we describe the Natanzon-Turaev compactification. From the Riemann-Hurwitz formula one deduces that the genus of the cover of S^2 in an element of $H^{n,w}$ is $g = \frac{1}{2}(w-2n+2)$. Fix a closed oriented connected (though [NT] does not require connected) surface Σ of genus g. Define $H(\Sigma, n)$ to be the set of equivalence classes of n-sheeted simple branched coverings $f : \Sigma \to S^2$, where the equivalence relation is: $f : \Sigma \to S^2$ and $f' : \Sigma \to S^2$ are equivalent if and only if there is a homeomorphism $\alpha : \Sigma \to \Sigma$ such that $f = f'\alpha$. $H(\Sigma, n)$ is the Hurwitz space $H^{n,w}$. Natanzon and Turaev construct a compactification $N(\Sigma, n)$ of $H(\Sigma, n)$. The points of $N(\Sigma, n)$ are equivalence classes of decorated functions, where Natanzon and Turaev define decorated functions and their equivalence as follows. We quote directly from [NT].

Definition 2.1. A decorated function (on the surface Σ) is a triple $(f, E, \{D_e\}_{e\in E})$ where $f : \Sigma \to S^2$ is a simple branched covering, E is a finite subset of S^2 disjoint from the set of branch points of f, and $\{D_e\}_{e\in E}$ are disjoint closed 2-discs embedded in S^2 such that: $e \in \operatorname{Int} D_e$ for all $e \in E$, each D_e contains at least two branch points of f, and the circles $\{\partial D_e\}_e$ do not contain branch points of f.

An isotopy of a decorated function $(f, E, \{D_e\}_e)$ is a family of decorated functions $(\varphi_t f, E, \{\varphi_t(D_e)\}_e)$ where $\{\varphi_t : S^2 \to S^2\}_{t\in[0,1]}$ is an isotopy of the identity map $\varphi_0 = \mathrm{id}_{S^2}$ such that for all $t \in [0, 1]$ the homeomorphism φ_t preserves (pointwise) E and the branch points of f lying outside $\bigcup_e D_e$.

We say that two decorated functions $(f, E, \{D_e\}_{e\in E})$ and $(f', E', \{D'_e\}_{e\in E'})$ are equivalent if $E = E'$ and f' may be obtained from f by an isotopy and/or composition with a homeomorphism $\Sigma \to \Sigma$. (The isotopy must also take each D_e to D'_e.)

Recall that the set of unordered sets of k not necessarily distinct points in complex projective one-space $\mathbb{CP}^1 = S^2$ is naturally identified with complex projective k-space $\mathbb{CP}^k = \mathbb{P}^k$. There is a natural map $q : H(\Sigma, n) = H^{n,w} \to \mathbb{P}^w$ which sends a cover $(f : \Sigma \to S^2) \in H(\Sigma, n)$ to its branch points. This extends to a map (also denoted q) $q : N(\Sigma, n) \to \mathbb{P}^w$ which sends a decorated function $(f, E, \{D_e\})$ to the set consisting of the branch points of f outside of $\bigcup_e D_e$ each counted once, plus the points of E each counted with multiplicity equal to the number of branch points of f inside the corresponding D_e. As mentioned in [NT] before Lemma 2.2 the extended mapping q is continuous and open.

Now we describe the algebraic compactification mentioned in [DE]. In that article the authors denote the compactification by $\overline{SH}_{k,b}$ (k corresponds to n and b to w in $H^{n,w}$). It has been known since Hurwitz that $q : H(\Sigma, n) \to \mathbb{P}^w$ is finite. Therefore the function field of $H(\Sigma, n)$ is a finite extension of the function field of $\mathbb{P}^w$. The compactification $\overline{SH}_{k,b}$ is defined to be the normalization of $\mathbb{P}^w$ in the function field of $H(\Sigma, n)$. The map $q : H(\Sigma, n) \to \mathbb{P}^w$ extends to a regular algebraic morphism $\pi : \overline{SH}_{k,b} \to \mathbb{P}^w$.

