ON THE HOROCYCLIC COORDINATE
FOR THE TEICHMÜLLER SPACE
OF ONCE PUNCTURED TORI

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Dedicated to Professor Hiroki Sato on the occasion of his sixtieth birthday

ABSTRACT. This paper deals with analytic and geometric properties of the Maskit embedding of the Teichmüller space of once punctured tori. We show that the image of this embedding has an inward-pointing cusp and study the boundary behavior of conformal automorphisms. These results are proved using Y.N. Minsky's Pivot Theorem.

Let $T_{1,1}$ be the Teichmüller space of once punctured tori. The horocyclic coordinate for $T_{1,1}$ is a holomorphic embedding of $T_{1,1}$ into the complex plane $\mathbb{C}$. A point of $T_{1,1}$ corresponds, via this coordinate, to a regular b-group representing a once punctured torus and a thrice punctured sphere. We denote by $\mathcal{M}$ the image of $T_{1,1}$ under this embedding. The purpose of this paper is to study analytic and geometric properties of the *Maskit embedding* $\mathcal{M}$.

It is known that $\mathcal{M}$ is simply connected, for $T_{1,1}$ is contractible (cf. Imayoshi and Taniguchi [1] and Nag [14]). Y.N. Minsky in [12] proved that $\mathcal{M}$ is a Jordan domain in the Riemann sphere $\hat{\mathbb{C}}$. Our main purpose here is to prove the following assertion:

Theorem 1. *The Maskit embedding $\mathcal{M}$ of $T_{1,1}$ has an inward-pointing cusp. In particular, $\mathcal{M}$ is not a quasi-disk.*

This is suggested by conjectures in an unpublished paper of D.J. Wright (see §5 of [19]). We prove this result by using Minsky's Pivot Theorem. As a corollary, we obtain a sufficient condition for the existence of finite and nonvanishing *angular derivatives* at appropriate boundary points, for conformal automorphisms of $\mathcal{M}$.

This paper is organized as follows: In §1, we recall the definition of the horocyclic coordinate for $T_{1,1}$, and standardize notation. We review Minsky's Pivot Theorem in §2. In §3, we prove the main theorem which asserts that $\mathcal{M}$ is not a quasi-disk. Section 4 deals with a result related to the opening of cusps for the maximally parabolic groups corresponding to points in $\partial\mathcal{M}$. In §5, we study the boundary behavior of conformal automorphisms on $\mathcal{M}$.

$\mu(\tfrac{0}{1}) = 2i$ $\mu(\tfrac{1}{1}) = 2+2i$

$\mu(\tfrac{1}{2}) = 1+\sqrt{3}i$

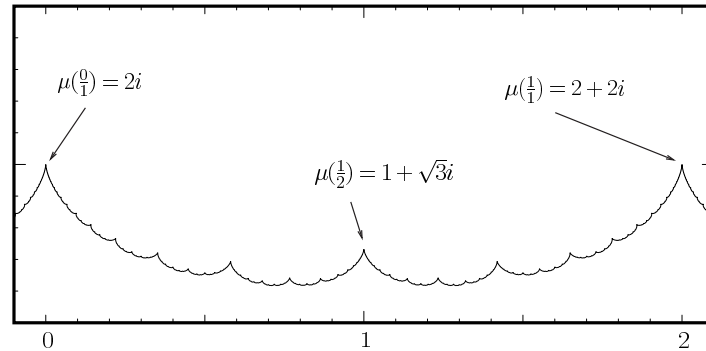0   1   2

FIGURE 1. The Maskit slice (the upper region): *Courtesy of David J. Wright*

Remark 1. After submitting this paper, the author obtained stronger results (cf.
[13]). Indeed, all points in $\partial\mathcal{M}$ corresponding to geometrically finite groups are
actually inward-pointing cusps. Furthermore, such points are not only inward-
pointing cusp-shaped but also sandwiched between two singularities of type $y^2 = ax^3$ for some $a > 0$.

1. PRELIMINARIES

We begin with a review of some standard material discussed more thoroughly in
Keen and Series [2], Kra [6], and Wright [19].

