AN EXPRESSION OF SPECTRAL RADIUS
VIA ALUTHGE TRANSFORMATION

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(Communicated by Joseph A. Ball)

Abstract. For an operator \( T \in B(H) \), the Aluthge transformation of \( T \) is defined by \( \tilde{T} = |T|^\frac{1}{2} U |T|^\frac{1}{2} \). And also for a natural number \( n \), the \( n \)-th Aluthge transformation of \( T \) is defined by \( \tilde{T}_n = (\tilde{T}_{n-1}) \) and \( \tilde{T}_1 = \tilde{T} \). In this paper, we shall show
\[
\lim_{n \to \infty} ||\tilde{T}_n|| = r(T),
\]
where \( r(T) \) is the spectral radius.

1. Introduction

As a characterization of the spectral radius, it is well known that \( \lim_{n \to \infty} ||T^n||^{\frac{1}{n}} = r(T) \). This result is very famous and quite useful. On the other hand, Aluthge [1] defined a transformation \( \tilde{T} \) of \( T \) by \( \tilde{T} = |T|^\frac{1}{2} U |T|^\frac{1}{2} \), where \( T = U |T| \) is the polar decomposition of \( T \). \( \tilde{T} \) is called the Aluthge transformation of \( T \). Many researchers have obtained their results by using Aluthge transformation, for example, [1], [2], [3], [4], [6], [7], [8]. It is easily obtained that \( ||T|| \geq ||\tilde{T}|| \geq r(T) = r(T) \).

Recently [9], as a generalization of Aluthge transformation, for each natural number \( n \), we defined a transformation \( \tilde{T}_n \) of \( T \) by
\[
\tilde{T}_n = (\tilde{T}_{n-1}) \quad \text{and} \quad \tilde{T}_1 = \tilde{T}.
\]
We call \( \tilde{T}_n \) the \( n \)-th Aluthge transformation of \( T \).

In this paper, we shall show another characterization of the spectral radius by using \( n \)-th Aluthge transformation as follows:

**Theorem 1.** Let \( T \in B(H) \). Then \( \lim_{n \to \infty} ||\tilde{T}_n|| = r(T) \).

2. Proof

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space \( H \). An operator \( T \) is said to be positive (denoted by \( T \geq 0 \)) if \( (Tx,x) \geq 0 \) for all \( x \in H \). To prove Theorem 1 we prepare the following results.

Received by the editors October 27, 2000.
2000 Mathematics Subject Classification. Primary 47A13, 47A30.
Key words and phrases. Aluthge transformation, Heinz inequality, spectral radius.

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**Theorem A** \((\star)\). Let \(A\) and \(B\) be positive operators, and \(X \in B(H)\). Then
\[
\|A^{\alpha}XB^{\alpha}\| \leq \|AXB\|^{\alpha}\|X\|^{1-\alpha}
\]
holds for all \(\alpha \in [0,1]\).

**Lemma 2.** For a natural number \(n\) and \(k = 0, 1, \ldots, n+1\), let
\[
(2.1)\quad nD_k = \frac{n!(n-2k+1)}{k!(n-k+1)!}
\]
Then the following assertions hold:

(i) \(nD_0 = 1\) for all natural numbers \(n\).

(ii) \(nD_k + nD_{k+1} = n+1D_{k+1}\) for all natural numbers \(n\) and \(k = 0, 1, \ldots, n\).

(iii) \(2n+1D_n = 2n+2D_{n+1}\) for all natural numbers \(n\).

(iv) \(\sum_{k=0}^\left\lfloor \frac{n}{2} \right\rfloor (n-2k+1) nD_k = 2^n\),
where \(\left\lfloor \frac{n}{2} \right\rfloor\) is the largest integer satisfying \(\frac{n}{2} \leq \frac{n}{2}\).

(v) \(\lim_{n \to \infty} \frac{(n-2k+1)nD_k}{2^n} = 0\) for all positive integers \(k\).

**Proof.** (i). By \((2.1)\), we have
\[
nD_0 = \frac{n!(n+1)}{0!(n+1)!} = 1.
\]

(ii). By \((2.1)\), we obtain
\[
nD_k + nD_{k+1} = \frac{n!(n-2k+1)}{k!(n-k+1)!} + \frac{n!(n-2k-1)}{(k+1)!(n-k)!}
= \frac{n!(n+1)(n-2k+1) + (n-k+1)(n-2k-1)}{(k+1)!(n-k+1)!}
= \frac{n!(n+1)(n-2k)}{(k+1)!(n-k+1)!}
= \frac{(n+1)(n-2k)}{(k+1)!(n-k+1)!} = n+1D_{k+1}.
\]

(iii). By (ii) and \(2n+1D_{n+1} = 0\), we have
\[
2n+2D_{n+1} = 2n+1D_n + 2n+1D_{n+1} = 2n+1D_n.
\]

(iv). We shall prove (iv) by induction on \(n\).

