

## PERIODS OF MIRRORS AND MULTIPLE ZETA VALUES

MICHAEL E. HOFFMAN

(Communicated by Michael Stillman)

ABSTRACT. In a recent paper, A. Libgober showed that the multiplicative sequence  $\{Q_i(c_1, \dots, c_i)\}$  of Chern classes corresponding to the power series  $Q(z) = \Gamma(1+z)^{-1}$  appears in a relation between the Chern classes of certain Calabi-Yau manifolds and the periods of their mirrors. We show that the polynomials  $Q_i$  can be expressed in terms of multiple zeta values.

### 1. THE MULTIPLICATIVE SEQUENCE

In [6], the Hirzebruch multiplicative sequence  $\{Q_i\}$  associated to the power series  $Q(z) = \Gamma(1+z)^{-1}$  is considered in connection with mirror symmetry. If  $e_i$  denotes the  $i$ th elementary symmetric function in the variables  $t_1, t_2, \dots$ , then

$$(1) \quad \sum_{i=0}^{\infty} Q_i(e_1, \dots, e_i) = \prod_{i=1}^{\infty} \frac{1}{\Gamma(1+t_i)}.$$

As shown in [6], the polynomials  $Q_i(c_1, \dots, c_i)$  in the Chern classes of certain Calabi-Yau manifolds  $X$  are related to the coefficients of the generalized hypergeometric series expansion of the period (holomorphic at a maximum degeneracy point) of a mirror of  $X$ . In particular, if  $X$  is a Calabi-Yau hypersurface of dimension 4 in a nonsingular toric Fano manifold, then

$$\int_X Q_4(c_1, c_2, c_3, c_4) = \frac{1}{24} \sum_{ijkl} K_{ijkl} \frac{\partial^4 c(0, \dots, 0)}{\partial \rho_i \partial \rho_j \partial \rho_k \partial \rho_l},$$

where the  $c(\rho_1, \dots, \rho_r)$  are coefficients in the expansion of the period and  $K_{ijkl}$  is the (suitably normalized) 4-point function corresponding to a mirror of  $X$ . In [6] it is shown that the polynomials  $Q_i$  have the form

$$Q_1(c_1) = \gamma c_1 \quad \text{and} \quad Q_i(c_1, \dots, c_i) = \zeta(i)c_i + \dots, \quad i > 1.$$

In this note we show that the polynomials  $Q_i$  have an explicit expression in terms of multiple zeta values (called multiple harmonic series in [3, 4]), which have previously appeared in connection with Kontsevich's invariant in knot theory [8, 5], and in quantum field theory [1].

---

Received by the editors November 23, 1999 and, in revised form, October 18, 2000.  
2000 *Mathematics Subject Classification*. Primary 14J32, 11M41; Secondary 05E05.  
*Key words and phrases*. Mirror symmetry, multiple zeta values, gamma function.

2. SYMMETRIC AND QUASI-SYMMETRIC FUNCTIONS

For the convenience of the reader, we collect in this section the definitions about symmetric and quasi-symmetric functions we will need. Let  $t_1, t_2, \dots$  be a countable sequence of indeterminates, each having degree 1, and let  $P \subset \mathbf{Q}[[t_1, t_2, \dots]]$  be the set of formal power series in the  $t_i$  having bounded degree. Then  $P$  is a graded  $\mathbf{Q}$ -algebra. Let  $f$  be an element of  $P$ . Then  $f$  is a symmetric function if

$$(2) \quad \text{coefficient of } t_{n_1}^{i_1} t_{n_2}^{i_2} \cdots t_{n_k}^{i_k} \text{ in } f = \text{coefficient of } t_{m_1}^{i_1} t_{m_2}^{i_2} \cdots t_{m_k}^{i_k} \text{ in } f$$

for any pair  $(n_1, n_2, \dots, n_k), (m_1, m_2, \dots, m_k)$  of  $k$ -tuples of distinct positive integers. Following [2], we say  $f$  is a quasi-symmetric function if equation (2) holds whenever  $n_1 < n_2 < \dots < n_k$  and  $m_1 < m_2 < \dots < m_k$ . The sets  $\text{Sym}$  and  $\text{QSym}$  of symmetric and quasi-symmetric functions respectively are both subalgebras of  $P$ , with  $\text{Sym} \subset \text{QSym}$ .

For a composition (ordered sequence of positive integers)  $I = (i_1, \dots, i_k)$ , the corresponding monomial quasi-symmetric function  $M_I \in \text{QSym}$  is defined by

$$M_I = \sum_{n_1 < n_2 < \dots < n_k} t_{n_1}^{i_1} t_{n_2}^{i_2} \cdots t_{n_k}^{i_k}.$$

Evidently  $\{M_I | I \text{ is a composition}\}$  is a basis for  $\text{QSym}$  as a vector space. For any composition  $I$ , there is a partition  $\pi(I)$  given by forgetting the ordering. For any partition  $\lambda$ , the monomial symmetric function  $m_\lambda$  is the sum of all the  $M_I$  over all distinct  $I$  such that  $\pi(I) = \lambda$ , e.g.,  $m_{21} = M_{(2,1)} + M_{(1,2)}$ . The monomial symmetric functions are a vector space basis for  $\text{Sym}$ . The elementary symmetric function  $e_k$  is  $M_{I_k} = m_{\pi(I_k)}$ , where  $I_k$  is the composition consisting of  $k$  1's, and the power-sum symmetric function  $p_k$  is  $M_{(k)} = m_{(k)}$ . If for a partition  $\lambda = \pi(\lambda_1, \lambda_2, \dots)$  we let  $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$  and  $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$ , then it is well known that  $\{e_\lambda | \lambda \text{ is a partition}\}$  and  $\{p_\lambda | \lambda \text{ is a partition}\}$  are vector space bases for  $\text{Sym}$ .