1.1. We recall the definition of the horocyclic coordinate for the Teichmüller space
$T_{1,1}$ of once punctured tori.

For $\mu \in \mathbb{C}$, we define $S, T_\mu \in \mathrm{SL}_2(\mathbb{C})$ by

$$S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad T_\mu = -i \begin{pmatrix} \mu & 1 \\ 1 & 0 \end{pmatrix}.$$

Set $G_\mu = \langle S, T_\mu \rangle$. We shall often identify G_μ with its image in $\mathrm{PSL}_2(\mathbb{C})$. The
domain $\mathcal{M}$ in $\mathbb{C}$ is defined as follows: a point $\mu \in \mathbb{C}$ is contained in $\mathcal{M}$ if and only if
$\mathrm{Im}\,\mu > 0$ and G_μ is a terminal regular b-group of type (1,1). Namely, the following
holds:

1. G_μ is a free group with two generators S and T_μ.
2. The connected components of the region of discontinuity of G_μ are one of two
   kinds:
   (a) A simply connected G_μ-invariant component Ω_μ for which the orbit space
       $\Sigma_\mu = \Omega_\mu/G_\mu$ is topologically equivalent to a once punctured torus.
   (b) Denumerably many non-invariant components $\Omega_{i,\mu}$ ($i = 1, 2, 3, \ldots$), con-
       jugate to one another under G_μ, with orbit spaces $\Omega_{i,\mu}/\mathrm{Stab}(\Omega_{i,\mu})$ con-
       formally equivalent to the thrice punctured sphere.

The shape of the boundary of $\mathcal{M}$ appears in Figure 1. It is known that G_μ is
an HNN-extension of the level 2 principal congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ by an
element T_μ, and that S is an accidental parabolic transformation of G_μ (cf. Kra
[6]).

For $\mu \in \mathcal{M}$, we denote by α_μ and β_μ the oriented simple closed curves on Σ_μ
corresponding to S and T_μ respectively. Then $(\Sigma_\mu, (\alpha_\mu, \beta_\mu))$ determines a point of
the Teichmüller space $T_{1,1}$. It is known that the mapping $\mu \mapsto [\Sigma_\mu, (\alpha_\mu, \beta_\mu)]$ is a

holomorphic bijection. The *horocyclic coordinate* for $T_{1,1}$ is defined as the inverse mapping of this correspondence. We often refer to $\mathcal{M}$ as the Maskit embedding of $T_{1,1}$ (cf. Kra [5] and Maskit [7]).

1.2. We review the canonical identification between the set of homotopy classes of simple closed curves of a once punctured torus and $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{1/0\}$. Henceforth, we will always express rational numbers in the form p/q where p and q are relatively prime integers and $q > 0$. Fix $\mu_0 \in \mathcal{M}$ and let $\Sigma = \Sigma_{\mu_0}$, $\alpha = \alpha_{\mu_0}$, and $\beta = \beta_{\mu_0}$. The fundamental group $\pi_1(\Sigma)$ of Σ with a suitable base point is generated by the equivalence classes of α and β. By abuse of language, the homotopy class of a closed curve γ on Σ is denoted by γ.

The enumeration is given by the inductive procedure as follows:

1. $\gamma(1/0) = \alpha^{-1}$ and $\gamma(n/1) = \alpha^{-n}\beta$ for $n \in \mathbb{Z}$.
2. For $p/q, r/s \in \hat{\mathbb{Q}}$ with $ps - rq = -1$ (which is equivalent to $|ps - rq| = 1$ and $p/q < r/s$), we define

$$\gamma((p+r)/(q+s)) = \gamma(r/s)\gamma(p/q).$$

Then it is known that $\gamma(p/q)$ can be defined for every $p/q \in \hat{\mathbb{Q}}$ and $\gamma(p/q)$ contains a simple closed curve whose homology class is $-p[\alpha] + q[\beta]$, where $[\gamma]$ is the homology class corresponding to $\gamma \in \pi_1(\Sigma)$ (cf. §2.4 of Keen and Series [2]). Notice that the ordered pair $([\alpha], [\beta])$ defines a positively oriented basis of the first homology group $H_1(\Sigma)$; we will need this remark in 3.2 below.

1.3. For $\mu \in \mathbb{C}$, we define the homomorphism χ_μ from $\pi_1(\Sigma)$ to G_μ sending α and β to S and T_μ respectively. Then, it is known from Jørgensen's theorem (cf. [10, Theorem 2.21]) that the homomorphism χ_μ is a faithful and discrete representation for every $\mu \in \overline{\mathcal{M}} \setminus \{\infty\}$. Set $W_{p/q,\mu} = \chi_\mu(\gamma(p/q))$ and $f_{p/q}(\mu) = \text{tr}^2(W_{p/q,\mu}) - 4$. As $\text{tr}^2(W_{p/q,\mu})$ is a polynomial of μ, $f_{p/q}(\mu)$ is holomorphic on $\mathbb{C}$.