The homeomorphism $g : N(\Sigma, n) \to \overline{SH}_{k,b}$ that we claim to exist will be defined to be the identity on $H(\Sigma, n) \subset N(\Sigma, n)$ and $H(\Sigma, n) \subset \overline{SH}_{k,b}$. It will also commute with the maps q and π to $\mathbb{P}^w$. What is left to define are the values of g on the points of $N(\Sigma, n)$ lying over points of $\mathbb{P}^w$ corresponding to nondistinct points in $\mathbb{P}^1$.

Denote by D the points of $\mathbb{P}^w$ corresponding to nondistinct sets of w points in $\mathbb{P}^1$. Let $p \in D$. Lemma 4.3 of [DE], which was proved for $\overline{H}_{k,b}$ can in the same way be proven for $\overline{SH}_{k,b}$ to obtain the following.

Lemma 2.1. *Pick a small connected neighborhood (say any small open ball) B of p in $\mathbb{P}^w$, and pick a point $r \in B - D$. The fundamental group of $B - D$ with base point r acts via monodromy on $\pi^{-1}(r)$. Define an equivalence relation on $\pi^{-1}(r)$ by saying that two points are equivalent if and only if they can be taken to each other by the monodromy action. For B sufficiently small the following are true.*

1. *Two points of $\pi^{-1}(r)$ lie in the same monodromy equivalence class iff they lie in the same connected component of $\pi^{-1}(B - D)$.*
2. *The closure of each connected component of $\pi^{-1}(B-D)$ in $\pi^{-1}(B)$ has exactly one point over p.*
3. *The closures of the connected components of $\pi^{-1}(B - D)$ in $\pi^{-1}(B)$ are all disjoint from each other.*

For our purposes the point of this lemma is the following obvious corollary.

Corollary 2.1. *For B sufficiently small there is a natural bijection between the points of $\pi^{-1}(p)$ and the connected components of $\pi^{-1}(B - D)$. The bijection is given by associating to each connected component X of $\pi^{-1}(B-D)$ the intersection of the closure of X in $\pi^{-1}(B)$ with $\pi^{-1}(p)$.*

Next we see that exactly the same result is true if we replace $\overline{SH}_{k,b}$ with $N(\Sigma, n)$ and $\pi : \overline{SH}_{k,b} \to \mathbb{P}^w$ with $q : N(\Sigma, n) \to \mathbb{P}^w$.

Proposition 2.1. *For B as in Lemma 2.1 sufficiently small there is a natural bijection between the points of $q^{-1}(p)$ and the connected components of $q^{-1}(B-D)$. The bijection is given by associating to each connected component X of $q^{-1}(B-D)$ the intersection of the closure of X in $q^{-1}(B)$ with $q^{-1}(p)$.*

Proof. The point p must consist of ℓ distinct points $p_1, \dots, p_\ell$ and ℓ' multiple points $e_1, \dots, e_{\ell'}$ of $\mathbb{P}^1$. Say the multiplicity of e_i is k_i.

Step 1. Given any connected component X of $q^{-1}(B - D)$ we can find an $(f : \Sigma \to S^2) \in X$ such that all the p_i's are branch points of f and none of the e_i's are branch points of f. We can then create a decorated function $(f, E, \{D_e\}_e)$ representing a point of $q^{-1}(p)$ as follows.

First, the existence of such an f is clear because $q(X) = B - D$. Because $f \in X$, for small B we have that if we let D_{e_i} be a small disk around e_i, then D_{e_i} contains exactly k_i branch points. Letting $E = \{e_1, \dots, e_{\ell'}\}$, $(f, E, \{D_{e_i}\}_{e_i})$ represents a point of $q^{-1}(p)$.

Step 2. Given any point x_0 of $q^{-1}(p)$ we can choose a decorated function $(f, E, \{D_e\}_e)$ representing x_0 that is obtained in the manner of Step 1, for some connected component of $q^{-1}(B - D)$ possibly depending on x_0.