(a) The case \(n = 1\). By \((2.1)\), we obtain
\[
\sum_{k=0}^{\left\lfloor \frac{1}{2} \right\rfloor} (1-2k+1)D_k = 2_1D_0 = 2.
\]

(b) Assume that
\[
(2.2)\quad \sum_{k=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} (n-2k)nD_k = 2^{n-1}.
\]
(c-1) The case \( n = 2m + 1 \) for \( m = 1, 2, \cdots \). Then \( \left[ \frac{n}{2} \right] = \left[ \frac{2m+1}{2} \right] = m \). Hence we obtain

\[
\sum_{k=0}^{m} (n - 2k + 1) n D_k \\
= (n + 1) n D_0 + \sum_{k=1}^{m} (n - 2k + 1) n D_k \\
= (n + 1) n D_0 + \sum_{k=1}^{m} (n - 2k + 1)(n D_{k-1} + n D_k) \text{ by (i) and (ii)} \\
= (n + 1) n D_0 + \sum_{k=1}^{m} (n - 2k + 1)n D_{k-1} + \sum_{k=1}^{m} (n - 2k + 1)n D_k \\
= \sum_{k=0}^{m-1} (n - 2k - 1) n D_k + \sum_{k=1}^{m} (n - 2k + 1)n D_{k-1} + \sum_{k=1}^{m} (n - 2k + 1)n D_k \\
= 2 \sum_{k=0}^{m-1} (n - 2k) n D_k + (n - 2m + 1)n D_{m} \text{ by } n = 2m + 1 \\
= 2 \sum_{k=0}^{m-1} (n - 2k) n D_k + 2 n D_{m} \\
= 2 \sum_{k=0}^{m} (n - 2k) n D_k = 2 \cdot 2^{n-1} = 2^n \text{ by (2.2)}. \\
\]

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(c-2) The case \( n = 2m + 2 \) for \( m = 0, 1, 2, \cdots \). Then \( \left[ \frac{n}{2} \right] = m + 1 \) and \( \left[ \frac{n-1}{2} \right] = m \). Hence we obtain

\[
\sum_{k=0}^{m+1} (n - 2k + 1) n D_k \\
= (n + 1) n D_0 + \sum_{k=1}^{m+1} (n - 2k + 1) n D_k \\
= (n + 1) n D_0 + \sum_{k=1}^{m+1} (n - 2k + 1)(n D_{k-1} + n D_k) \text{ by (i) and (ii)} \\
= (n + 1) n D_0 + \sum_{k=1}^{m+1} (n - 2k + 1)n D_{k-1} + \sum_{k=1}^{m+1} (n - 2k + 1)n D_k \\
= \sum_{k=0}^{m} (n - 2k - 1) n D_k + \sum_{k=0}^{m+1} (n - 2k + 1)n D_{k} \\
= 2 \sum_{k=0}^{m} (n - 2k) n D_k + (n - 2(m + 1) + 1)n D_{m+1} \\
= 2 \cdot 2^{n-1} = 2^n \text{ by (2.2) and } n-1 D_{m+1} = 2m+1 D_{m+1} = 0. \\
\]
(v). We remark that
\[
\lim_{n \to \infty} \frac{n^{\alpha}}{2^n} = 0 \quad \text{holds for fixed } \alpha \geq 0.
\]

(a) The case \(k = 0\). We have
\[
\lim_{n \to \infty} \frac{(n + 1)_n D_0}{2^n} = 2 \lim_{n \to \infty} \frac{(n + 1)}{2^{n+1}} = 0 \quad \text{by (2.3)}.
\]

(b) The case \(k = 1\). We have
\[
\lim_{n \to \infty} \frac{(n - 1)_n D_1}{2^n} = \frac{1}{2} \lim_{n \to \infty} \frac{(n - 1)^2}{2^{n-1}} = 0 \quad \text{by (2.3)}.
\]

(c) The case \(k \geq 2\). For sufficiently large \(n\),
\[
0 \leq \frac{(n - 2k + 1)_n D_k}{2^n} = \frac{n!(n - 2k + 1)^2}{2^nk!(n - k + 1)!} \\
= \frac{n(n - 1) \cdots (n - k + 2)(n - 2k + 1)^2}{2^nk!} \\
= \frac{n^{k+1} \cdot (1 - \frac{1}{n}) \cdots (1 - \frac{k-2}{n})(1 - \frac{2k-1}{n})^2}{2^nk!} \leq \frac{n^{k+1}}{2^n}.
\]

Hence we obtain (v) by (2.3).