By a *maximally parabolic group* we mean a Kleinian group with the largest number of non-conjugate rank 1 maximal parabolic subgroups (cf. Definition 3.2 of [3]). According to Theorem 5.1 of Keen and Series [2] and Theorem III of Keen, Maskit and Series [3], for any $p/q \in \mathbb{Q}$, there exists a unique $\mu(p/q) \in \partial\mathcal{M} \setminus \{\infty\}$ such that $G_{\mu(p/q)}$ is a maximally parabolic group with three rank 1 maximal parabolic groups $\langle S \rangle$, $\langle W_{p/q,\mu(p/q)} \rangle$, and the cyclic group generated by the commutator of S and $T_{\mu(p/q)}$.

Denote by $\dot{f}_{p/q}$ the derivative of $f_{p/q}$ at $\mu(p/q)$. The following table gives $f_{p/q}(\mu)$, $\mu(p/q)$, and $\dot{f}_{p/q}$ for p/q with $0 \leq p/q \leq 1$ and $q \leq 3$ (see also the table in [19]).

p/q	$f_{p/q}(\mu)$	$\mu(p/q)$	$\dot{f}_{p/q}$
0/1	$-\mu^2 - 4$	$2i$	$-4i$
1/3	$-(\mu^3 - 2\mu^2 + 3\mu - 2)^2 - 4$	$0.5812 + 1.6939i$	$-3.6768 - 1.7371i$
1/2	$(\mu^2 - 2\mu + 2)^2 - 4$	$1 + \sqrt{3}i$	$-8\sqrt{3}i$
2/3	$-(\mu^3 - 4\mu^2 + 7\mu - 4)^2 - 4$	$1.4188 + 1.6939i$	$3.6768 - 1.7371i$
1/1	$-(\mu - 2)^2 - 4$	$2 + 2i$	$-4i$

2. MINSKY'S PIVOT THEOREM

In this section, we recall Minsky's Pivot Theorem (see Theorem 4.1 in [12]). This theorem enables us to control the complex translation length of certain loxodromic elements of a marked punctured torus group by the structure of the ends

of its quotient 3-manifold. In §4 of [12], a simple closed curve associated to such a loxodromic element is called a "pivot"; we will not need to use this notion here.

To simplify the argument, we state his theorem in the following special case: Let ρ be a marked punctured torus group, that is, ρ is a faithful and discrete representation from $\pi_1(\Sigma)$ to $\mathrm{PSL}_2(\mathbb{C})$ sending the commutator of α and β to a parabolic element. Suppose that the image of ρ is a terminal regular b-group with accidental parabolic transformation $\rho(\alpha)$—note that χ_μ satisfies this condition for any $\mu \in \mathcal{M}$—and denote by Σ_ρ the associate torus.

Let $A, B \in H_1(\Sigma_\rho)$ be the homology classes of curves corresponding to $\rho(\alpha)$ and $\rho(\beta)$ respectively. As indicated above, we will regard (A, B) as a positively oriented homology basis. Let $(1, \tau)$ be the period matrix of the compactification of Σ_ρ associated to the oriented basis (A, B).

We define the end invariants ν_- and ν_+ for ρ after making a normalization that depends on $p/q \in \hat{\mathbb{Q}}$. For $p/q \in \mathbb{Q}$, take $h \in \mathrm{PSL}_2(\mathbb{Z})$ satisfying $h(p/q) = \infty$. The *end invariants* (ν_-, ν_+) associated to ρ (with respect to $p/q \in \mathbb{Q}$) are defined[1] by $\nu_- = h(\infty)$ and $\nu_+ = h(\tau)$,—note that $\alpha = \gamma(1/0)^{-1}$ and that $\rho(\alpha)$ is an accidental parabolic transformation. Then $\nu_\pm$ are uniquely determined up to integer translations because such $h \in \mathrm{PSL}_2(\mathbb{Z})$ is determined up to composing translations. Therefore the quantity $\nu_+ - \overline{\nu_-}$ is independent of the choice of a transformation h.

Finally, let $\lambda = l + i\theta$ be the complex translation length of $\rho(\gamma(p/q))$ with normalizations $l \geq 0$ and $-\pi < \theta \leq \pi$, that is, $\mathrm{tr}^2(\rho(\gamma(p/q))) = 4\cosh^2(\lambda/2)$. Then we have the following theorem:

Pivot Theorem. *There exist universal constants ϵ_1 and $C_1 > 0$ such that, if $l \leq \epsilon_1$, then*

$$d_{\mathbb{H}}\left(\frac{2\pi i}{\lambda}, \nu_+ - \overline{\nu_-} + i\right) < C_1,$$

where d_X is the Poincaré hyperbolic distance on a domain X.