It is simply a matter of using an isotopy φ_t to shrink the disks D_e until they are small enough so that $\varphi_t f \in q^{-1}(B - D)$.

Step 3. Two decorated functions $(f, E, \{D_e\}_e)$ and $(f', E, \{D'_e\})$ obtained as in steps 1 and 2 are equivalent iff f and f' lie in the same connected component of $q^{-1}(B - D)$.

By applying isotopies we can assume the disks $D_e = D'_e$ for all $e \in E$. If f and f' lie in the same connected component X, then a path in X connecting them can be used to create the desired equivalence as in [NT], proof of Theorem 3.9, the paragraph beginning "Since the Hurwitz space $H_{g,n}$ is connected ... ". On the other hand suppose $(f, E, \{D_e\}_e)$ and $(f', E, \{D_e\}_e)$ are equivalent. Since as points of $H(\Sigma, n)$ f and f' are well defined only up to homeomorphisms $\alpha : \Sigma \to \Sigma$ we may assume that $(f, E, \{D_e\}_e)$ and $(f', E, \{D_e\}_e)$ are isotopic. Let $a_0, a_1 \in B - D$ correspond to the branch points of f and f' respectively. Following the branch points of f along the isotopy gives a path in $\mathbb{P}^w - D$ starting at a_0 and ending at a_1 that lifts to a path in $H(\Sigma, n)$ joining f to f'. Under the isotopy the branch points not in any D_e never move and the branch points in each D_e always stay inside disks centered at e. At all stages of the isotopy these disks must remain disjoint from each other and from the branch points not in any D_e. This says that the branch points in each D_e may loop around each other, but they may not loop around branch points in other D_e's or branch points not in any D_e. This says that the resulting path in $\mathbb{P}^w - D$ joining a_0 to a_1 is homotopic to a path in $B - D$ joining a_0 to a_1 that now lifts to a path in $q^{-1}(B - D)$ joining f to f' so that f and f' lie in the same connected component of $q^{-1}(B - D)$.

From steps 1–3 we see that the number of points in $q^{-1}(p)$ equals the number of connected components of $q^{-1}(B - D)$. To complete the proof it is enough to show that no connected component of $q^{-1}(B-D)$ can have more than one point of $q^{-1}(p)$ in its closure. Assume to the contrary that some connected component had at least two points of $q^{-1}(p)$ in its closure. From [NT], Lemma 2.2, we know that $N(\Sigma, n)$ is Hausdorf. We can find disjoint open neighborhoods of these two points. After possibly shrinking B the intersection of $q^{-1}(B - D)$ with these two disjoint open neighborhoods would lead to two connected components, a contradiction. $\square$

We can now define the map g on points of $N(\Sigma, n)$ lying over points of $D \subset \mathbb{P}^w$. Given $p \in D$ and $x \in q^{-1}(p)$, for a sufficiently small open ball B around p, x will be the only point of $q^{-1}(p)$ lying in the closure of some connected component X of $q^{-1}(B - D)$. Similarly, the closure of X in $\pi^{-1}(B)$ contains a unique point y of $\pi^{-1}(p)$. Define $g(x) = y$.

Proposition 2.2. *The map g is a homeomorphism.*

Proof. It is clear that g is bijective. We show that g is continuous. The proof that g^{-1} is continuous is similar.