Lemma 3. Let \(T \in B(H)\). Then
\[
\|T^n\| \leq \|T^{n+1}\|^{\frac{1}{2}} \|T^{n-1}\|^{\frac{1}{2}}
\]
holds for all natural numbers \(n\).

Proof. Let \(T = U|T|\) be the polar decomposition of \(T\). Then we have
\[
\|\widetilde{T^n}\| = \|(T^n)^{\frac{1}{2}} U|T|^{\frac{1}{2}}\| = \|T^{\frac{1}{2}} (U|T|)^{n-1} U|T|^{\frac{1}{2}}\| \\
\leq \|T^{\frac{1}{2}} (U|T|)^{n-1} U|T|^{\frac{1}{2}}\| \|U|T|\|^{n-1} U\|^{\frac{1}{2}} \quad \text{by Theorem A} \\
= \|T^{n+1}\|^{\frac{1}{2}} \|T^n\|^{\frac{1}{2}} \|T^{n-1}\|^{\frac{1}{2}}.
\]

Lemma 4. Let \(T \in B(H)\) and \(m = \lfloor \frac{n}{2} \rfloor\). Then
\[
\|\widetilde{T^n}\| \leq \|T^{n+1}\|^{\frac{1}{2}} \|T^{n-1}\|^{\frac{1}{2}} \|T^{n-2}\|^{\frac{1}{2}} \cdots \|T^m\|^{\frac{1}{2}},
\]
where \(m = \lfloor \frac{n-1}{2} \rfloor\).

Proof. We shall prove Lemma 4 by induction on \(n\).
(a) \(\|\widetilde{T^n}\| \leq \|T^2\|^{\frac{1}{2}}\) holds by Lemma 3.
(b) Assume that
\[
\|\widetilde{T_{n-1}}\| \leq \|T^n\|^{\frac{1}{2}} \|T^{n-2}\|^{\frac{1}{2}} \cdots \|T^{n-2k}\|^{\frac{1}{2}} \cdots \|T^{n-2m}\|^{\frac{1}{2}},
\]
where \(m = \lfloor \frac{n-1}{2} \rfloor\).
(c-1) The case \(n = 2m + 1\) for \(m = 1, 2, \ldots\). Then \(\frac{m-1}{2} = m\). Hence by (2.3), we have

\[
\|\hat{T}_n\| = \|\hat{(T)}_{n-1}\|
\]

\[
\leq \|\hat{T}_n\| \frac{D_{n-1}}{2^{n-1}} \|\hat{T}_n-1\| \frac{D_{n-2}}{2^{n-2}} \ldots \|\hat{T}_3\| \frac{D_{n-3}}{2^{n-3}} \|\hat{T}_2\| \frac{D_{n-4}}{2^{n-4}} \|\hat{T}_1\| \frac{D_{n-5}}{2^{n-5}} \ldots \|\hat{T}_1\| \frac{D_{n-2m-1}}{2^{n-2m-1}}
\]

\[
\leq \left(\|\hat{T}_n^{m+1}\| \|\hat{T}_n^{m-1}\| \|\hat{T}_n^{m-2}\| \ldots \|\hat{T}_n^{m-3}\| \right)^{\frac{1}{2m+1}} \left(\|\hat{T}_n^{m}\| \|\hat{T}_n^{m-1}\| \|\hat{T}_n^{m-2}\| \ldots \|\hat{T}_n^{m-3}\| \right)^{\frac{1}{2m+1}} \ldots \left(\|\hat{T}_n^1\| \|\hat{T}_n^{m}\| \|\hat{T}_n^{m-1}\| \|\hat{T}_n^{m-2}\| \ldots \|\hat{T}_n^{m-3}\| \right)^{\frac{1}{2m+1}}
\]

\[
= \|\hat{T}_n^{m+1}\| \|\hat{T}_n^{m-1}\| \|\hat{T}_n^{m-2}\| \ldots \|\hat{T}_n^{m-3}\| \|\hat{T}_n^1\| \|\hat{T}_n^{m}\| \|\hat{T}_n^{m-1}\| \|\hat{T}_n^{m-2}\| \ldots \|\hat{T}_n^{m-3}\| \|\hat{T}_n^1\|
\]

by (i) and (ii) of Lemma 2 and the last inequality holds by Lemma 3.