Remark 2. The assumption $l \leq \epsilon_1$ means that $\gamma(p/q)$ is a pivot of ρ in the sense of Minsky (see §4 of [12]).

3. The main theorem

Henceforth, we fix $p/q \in \mathbb{Q}$. This section deals with the following theorem:

Theorem 2. *If $\dot{f}_{p/q} \neq 0$, then $\mu(p/q)$ is an inward-pointing cusp of $\mathcal{M}$. In particular, $\mathcal{M}$ is not a quasi-disk.*

Here, we say that a boundary point e_0 of a Jordan domain E is called an *inward-pointing cusp* if there exists a disk $B \subset \mathbb{C}$ such that $0 \in \partial B$ and $e_0 + t^2 \in E$ for all $t \in B$. If a domain E has an inward-pointing cusp, then E is not linearly connected. So E is not a quasi-disk (see page 104 of [16] and Figure 2). In view of the example tabulated above, Theorem 2 implies Theorem 1.

We summarize the proof of Theorem 2 as follows: Take the double covering space of $\mathbb{C}$ defined by the projection $\Pi(t) = \mu(p/q) + t^2$. The lift $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ is mapped conformally to $\mathcal{M}$ by its projection. Theorem 2 follows from the existence of an

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[1]In our case, the negative end invariant ν_- is always a rational number (for every normalization by h above). However, general once punctured torus groups have complex valued negative end invariant; e.g., both end invariants of all quasifuchsian groups are in $\mathbb{H}$. For detail, see §3 of Minsky [12].
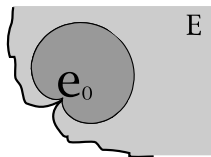
FIGURE 2. An inward-pointing cusp

open ball in $\widetilde{\mathcal{M}}$ tangent at the origin. We prove the existence of such a ball by studying the behavior of the complex translation length of $W_{p/q,\mu}$ using Minsky's Pivot Theorem.

3.1. Recall that $\mathcal{M}$ is a Jordan domain (see §12 of [12]). Set $\Pi(t) = \mu(p/q) + t^2$ as above. The condition $\mu(p/q) \in \partial \mathcal{M}$ guarantees the existence[2] of a simply connected Jordan domain $\widetilde{\mathcal{M}}$ in $\hat{\mathbb{C}}$ with $0 \in \partial \widetilde{\mathcal{M}}$ such that $\Pi|_{\widetilde{\mathcal{M}}}$ is a conformal mapping from $\widetilde{\mathcal{M}}$ onto $\mathcal{M}$.

Denote by $\lambda_{p/q}(t)$, $t \in \widetilde{\mathcal{M}}$, the complex translation length of $W_{p/q,\Pi(t)}$. By applying the identity $\cosh^2 - \sinh^2 = 1$ we obtain

$$(1) \qquad f_{p/q}(\Pi(t)) = 4\sinh^2(\lambda_{p/q}(t)/2), \quad \text{for} \quad t \in \widetilde{\mathcal{M}}.$$

Since $W_{p/q,\mu}$ is loxodromic on $\mathcal{M}$, $\lambda_{p/q}$ can be taken as a holomorphic function satisfying $\mathrm{Re}\lambda_{p/q} > 0$ on $\widetilde{\mathcal{M}}$. Furthermore, since $f_{p/q}(\Pi(0)) = 0$ and the holomorphic function $\sinh^2(z)$ is even, equation (1) implies that $\lambda_{p/q}$ can be extended holomorphically at $t = 0$. We take the branch of $\lambda_{p/q}$ as $\lambda_{p/q}(0) = 0$. We can summarize the result of this subsection in the following:

Lemma 1. *There exist a Jordan domain $\widetilde{\mathcal{M}}$ in $\mathbb{C}$ and a holomorphic function $\lambda_{p/q}$ satisfying the following conditions:*

(a) *$\Pi|_{\widetilde{\mathcal{M}}}$ is a conformal mapping from $\widetilde{\mathcal{M}}$ onto $\mathcal{M}$ and $0 \in \partial\widetilde{\mathcal{M}}$.*
(b) *$\lambda_{p/q}$ is holomorphic on a neighborhood of $\widetilde{\mathcal{M}} \cup \{0\}$ with $\lambda_{p/q}(0) = 0$ and $\mathrm{Re}\lambda_{p/q} > 0$ on $\widetilde{\mathcal{M}}$.*
(c) *$f_{p/q}(\Pi(t)) = 4\sinh^2(\lambda_{p/q}(t)/2)$ for $t \in \widetilde{\mathcal{M}}$.*

