ALMOST POSITIVE CURVATURE ON THE GROMOLL-MEYER 7-SPHERE

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Abstract. D. Gromoll and W. Meyer have represented a certain exotic 7-sphere $M$ as a biquotient of the compact Lie group $Sp(2)$. Thus any invariant normal homogeneous metric on $Sp(2)$ induces a metric of nonnegative sectional curvature on $M$. We show that the simplest such metrics (except the bi-invariant one) induce metrics which have in fact strictly positive curvature outside a subset of $M$ with measure zero.

There are only very few compact manifolds known which allow metrics of strictly positive sectional curvature. But recently it has been shown ([PW], [Wk]) that much more spaces satisfy a condition which seems to be only slightly weaker: A Riemannian manifold $M$ is said to have almost positive curvature if it has positive curvature on an open subset $M_0 \subset M$ such that $M \setminus M_0$ is a set of measure zero.

D. Gromoll and W. Meyer [GM] constructed a metric of nonnegative sectional curvature on the exotic 7-sphere $M = G/U$ where $G = Sp(2)$ and $U = \{((q_1,q_2), q); \ q \in Sp(1)\} \subset G \times G$.

In fact, a subgroup $U \subset G \times G$ acts on $G$ by left and right multiplication: $(u_1, u_2).g := u_1 gu_2^{-1}$. If this action is free, the orbit space $G/U$ is a smooth manifold, called a biquotient. Any normally homogeneous metric on $G$ has nonnegative curvature, and if this metric is also $U$-invariant, it induces a metric on the orbit space which has also nonnegative curvature by O'Neill’s formulas for Riemannian submersions. For the bi-invariant metric and many other normal homogeneous metrics on $Sp(2)$, the curvature on $M = Sp(2)/U$ is even strictly positive near the point $U.e$ where $e \in Sp(2)$ is the identity, but this cannot hold on the whole manifold ([E1]). How large is the subset $M_0 \subset M$ where the curvature is strictly positive? It is known ([W]) that for the bi-invariant metric $M \setminus M_0$ contains an open subset, so this metric does not have almost positive curvature in the above sense. However the property does hold for the simplest normally homogeneous metrics on $Sp(2)$ which are not bi-invariant. Using arguments taken from [E1] we will show that $M \setminus M_0$ is essentially a hypersurface. F. Wilhelm [W] has shown almost positivity for another set of metrics on $M$, but his computations are much more involved.

Let $K = Sp(1) \times Sp(1) \subset Sp(2) = G$. Then $G$ is equivariantly diffeomorphic to the homogeneous space $(G \times K)/K$ where $K$ sits diagonally in $G \times K$. A bi-invariant metric on $G \times K$ thus induces a normally homogeneous metric on $G$. Note

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that \( G/K = \mathbb{H}P^1 = S^4 \) is a symmetric space. Such metrics are described in detail in [F2]. They are induced by certain \( Ad(K) \)-invariant inner products on the Lie algebra \( \mathfrak{g} \) and have nonnegative curvature (by O'Neill's formula). Moreover, the 2-planes with curvature zero are those spanned by two orthogonal vectors \( X, Y \in \mathfrak{g} \) with

\[
[X, Y] = [X_t, Y_t] = [X_p, Y_p] = 0
\]

where \( X_t \) and \( X_p \) are the components of \( X \) with respect to the Cartan decomposition \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \). Since \( G/K \) is a rank-one symmetric space, there are no vanishing commutators in \( \mathfrak{p} \); thus we may assume that \( Y \) has no \( \mathfrak{p} \)-component, i.e. \( Y = (y z 0 0) \in \mathfrak{k} \) where \( y, z \) are imaginary quaternions. Let \( X_p = (0 -x 0) \) for some nonzero \( x \in \mathbb{H} \). Then \( [X_p, Y] = 0 \) iff \( zx = xy \) or

\[
z = xyx^{-1}.
\]

The infinitesimal action of the Lie algebra \( \mathfrak{u} \) of \( U \) on \( G \) is given as follows: For any \( g = (a b c d) \in Sp(2) \) we have \( V_g := g^{-1}(u, g) = \{v_g; v \in \mathbb{R}^3\} \) where \( \mathbb{R}^3 \subset \mathbb{H} \) denotes the set of imaginary quaternions (the Lie algebra of \( Sp(1) \)) and where

\[
v_g = Ad(g^*)(\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}) = \begin{pmatrix} \tilde{a}v - v & \tilde{a}v \tilde{b} - v \\ \tilde{b}v a - v & \tilde{b}v b - v \end{pmatrix}.
\]

In order to have zero curvature at the point \( U.g \in G/U \) we need to find perpendicular \( X, Y \perp V_g \) satisfying (1), thus spanning a horizontal zero curvature plane at \( g \), and in fact this condition is also sufficient (cf. [E1], p. 31, and [GM]).