Let $U \subset \overline{SH}_{k,b}$ be an open set. We wish to show that $g^{-1}(U) \subset N(\Sigma, n)$ is open. For this it is sufficient to show that for each $x \in g^{-1}(U)$ we can find an open set V_x of $N(\Sigma, n)$ with $x \in V_x \subset g^{-1}(U)$. Clearly this can be done for $x \in H(\Sigma, n)$ so we assume $x \in N(\Sigma, n) - H(\Sigma, n)$. Set $y = g(x)$. For a sufficiently small ball B around $\pi(y)$ the connected component Y of $\pi^{-1}(B)$ containing y will be an open neighborhood of y contained in U. The connected component X of $q^{-1}(B)$ containing x will be an open neighborhood of x. We wish to show $g(X) \subset Y$, so that $X \subset g^{-1}(U)$. Clearly $g(X \cap H(\Sigma, n)) = Y \cap H(\Sigma, n)$, in fact $X \cap H(\Sigma, n) = Y \cap H(\Sigma, n)$. Pick any $x_0 \in X - H(\Sigma, n)$. To compute $g(x_0)$ we find a sufficiently small ball B_0 around $q(x_0)$, we can assume $B_0 \subset B$, so that Corollary 2.1 and Proposition 2.1 apply. Say X_0 is the connected component of $q^{-1}(B_0 - D)$ with x_0 in its closure. Then $g(x_0)$ will be the point over $q(x_0)$ in the closure of X_0 in $\pi^{-1}(B_0)$. But $B_0 \subset B$ says $X_0 \subset X \cap H(\Sigma, n) = Y \cap H(\Sigma, n)$, so $g(x_0) \in Y$. $\square$

As pointed out in [DE], section 4.4, $\overline{SH}_{k,b}$ is a projective variety and it is certainly normal. In view of the homeomorphism g we could define a complex structure on

$N(\Sigma, n)$ to be the corresponding complex structure on $\overline{SH}_{k,b}$. $N(\Sigma, n)$ would then be a normal projective variety. Since it is normal its singularities have complex codimension at least 2. As we shall see in the next section $N(\Sigma, n)$ can have singularities.

3. A singular example

We shall study $N(\Sigma, n)$ when $\Sigma = S^2 = \mathbb{P}^1$ and $n = 3$. Thus we are studying degree three covers of S^2 simply branched at four points. We have the map $q : N(S^2, 3) \to \mathbb{P}^4$. We will show that over points of $\mathbb{P}^4$ corresponding to two distinct points of $\mathbb{P}^1$ each taken with multiplicity 2, $N(S^2, 3)$ has two points—one nonsingular and one singular.

Let $D \subset \mathbb{P}^4$ be the discriminant locus consisting of nondistinct points and fix $O \in D$ where O corresponds to two distinct points each with multiplicity 2. Locally near O, D consists of two smooth branches crossing transversally. Each branch corresponds to allowing one of the two multiplicity 2 points to become two distinct points. Pick a point $P \in \mathbb{P}^4 - D$ near O. By standard techniques from Hurwitz space theory (see [F], proof of Proposition 1.5, [A], proof of Theorem 2.7, or [DE], section 4.2 shortly before Lemma 4.2) the fiber of q over P corresponds to equivalence classes of ordered 4-tuples of simple transpositions $[\sigma_1, \sigma_2, \sigma_3, \sigma_4]$, $\sigma_i \in S_3$ (the symmetric group on three letters), such that the product $\sigma_1\sigma_2\sigma_3\sigma_4 = (1)$ and $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ generates a transitive subgroup of S_3, where the equivalence relation is $[\sigma_1, \sigma_2, \sigma_3, \sigma_4]$ is equivalent to $[\tau_1, \tau_2, \tau_3, \tau_4]$ iff there exists an $\alpha \in S_3$ such that $\sigma_i = \alpha\tau_i\alpha^{-1}$, $i = 1, \ldots, 4$. Each σ_i represents the ramification point over one of the four points of $\mathbb{P}^1$ represented by P. One computes that over P there are four points which we may represent as: $[(12), (12), (23), (23)]$, $[(12), (23), (12), (13)]$, $[(12), (23), (13), (23)]$, and $[(12), (23), (23), (12)]$.