3.2. For $\mu \in \mathcal{M}$, let $(\nu_-(\mu), \nu_+(\mu))$ be the end invariants associated to χ_μ with respect to $p/q \in \mathbb{Q}$. As noted as above, the pair $([\alpha_\mu], [\beta_\mu])$ is a positively oriented basis of $H_1(\Sigma_\mu)$. By the definition of end invariants, ν_- is a rational number independent of the choice of $\mu \in \mathcal{M}$. From §4 of Kra [5], ν_+ is a conformal mapping from $\mathcal{M}$ to $\mathbb{H}$. Since $\widetilde{\mathcal{M}}$ is a Jordan domain, ν_+ can be extended on the closure of $\widetilde{\mathcal{M}}$.

Set $\Phi = \nu_+^{-1}$. By definition, $\Phi(\infty) = \mu(p/q)$. Define the simple path σ on $\widetilde{\mathcal{M}}$ by $\sigma(s) = (\Pi|_{\widetilde{\mathcal{M}}})^{-1} \circ \Phi(s + \nu_- + i)$ for $s \in \mathbb{R}$. Then we have the following assertion.

Lemma 2. *The limits of $\sigma(s)$ exist as $s \to \pm\infty$ and are equal to 0.*

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[2] Strictly speaking, the lift $\Pi^{-1}(\mathcal{M})$ consists of two components. Here, we can frequently choose one of the two to get $\widetilde{\mathcal{M}}$, but that choice should be fixed throughout this paper.

3.3. Our aim in this section is to prove the following geometric properties of the image of $\lambda_{p/q}$ using the Pivot Theorem.

Proposition 3. *There exist disjoint open disks B_1 and B_2 such that*
  (i) *$0 \in \partial B_j$ for $j = 1, 2$ and the centers of the disks lie on the real axis,*
 (ii) *the curve $\{\lambda_{p/q}(\sigma(s)) \mid s \in \mathbb{R}\}$ is disjoint from the disks, and*
(iii) *the mapping $s \mapsto \lambda_{p/q}(\sigma(s))$ tends to 0 from both sides of $\mathbb{C} \setminus B_1 \cup B_2$ as $s \to +\infty$ and $-\infty$ (see Figure 3).*
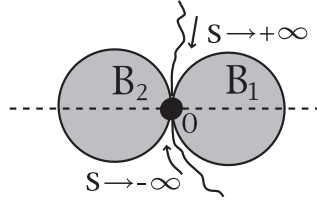

FIGURE 3.

Proof. By applying the condition $\lambda_{p/q}(0) = 0$ and Lemma 2, we find $s_0 > 0$ such that $\operatorname{Re} \lambda_{p/q}(\sigma(s)) < \epsilon_1$ and $|\operatorname{Im} \lambda_{p/q}(\sigma(s))| < \pi$ whenever $|s| > s_0$. By virtue of the Pivot Theorem, we have

$$C_1 > d_{\mathbb{H}}\left(\frac{2\pi i}{\lambda_{p/q}(\sigma(s))}, \nu_+(\Pi(\sigma(s))) - \overline{\nu_-(\Pi(\sigma(s)))} + i \right)$$

$$= d_{\mathbb{H}}\left(\frac{2\pi i}{\lambda_{p/q}(\sigma(s))}, s + 2i \right),$$

whenever $|s| > s_0$. This implies that, for $|s| > s_0$,

(2) $0 < \operatorname{Re} \lambda_{p/q}(\sigma(s)) < C'|\lambda_{p/q}(\sigma(s))|^2, |2\pi \operatorname{Im}\lambda_{p/q}(\sigma(s))/|\lambda_{p/q}(\sigma(s))|^2 - s| < C'$

for some $C' > 0$. By (c) of Lemma 1, there exists $C'' > 0$ such that $|\lambda_{p/q}(\sigma(s))| > C''$ if $|s| \le s_0$. Set $\delta = \min\{1/4C', C''/3\}$ and $B_j = \{|w - (-1)^j \delta| < \delta\}$, $j = 1, 2$. Then, it follows from (2) that the disks $\{B_j\}_{j=1}^2$ have the desired properties. $\square$

3.4. We now prove Theorem 2.

Proof. It suffices to show the following assertion: Suppose that $\dot{f}_{p/q} \ne 0$. Then there exist disjoint disks D_1 and D_2 satisfying the following conditions:
  (I) $0 \in \partial D_j$ for $j = 1, 2$.
 (II) $D_1 \subset \widetilde{\mathcal{M}}$ and $D_2 \cap \widetilde{\mathcal{M}} = \emptyset$.
Indeed, if this assertion is true, then

$$\Pi(D_1) = \{\mu(p/q) + t^2 \mid t \in D_1\} \subset \mathcal{M}.$$

This implies Theorem 2.