3. THE FORMULA FOR THE  $Q_i$

In [4] it is shown (Theorem 5.1) that the homomorphism  $\zeta : \text{Sym} \rightarrow \mathbf{R}$  such that  $\zeta(p_1) = \gamma$  and  $\zeta(p_i) = \zeta(i)$  for  $i > 2$  satisfies

$$(3) \quad \zeta \left( \sum_{i \geq 0} e_i z^i \right) = \frac{1}{\Gamma(1+z)}.$$

Our main result expresses the polynomials  $Q_i$  in terms of  $\zeta$ .

**Theorem.** *For any partition  $\lambda$  of  $i$ , the coefficient of  $e_\lambda$  in  $Q_i(e_1, \dots, e_i)$  is  $\zeta(m_\lambda)$ .*

*Proof.* Using equations (1) and (3), we have

$$\sum_{i \geq 0} Q_i(e_1, e_2, \dots) = \prod_{i=1}^{\infty} \frac{1}{\Gamma(1+t_i)} = \prod_{i=1}^{\infty} \sum_{j=0}^{\infty} \zeta(e_j) t_i^j = \sum_{\lambda} \zeta(e_\lambda) m_\lambda.$$

Now the transition matrix  $M$  from the basis  $\{e_\lambda\}$  of  $\text{Sym}$  to the basis  $\{m_\lambda\}$ , i.e.

$$e_\lambda = \sum_{\mu} M_{\lambda\mu} m_\mu,$$

is known to be symmetric (see Ch. I, §6 of [7]), so we have

$$\sum_{\lambda} \zeta(e_{\lambda})m_{\lambda} = \sum_{\lambda} \sum_{\mu} M_{\lambda\mu} \zeta(m_{\mu})m_{\lambda} = \sum_{\mu} \zeta(m_{\mu}) \sum_{\lambda} M_{\mu\lambda} m_{\lambda} = \sum_{\mu} \zeta(m_{\mu})e_{\mu},$$

and the result follows.

4. MULTIPLE ZETA VALUES

The homomorphism  $\zeta : \text{Sym} \rightarrow \mathbf{R}$  above is the restriction to  $\text{Sym}$  of a homomorphism defined in [4] from  $\text{QSym}$  to  $\mathbf{R}$ . The definition in [4] is motivated by the multiple zeta values introduced in [3] and [8], i.e.

$$(4) \quad \zeta(i_1, i_2, \dots, i_k) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \dots n_k^{i_k}},$$

where  $i_1, i_2, \dots, i_k$  are positive integers with  $i_1 > 1$ . To explain the connection, let  $\mathfrak{H}^1$  be the rational vector space of polynomials in the noncommuting variables  $z_1, z_2, \dots$ ; then  $\mathfrak{H}^1$  becomes isomorphic to  $\text{QSym}$  if it is given the (commutative) multiplication  $*$  defined by the inductive rule

$$z_i w_1 * z_j w_2 = z_i (w_1 * z_j w_2) + z_j (z_i w_1 * w_2) + z_{i+j} (w_1 * w_2)$$

for any words  $w_1, w_2$  in the  $z_i$ . (In fact the isomorphism is simply the map that sends  $z_{i_1} \dots z_{i_k} \in \mathfrak{H}^1$  to  $M_{(i_k, \dots, i_1)} \in \text{QSym}$ : see [4] for details.) The algebra  $\text{Sym}$  of symmetric functions can be identified with the subspace of  $\mathfrak{H}^1$  generated by linear combinations of monomials invariant under permutation of subscripts, e.g.,  $z_2^2 = m_{22}$  and  $z_1 z_2 + z_2 z_1 = m_{21}$ ;  $z_i$  and  $z_1^i$  correspond to  $p_i$  and  $e_i$  respectively. As an algebra  $\mathfrak{H}^1$  is generated by Lyndon words in the  $z_i$ , i.e., monomials  $w$  such that for any nontrivial decomposition  $w = uv$  one has  $v > w$ , where the  $z_i$  are ordered as  $z_1 > z_2 > \dots$  and this order is extended to monomials lexicographically. Then the only Lyndon word that starts with  $z_1$  is  $z_1$  itself, and we can define a homomorphism  $\zeta : \mathfrak{H}^1 \rightarrow \mathbf{R}$  by giving its value on Lyndon words  $w = z_{i_1} z_{i_2} \dots z_{i_k}$ :

$$\zeta(w) = \begin{cases} \gamma, & w = z_1, \\ \zeta(i_1, i_2, \dots, i_k), & \text{otherwise.} \end{cases}$$

By the results of [4],  $\zeta(z_{i_1} z_{i_2} \dots z_{i_k})$  coincides with  $\zeta(i_1, i_2, \dots, i_k)$  as defined by equation (4) whenever  $i_1 > 1$ .