We may assume that we have set things up so that one branch of D near O corresponds to the first two points becoming one double point and the other branch represents the last two points becoming one double point. Again from standard Hurwitz space techniques (see [F], proof of Proposition 1.5, [A], proof of Theorem 2.7, or [DE], end of section 4.2) we see that the monodromy action on the inverse image of P generated by a loop based at P going around branch 1 is generated by $[\sigma_1, \sigma_2, \sigma_3, \sigma_4] \mapsto [\sigma_2, \sigma_2^{-1}\sigma_1\sigma_2, \sigma_3, \sigma_4]$ and around branch 2 it is $[\sigma_1, \sigma_2, \sigma_3, \sigma_4] \mapsto [\sigma_1, \sigma_2, \sigma_4, \sigma_4^{-1}\sigma_3\sigma_4]$. One computes that in both cases $[(12), (12), (23), (23)]$ does not move but that the other three points are permuted cyclically. Remember that after applying the monodromy transformation you might need to conjugate by an appropriate element of S_3 to get the ordered 4-tuple to be one of the four we have chosen to represent the fiber.

Thus over a small neighborhood of O in $\mathbb{P}^w$, $N(S^2, 3)$ has two components. One is a single sheet mapping isomorphically onto the small neighborhood. This gives the nonsingular point of $q^{-1}(O)$. The other component consists of three sheets all coming together and ramifying to order 3 over each branch of D. We now concentrate on that component; call it X.

Choose local coordinates u, v, x, y on $\mathbb{P}^4$ near O so that D has local equation $xy = 0$. The ramification to order 3 along both $x = 0$ and $y = 0$ says that in X, xy has a cube root. In $\mathbb{C}^5$ with coordinates u, v, x, y, z take the hypersurface $X' = \{xy = z^3\}$. X' maps to $\mathbb{P}^4$ by $(u, v, x, y, z) \mapsto (u, v, x, y)$ (locally near O of course). One easily computes that the singularities of X' are $x = y = z = 0$.

X' is normal because it is a hypersurface with singularities in codimension greater than 1; see [Ha], Proposition II.8.23. X' also has the appropriate monodromy along $xy = 0$. By uniqueness of normalization X' near $(0, \ldots, 0)$ is isomorphic to X near $q^{-1}(O) \cap X$. Thus X is singular.

Even if we get a loop backwards in the monodromy the only other possibility is $z^3 = x^2y$ which also has a singular normalization.

As a final remark we note that since $\overline{SH}_{k,b}$ is normal any nonsingular variety Z finite over $\mathbb{P}^w$ compactifying $H^{n,w}$ would be isomorphic to $\overline{SH}_{k,b}$. Since a nonsingular variety is normal such a Z would have to be the normalization of $\mathbb{P}^w$ in the function field of $H^{n,w}$, hence equal to $\overline{SH}_{k,b}$. Thus we cannot make $N(S^2, 3)$ nonsingular by finding a different complex structure to put on it.

References

[A] E. Arbarello, *On subvarieties of the moduli space of curves of genus g defined in terms of Weierstrass points*, Lincei-Mem. Sci. Fis., ecc.-1978-S. VIII, **Vol. XV**, Sez. I, 1. MR **80f:**14013

[DE] S. Diaz and D. Edidin, *Towards the Homology of Hurwitz Spaces*, J. Differential Geometry **Vol. 43**, No. 1, (1996), 66–98. MR **98e:**14028

[F] W. Fulton, *Hurwitz schemes and the irreducibility of moduli of algebraic curves*, Ann. Math. **90** (1969), 542–575. MR **41:**5375

[Ha] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977. MR **57:**3116

[HM] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Inventiones Mathematicae **67** (1982), 23–86. MR **83i:**14018

[Hu] A. Hurwitz, *Über Riemann'sche Flächen mit gegeberen Verzweigungspunkten*, Math. Ann. **39** (1891), 1–61.

[M] S. Mochizuki, *The geometry of the compactification of the Hurwitz scheme*, Publications RIMS, Kyoto University, **31** (1995), 355–441. MR **96j:**14017

[NT] S. Natanzon and V. Turaev, *A compactification of the Hurwitz space*, Topology **38**, No. 4, (1999), 889–914. MR **2000b:**57004

Department of Mathematics, Syracuse University, Syracuse, New York 13244
E-mail address: spdiaz@mailbox.syr.edu