(c-2) The case \(n = 2m + 2\) for \(m = 0, 1, 2, \ldots\). Then \(\frac{m+1}{2} = m + 1\) and \(\frac{m-1}{2} = m\). Hence by (2.4), we have

\[
\|\hat{T}_n\| = \|\hat{(T)}_{n-1}\|
\]

\[
\leq \|\hat{T}_n\| \frac{D_{n-1}}{2^{n-1}} \|\hat{T}_n-1\| \frac{D_{n-2}}{2^{n-2}} \ldots \|\hat{T}_3\| \frac{D_{n-3}}{2^{n-3}} \|\hat{T}_2\| \frac{D_{n-4}}{2^{n-4}} \|\hat{T}_1\| \frac{D_{n-5}}{2^{n-5}} \ldots \|\hat{T}_1\| \frac{D_{n-2m-1}}{2^{n-2m-1}}
\]

\[
\leq \left(\|\hat{T}_n^{m+1}\| \|\hat{T}_n^{m-1}\| \|\hat{T}_n^{m-2}\| \ldots \|\hat{T}_n^{m-3}\| \right)^{\frac{1}{2m+1}} \left(\|\hat{T}_n^{m}\| \|\hat{T}_n^{m-1}\| \|\hat{T}_n^{m-2}\| \ldots \|\hat{T}_n^{m-3}\| \right)^{\frac{1}{2m+1}} \ldots \left(\|\hat{T}_n^1\| \|\hat{T}_n^{m}\| \|\hat{T}_n^{m-1}\| \|\hat{T}_n^{m-2}\| \ldots \|\hat{T}_n^{m-3}\| \right)^{\frac{1}{2m+1}}
\]

\[
= \|\hat{T}_n^{m+1}\| \|\hat{T}_n^{m-1}\| \|\hat{T}_n^{m-2}\| \ldots \|\hat{T}_n^{m-3}\| \|\hat{T}_n^1\| \|\hat{T}_n^{m}\| \|\hat{T}_n^{m-1}\| \|\hat{T}_n^{m-2}\| \ldots \|\hat{T}_n^{m-3}\| \|\hat{T}_n^1\|
\]

by (i), (ii) and (iii) of Lemma 2 and the last inequality holds by Lemma 3.
Lemma 5. Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence satisfying \( \lim_{n \to \infty} a_n = a \), and for each natural number \( n \), let \( \{\alpha_{n,k}\}_{k=1}^{n} \) be a positive sequence satisfying
\[
\alpha_{n,1} + \cdots + \alpha_{n,k} + \cdots + \alpha_{n,n} = 1 \quad \text{for all natural numbers } n \quad \text{and } \lim_{n \to \infty} \alpha_{n,k} = 0 \quad \text{for fixed } k = 1, 2, \ldots.
\]
Then
\[
\lim_{n \to \infty} \left( \alpha_{n,1}a_1 + \cdots + \alpha_{n,k}a_k + \cdots + \alpha_{n,n}a_n \right) = a.
\]

Proof. For any \( \varepsilon > 0 \), there exists \( k > 0 \) such that \( |a_n - a| < \varepsilon \) and \( \alpha_{n,1}|a_1 - a| + \cdots + \alpha_{n,k}|a_k - a| < \varepsilon \) for all natural numbers \( n > k \) by the assumptions \( \lim_{n \to \infty} a_n = a \) and \( \lim_{n \to \infty} \alpha_{n,k} = 0 \). Then we have
\[
\left| \left( \alpha_{n,1}a_1 + \cdots + \alpha_{n,k}a_k + \cdots + \alpha_{n,n}a_n \right) - a \right| = \left( \alpha_{n,1}(a_1 - a) + \cdots + \alpha_{n,k}(a_k - a) \right. \\
+ \left. \alpha_{n,k+1}(a_{k+1} - a) + \cdots + \alpha_{n,n}(a_n - a) \right) \leq \alpha_{n,1}|a_1 - a| + \cdots + \alpha_{n,k}|a_k - a| \\
+ \alpha_{n,k+1}|a_{k+1} - a| + \cdots + \alpha_{n,n}|a_n - a| < \varepsilon
\]
by \( (2.5) \) and \( \varepsilon \).


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