Let us show the assertion above. By (1), we have

(3) $(d\lambda_{p/q}/dt|_{t=0})^2 = \dot{f}_{p/q}.$

Indeed, since $f_{p/q}(\mu(p/q)) = 0$ and $\Pi(0) = \mu(p/q)$, the left-hand side of the equation in (c) becomes $f_{p/q} \circ \Pi(t) = \dot{f}_{p/q} t^2 + o(t^2)$ near $t = 0$, while the right-hand side of

it forms $(d\lambda_{p/q}/dt|_{t=0})^2 t^2 + o(t^2)$. By assumption, there exists a neighborhood U of $t = 0$ in which $\lambda_{p/q}$ is univalent.

Let $\{B_1\}_{j=1}^2$ be disks satisfying the conditions in Proposition 3. We may assume that $B_j \subset \lambda_{p/q}(U)$ and B_2 is contained in the left half-plane. By applying (i) of Proposition 3 and the equation (3), we conclude that each $(\lambda_{p/q}\mid_U)^{-1}(\partial B_j)$ is a smooth path through $t = 0$ whose normal vector at the origin is parallel to the vector $(\dot{f}_{p/q})^{1/2}$ as real vectors. Take disks D_j contained in $(\lambda_{p/q}\mid_U)^{-1}(B_j)$ and tangent to $(\lambda_{p/q}\mid_U)^{-1}(\partial B_j)$. It is easy to see that the disks $\{D_j\}_{j=1}^2$ satisfy the conditions (I) above.

We shall show that the disks satisfy the condition (II). Lemma 2 asserts that the curve $\xi = \{\sigma(s) \mid s \in \mathbb{R}\} \cup \{0\}$ is a Jordan curve contained in $\widetilde{\mathcal{M}} \cup \{0\}$. Since $\widetilde{\mathcal{M}}$ is simply connected, the curve ξ bounds the simply connected domain E included in $\widetilde{\mathcal{M}}$. By virtue of (i) and (ii) of Proposition 3, we deduce that $D_j \cap \xi = \emptyset$ for $j = 1, 2$ and $D_1 \cap D_2 = \emptyset$. Since $\lambda_{p/q}(D_2) \subset B_2$, the real part of $\lambda_{p/q}$ is negative on B_2. It follows from (c) of Lemma 1 that D_2 is disjoint from the closure of $\widetilde{\mathcal{M}}$. This implies $D_1 \subset E \subset \widetilde{\mathcal{M}}$. $\qquad\square$

Remark 3. From the proof above, the centers of each disk D_j lie on the line parameterized by $t = l(\overline{\dot{f}_{p/q}})^{1/2}$, $l \in \mathbb{R}$. This remark is needed for proving Theorem 3 later.

4. Opening up cusps

Fix $p/q \in \mathbb{Q}$, and assume that $\dot{f}_{p/q} \neq 0$. We set $v_{p/q} = -\overline{\dot{f}_{p/q}}/|\dot{f}_{p/q}|$. For example, $v_{n/1} = -i$ with $n \in \mathbb{Z}$. Recall that $G_{\mu(p/q)}$ is a maximally parabolic group with accidental parabolic transformations $W_{p/q,\mu}$ and S (see §1.3). Then, we observe the following phenomenon (cf. Figure 4).

Theorem 3. *Let $p/q \in \mathbb{Q}$ as above. For $v \neq v_{p/q}$ on the unit circle, there exists $d_0 = d_0(p/q, v) > 0$ such that $\mu(p/q) + dv \in \mathcal{M}$ whenever $0 < d < d_0$. In particular, $G_{\mu(p/q)+dv}$ is a terminal regular b-group with accidental parabolic transformation S.*

Proof. By Remark 3 above, the region $\widetilde{\mathcal{M}}$ contains a disk

$$D_1 := \{t \in \mathbb{C} \mid |t - (\overline{\dot{f}_{p/q}})^{1/2}\delta'| < |\dot{f}_{p/q}|^{1/2}\delta'\}$$

for some $\delta' > 0$[3]. Set $\delta = \delta'/|\dot{f}_{p/q}|^{1/2}$. Then

$$\mathcal{M} \supset \Pi(D_1) = \{\mu(p/q) + \overline{\dot{f}_{p/q}}t^2 \mid |t - \delta| < \delta\}.$$

This completes the proof of Theorem 3. $\qquad\square$

Remark 4. The author made the limit sets in Figure 4 by using Professor Masaaki Wada's program "OPTI3.0" [18].