Since the power-sum symmetric functions  $p_i$  generate the algebra  $\text{Sym}$ , we can compute  $\zeta(m_{\lambda})$  by first expressing  $m_{\lambda}$  in terms of power-sum functions (see [7], p. 109 for an explicit formula) and then applying the homomorphism  $\zeta$ . Hence the coefficient of each monomial  $c_{\lambda}$  in  $Q_i(c_1, \dots, c_i)$  is a polynomial in the numbers  $\gamma$  and  $\zeta(i)$ ,  $i \geq 2$ . For example, since  $m_3 = p_3$ ,  $m_{21} = p_1 p_2 - p_3$  and  $m_{111} = \frac{1}{6}(p_1^3 - 3p_1 p_2 + 2p_3)$ , we have

$$Q_3(c_1, c_2, c_3) = \zeta(3)c_3 + (\gamma\zeta(2) - \zeta(3))c_1 c_2 + \frac{1}{6}(\gamma^3 - 3\gamma\zeta(2) + 2\zeta(3))c_1^3,$$

which corrects equation (1.5) of [6]. Similarly we can obtain equations (1.3), (1.4) (the coefficient of  $c_1^2$  should be  $\frac{1}{2}(\gamma^2 - \zeta(2))$ ), and (1.6) of [6].

If  $m_{\lambda}$  is a monomial symmetric function such that the partition  $\lambda$  involves no 1's, there is an alternative method of computing  $\zeta(m_{\lambda})$ : in this case  $\zeta(m_{\lambda})$  is just a symmetric sum of ordinary multiple zeta values, e.g.  $\zeta(m_{32}) = \zeta(2, 3) + \zeta(3, 2)$ ,

and such a symmetric sum can be written as a sum of products of the numbers  $\zeta(i)$  by Theorem 2.2 of [3]; one can use Euler's formula for  $\zeta(2n)$  to simplify further. Since  $c_1 = 0$  on a Calabi-Yau manifold, only terms  $\zeta(m_\lambda)c_\lambda$  with  $\lambda$  having no 1's appear in  $Q_i$ , and we have in this case

$$Q_2(c_2) = \zeta(2)c_2,$$

$$Q_3(c_2, c_3) = \zeta(3)c_3,$$

$$Q_4(c_2, c_3, c_4) = \zeta(4)c_4 + \zeta(2, 2)c_2^2 = \zeta(4)c_4 + \frac{3}{4}\zeta(4)c_2^2,$$

$$Q_5(c_2, c_3, c_4, c_5) = \zeta(5)c_5 + (\zeta(3, 2) + \zeta(2, 3))c_2c_3 = \zeta(5)c_5 + (\zeta(2)\zeta(3) - \zeta(5))c_2c_3,$$

and

$$\begin{aligned} Q_6(c_2, c_3, c_4, c_5, c_6) &= \zeta(6)c_6 + (\zeta(4, 2) + \zeta(2, 4))c_2c_4 + \zeta(2, 2, 2)c_3^3 + \zeta(3, 3)c_3^2 \\ &= \zeta(6)c_6 + \frac{3}{4}\zeta(6)c_2c_4 + \frac{3}{16}\zeta(6)c_2^3 + \frac{1}{2}(\zeta(3)^2 - \zeta(6))c_3^2. \end{aligned}$$

#### REFERENCES

1. D. J. Broadhurst and K. Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, Phys. Lett. B **393** (1997), 403–412. MR **98g**:11101
2. I. M. Gessel, *Multipartite P-partitions and inner products of skew Schur functions*, Combinatorics and Algebra, Contemp. Math. vol. 34, Amer. Math. Soc., Providence, RI, 1984, pp. 289–301. MR **86k**:05007
3. M. E. Hoffman, *Multiple harmonic series*, Pacific J. Math. **152** (1992), 275–290. MR **92i**:11089
4. M. E. Hoffman, *The algebra of multiple harmonic series*, J. Algebra **194** (1997), 477–495. MR **99e**:11119
5. T. Q. T. Le and J. Murakami, *Kontsevich's integral for the Homfly polynomial and relations between values of the multiple zeta functions*, Topology Appl. **62** (1995), 193–206. MR **96c**:57017
6. A. Libgober, *Chern classes and the periods of mirrors*, Math. Res. Lett. **6** (1999), 141–149. MR **2000h**:32017
7. I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford University Press, New York, 1995. MR **96h**:05207
8. D. Zagier, *Values of zeta functions and their applications*, First European Congress of Mathematics, vol. 2, Birkhauser, Boston, 1994, pp. 497–512. MR **96k**:11110

UNITED STATES NAVAL ACADEMY, ANNAPOLIS, MARYLAND 21402  
*E-mail address*: meh@usna.edu