**Theorem.** Let \( g = (a b c d) \in Sp(2) \) with \( a, b \neq 0 \). There exists a zero curvature plane at \( U.g \in G/U \) iff

\[
det(I - Ad(b^{-1}) - Ad(a^{-1})) = 0.
\]

**Proof.** Let \( X, Y \perp V_g \) with (1), spanning a zero curvature plane. Our first claim is that \( X_t \) and \( Y_t \) are linearly dependent. In fact, since \( [X_t, Y_t] = 0 \), we may assume \( X_t = (\tilde{x} 0 0 0) \) and \( Y_t = (0 y 0 0) \) for \( x, y \in \mathbb{R}^3 \). Thus \( \langle v_g, X \rangle = \langle \tilde{a}v - v, x \rangle = \langle v, ax\tilde{a} - x \rangle \) and likewise \( \langle v_g, Y \rangle = \langle v, by\tilde{b} - y \rangle \). This vanishes for all \( v \in \mathbb{R}^3 \) if \( ax\tilde{a} = x \) and \( by\tilde{b} = y \). If both \( x, y \) are nonzero, we have \( |a|^2 = |b|^2 = 1 \) which is impossible since \( |a|^2 + |b|^2 = 1 \) (recall that \( g \) is unitary).

Thus we may assume \( X_t = 0 \) and hence by (2)

\[
X = \begin{pmatrix} 0 & -\tilde{x} \\ x & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} y & 0 \\ 0 & xyx^{-1} \end{pmatrix}.
\]

Now

\[
\langle v_g, X \rangle = 2\langle \tilde{b}v a, x \rangle = 2\langle v, bx\tilde{a} \rangle,
\]

and this vanishes if \( bx\tilde{a} \) is perpendicular to \( \mathbb{R}^3 \subset \mathbb{H} \), hence a real number. Thus if \( a \neq 0 \), we get

\[
bx = ta
\]

for some nonzero \( t \in \mathbb{R} \). Moreover, \( \langle v_g, Y \rangle = \langle v, ay\tilde{a} - y + bx y x^{-1}\tilde{b} - xyx^{-1} \rangle \) vanishes for all \( v \in \mathbb{R}^3 \) iff

\[
ay\tilde{a} - y + bx y x^{-1}\tilde{b} - xyx^{-1} = 0.
\]
By (5) we have \( bxy - \bar{b} = |b|^2 bxy(bx)^{-1} = |b|^2 aya^{-1} \) if also \( b \neq 0 \). Hence
\[
ay + bxy - \bar{b} = |a|^2 aya^{-1} + |b|^2 aya^{-1} = aya^{-1} = Ad(a)y.
\]
Further (5) implies \( Ad(x) = Ad(b^{-1}a) \). Therefore \( \langle v_g, Y \rangle = 0 \) iff
\[
Ad(a)y - Ad(b^{-1})Ad(a)y - y = 0.
\]
Thus \( Ad(a)y \neq 0 \) is in the kernel of \( I - Ad(b^{-1}) - Ad(a^{-1}) \) which implies that the determinant of that matrix vanishes.

Vice versa, if \( \det(I - Ad(b^{-1}) - Ad(a^{-1})) = 0 \), we find a nonzero \( y \in \mathbb{R}^3 \) such that \( Ad(a)y \) is in the kernel of this matrix. Now putting \( x = b^{-1}a \) and defining \( X, Y \) by (4), we obtain a horizontal zero curvature plane at \( g \).

Remarks. 1. We can determine the horizontal zero curvature planes also in the cases \( a = 0 \) or \( b = 0 \), using (6). E.g. if \( b = 0 \), then (6) becomes \( aya^{-1} - y - xyy^{-1} = 0 \) which is solvable precisely for those \( a \) such that \( Ad(a) \) turns some vector \( y \in \mathbb{R}^3 \) by the angle \( \pi/3 \); then \( |Ad(a)y - y| = |y| \), and we find some \( x \in \mathbb{H} \) with \( Ad(x)y = Ad(a)y - y \). Thus a horizontal zero curvature plane at such \( g \) exists if and only if the (minimal) rotation angle of \( Ad(a) \) is \( \geq \pi/3 \).

2. Note that equation (*) for \( g \) in the Theorem is invariant under the action of \( U \) and thus determines a hypersurface (possibly with singularities) in \( G/U \). In fact, if \( u = (q_1, q) \in U \), then
\[
u.g = \left( \begin{array}{cc} qaq^{-1} & qbq^{-1} \\ cq^{-1} & dq^{-1} \end{array} \right).
\]
Thus \( a \) and \( b \) become conjugated by \( q \) which does not change the determinant equation.

References


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