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[3]We re-choose the branch of $(\dot{f}_{p/q})^{1/2}$ as satisfying this condition, if necessary.
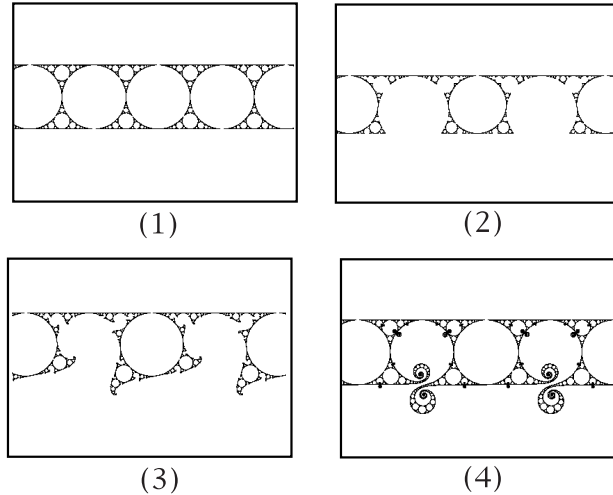
FIGURE 4. Opening up a cusp: Figures are the limit sets of G_μ corresponding to (1) $\mu = 2i$, (2) $\mu \approx 2.22222222\,i$, (3) $\mu \approx 0.2 + 2i$, and (4) $\mu \approx 0.00768824 + 1.97746809\,i$, respectively.

5. BOUNDARY BEHAVIOR OF AUTOMORPHISMS OF $\mathcal{M}$

In this section, we study the boundary behavior of conformal automorphisms of $\mathcal{M}$. Before proving the result of this section, we give the notation of the angular derivative and recall a proposition on the angular derivative of conformal mappings at boundary.

Let D be a domain in $\mathbb{C}$ and f a holomorphic function on D. We say that f has *angular derivative A* at $z_0 \in \partial D$ if D contains a triangle (called a *Stolz domain*) with vertex at z_0 and if when z tends to z_0 from inside of any fixed Stolz domain with vertex z_0, the quotient $(f(z) - f(z_0))/(z - z_0)$ tends to A.

Concerning with existence of angular derivatives, the following is classical[4] (cf. Theorem IX.9 in Tsuji [17]):

Proposition 4. *Let M be a Jordan domain in w-plane such that $0 \in \partial M$. Suppose that there exist a neighborhood U of the origin and two disjoint open disks B_i $(i = 1, 2)$ tangent at 0 such that $\overline{B_1} - \{0\} \subset M$ and $\partial M \cap (B_1 \cup B_2) = \emptyset$. Then the conformal mapping w from $\mathbb{H}$ to M with $w(0) = 0$ has a non-vanishing angular derivative at the boundary point 0.*

We show the following lemma.

Lemma 5. *Let M be a Jordan domain such that $\{|w - ir| < r\} \subset M$ and $\{|w + ir| \leq r\} \cap M = \emptyset$ for some $r > 0$, and let h be a conformal mapping from M to $\mathbb{H}$ with $h(0) = 0$. Set $H(\alpha) = \{w \in \mathbb{C} \mid |\arg(w/i)| < \alpha, |w| < r\cos\alpha\}$. Then for $0 < \alpha < \pi/2$, the image $h(H(\alpha))$ is contained in some Stolz domain in $\mathbb{H}$ whose vertex is at $z = 0$.*

Proof. Let $g = h^{-1}$. By Proposition 4, g has an angular derivative $A \neq 0$ at the origin. Thus, $|g(iy) - iAy| < A(\sin\alpha)y$ for all $0 < y < \delta$, where $\delta > 0$ is sufficiently

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[4]Actually, a more general result is stated in [17].

small. In particular, the mapping g satisfies $g(iy) \in H(\alpha)$ for $0 < y < \delta$. Since $\{|w - ir| < r\} \subset M$, there exists $C > 0$ such that for any $w \in H(\alpha)$, the Poincaré distance between w and the image of the imaginary axis under g is less than C. Thus, $h(H(\alpha))$ is contained in a Stolz domain $\bigcup_{y>0}\{z \in \mathbb{H} \mid d_{\mathbb{H}}(iy, z) < C\}$. $\square$

Here, we note that any automorphism of a Jordan domain can be extended homeomorphically on its closure from Carathéodory's theorem.

Proposition 6. *Let M be a Jordan domain in $\mathbb{C}$ with inward-pointing cusps z_1, z_2 $\in \partial M$, and let $\tau : M \to M$ be a conformal automorphism with $\tau(z_1) = z_2$. Then τ has a nonvanishing angular derivative at z_1.*

Proof. By definition, there exist $\alpha_i \in \mathbb{C} - \{0\}$ such that the image of the mappings $g_i(t) = z_i + \alpha_i t^2$ of $B = \{w \mid |w - i| < 1\}$ are contained in M. Let $\tilde{M}_i$ $(i = 1, 2)$ be the component $g_i^{-1}(M)$ containing B. Since M is simply connected and $g_i(-t) = g_i(t)$, g_i is conformal on $\tilde{M}_i$ and $\tilde{M}_i \cap (-B) = \emptyset$ where $-B = \{w \mid -w \in B\}$.

Let

$$\tilde{\tau} = g_2^{-1} \circ \tau \circ g_1.$$

Take h_i conformal mappings from $\mathbb{H}$ to $\tilde{M}_i$ so that $h_i(0) = 0$ and $h_2 \circ h_1^{-1} = \tilde{\tau}$. By virtue of Proposition 4, each h_i has the form

(4) $$h_i(w) = A_i w + O(|w|^2)$$

for some $0 < A_i < \infty$ within any Stolz domain in $\mathbb{H}$ with vertex at the origin.

Let $\mathcal{S}_1$ be a Stolz domain in M whose vertex is at $z = z_1$. Then by the definition of g_1, $g_1^{-1}(\mathcal{S}_1)$ contains a Stolz domain $\tilde{\mathcal{S}}_1$ in $\tilde{M}_1$ with vertex at $t = 0$ so that $g_1(\tilde{\mathcal{S}}_1) \cap U = \mathcal{S}_1 \cap U$ for a sufficiently small neighborhood U of z_1 (indeed, the angle of the vertex is the half of that of $\mathcal{S}_1$). From Lemma 5, we can suppose that $\tilde{\mathcal{S}}_1$ is contained in the image of some Stolz region with vertex at $w = 0$ in $\mathbb{H}$ under h_1. Hence by (4) if t tends to 0 within $\tilde{\mathcal{S}}_1$, $\tilde{\tau}$ has a nonvanishing angular derivative at $t = 0$:

$$\tilde{\tau}(t) = h_2 \circ h_1^{-1}(t) = (A_2/A_1)t + O(|t|).$$

Therefore $\tau = g_2 \circ \tilde{\tau} \circ g_1^{-1}$ also has a nonvanishing angular derivative at $z = 0$ as

$$\tau(z) = z_2 + (A_2/A_1)^2(\alpha_2/\alpha_1)(z - z_1) + O(|z - z_1|),$$

when $z \to z_1$ within $\mathcal{S}_1 = g_1(\tilde{\mathcal{S}}_1)$. $\square$

From Theorem 2 and the proofs of Theorem 3 and Proposition 6, we have

Corollary 1. *For p/q and $r/s \in \mathbb{Q}$, assume that $\dot{f}_{p/q}, \dot{f}_{r/s} \neq 0$. Then every automorphism τ of $\mathcal{M}$ with $\tau(\mu(p/q)) = \mu(r/s)$ has a nonvanishing angular derivative at $\mu = \mu(p/q)$. Further this angular derivative is a positive multiple of the quotient $\dot{f}_{r/s}/\dot{f}_{p/q}$.*

Concluding remarks. 1. Any Teichmüller modular transformation acting on $\mathcal{M}$ preserves the set $\{\mu(p/q)\}_{p/q \in \mathbb{Q}} \cup \{\infty\}$. Hence if $\dot{f}_{p/q} \neq 0$ for all p/q, every Teichmüller transformation acting on $\mathcal{M}$ has a nonvanishing angular derivative at all $\mu(p/q)$ except at most one $\mu(r/s)$, which corresponds to ∞.

2. At Shizuoka conference (Japan, January, 2000, organized by Professor H. Sato), Professor Irwin Kra asked the author "Can 'a nonvanishing angular derivative' be replaced by 'a derivative' in Corollary 1?". However the author does not know the answer.

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