PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 5, Pages 1539–1550
S 0002-9939(01)06184-6
Article electronically published on October 5, 2001

ON A THEOREM OF JAWOROWSKI ON LOCALLY EQUIVARIANT CONTRACTIBLE SPACES

SERGEI M. AGEEV AND DUŠAN REPOVŠ

(Communicated by Alan Dow)

ABSTRACT. Ancel's method of fiberwise trivial relations is applied to the problem of characterization of absolute equivariant extensors. We obtain a generalization of Jaworowski's theorem on characterization of equivariant extensors lying in $\mathbb{R}^n$ to the case when the space is infinite-dimensional, has infinitely many orbit types and the acting compact group G is not necessarily a Lie group.

1. INTRODUCTION

Spaces which possess absolute extension property for partially defined maps, are called *absolute (neighborhood) extensors* (denoted by A[N]E) and they play an important role in topology. When a compact group G acts on the space, similar objects with respect to G-maps are called *equivariant absolute* (*neighborhood*) *extensors* (denoted by G-A[N]E). We recall that a metric space X is said to be a G-AE (resp. G-ANE) space, if for every partial G-map $Z \supset A \xrightarrow{\varphi} X$, defined on a closed G-subspace A of a metric G-space Z, there exists a G-extension $\widehat{\varphi}\colon Z \to X$ onto the entire space Z (resp. $\widehat{\varphi}\colon U \to X$ onto a G-neighborhood U of the subspace A). Observe that the following notions are identical:

(a) equivariant absolute (neighborhood) extensors and absolute (neighborhood) extensors for spaces with trivial G-action; and
(b) equivariant absolute (neighborhood) extensors and equivariant absolute (neighborhood) retracts for metric G-spaces.

The recognition of G-A[N]E is a difficult problem. One possible approach is to find sufficiently simple (for verification) properties which guarantee that $X \in G$-A[N]E. Working in this direction, Jaworowski [Ja] proved a theorem on characterization of equivariant extensors lying in a finite-dimensional Euclidean space $\mathbb{R}^n$ with orthogonal action of a compact Lie group G in terms of local G-contractibility, which can be reformulated as follows.

Theorem 1.1. *Every invariant subspace $X \subset \mathbb{R}^n$ is a G-ANE (resp. G-AE) space if and only if X is a locally G-contractible (resp. locally G-contractible and G-contractible) space.*

Received by the editors May 17, 2000 and, in revised form, November 10, 2000.
1991 *Mathematics Subject Classification.* Primary 54C55, 57S10; Secondary 54C20, 54C60, 54F65, 55M15.
Key words and phrases. Local equivariant contractibility, absolute equivariant extensor, fiberwise trivial relation, countable dimensional orbit space, selections of multivalued maps.

We recall that a G-contractible space X is by definition a space which has the G-homotopy type of a point (i.e. there exists a fixed point $x_0 \in X^G$ and a G-homotopy $F_t\colon X \to X$ such that $F_0 = \mathrm{Id}$ and $F_1(X) = x_0$). Introducing the concept of local G-contractibility of a space is a more delicate problem even for a compact Lie group G.

By a theorem of Palais [Br] there exists a neighborhood G-retraction $r\colon U \to G(x)$ onto an arbitrary orbit $G(x)$. Thus this invariant neighborhood U of $G(x)$ is G-homeomorphic to the cross-product $G \times_{G_x} S$, where $S \rightleftharpoons r^{-1}(x) \subset U$ is a G_x-space (cf. [Br]). This invariant neighborhood U is called an *exact tube around the orbit* $G(x)$ and the G_x-space S is called an *exact slice at the point* x.

We shall say that:

(c) X *is locally G-contractible at the point* $x \in X^G$, if for every $\epsilon > 0$ there exist an invariant neighborhood $\mathcal{O}(x)$ and a G-homotopy $F\colon \mathcal{O}(x) \to \mathrm{N}(x;\epsilon)$ such that $F_0 = \mathrm{Id}$ and $F_1(\mathcal{O}(x)) = x$;

(d) X is *locally G-contractible at the point* $x \notin X^G$ if the slice S of every tube $\mathcal{U} = G \times_{G_x} S$ around the orbit $G(x)$ is locally G_x-contractible at the point $x \in S^{G_x}$.

The following theorem describes the local G-contractibility in different terms.

Theorem 1.2. *Let G be any compact Lie group and X any metric G-space. Then the following conditions are equivalent:*

(1) *X is a locally G-contractible space at every point;*
(2) *for every point $x \in X$ there exists a tube $\mathcal{U} = G \times_{G_x} S$ around the orbit $G(x)$ whose slice S is locally G_x-contractible at x;*
(3) *for every orbit $G(x)$ and every $\epsilon > 0$ there exist a G-neighborhood $\mathcal{U} \supset G(x)$ and a G-homotopy $F_t\colon \mathcal{U} \to X$ such that $F_0 = \mathrm{Id}, F_1(\mathcal{U}) \subset G(x)$ and $\bigcup\{F_t(x) | 0 \le t \le 1\} \subset \mathrm{N}(x;\epsilon)$; and*
(4) *for every point x and every $\epsilon > 0$ there exists a G_x-neighborhood $\mathcal{U}$ of x which is G_x-contractible to the point x inside the neighborhood $N(x;\epsilon)$.*

Remarks. (i) The initial definition of local G-contractibility, which implicitly appears in [Ja], agrees with the property (4) above.

(ii) This theorem fails beyond the class of Lie groups because there is no tubular structure of invariant neighborhoods [Vi]. A version of the concept of local G-contractibility for arbitrary compact groups will be given later.

Proof of Theorem 1.2. (2) $\Longrightarrow$ (3). For every x and $\epsilon > 0$, there exists a tube $\mathcal{V} = G \times_{G_x} S$ around $G(x)$ such that S is locally G_x-contractible at x and $S \subset \mathrm{N}(x;\epsilon)$. Hence, by (2), there exists a thinner tube $\mathcal{V}_1 = G \times_{G_x} S_1$ around $G(x)$, the slice S_1 of which admits a G_x-contraction $\Phi_t\colon S_1 \to S$ into x. Then the desired G-neighborhood $\mathcal{U}$ coincides with $\mathcal{V}_1$ and the desired G-homotopy $F_t\colon \mathcal{V}_1 \to X$ is defined by the formula $F_t([g, s_1]) = g \cdot \Phi_t(s_1)$.

(3) $\Longrightarrow$ (4). Fix x and $\epsilon > 0$. Since the smooth manifold $G(x)$ belongs to G_x-ANE, there exists $\delta > 0$ such that $\mathrm{N}(x;\delta) \cap G(x)$ admits a G_x-contraction in x inside $\mathrm{N}(x;\epsilon) \cap G(x)$. By (3), there exist a tube $\mathcal{U} = G \times_{G_x} S$ around $G(x)$ and a G-homotopy $\Phi_t\colon \mathcal{U} \to X$, such that $\Phi_0 = \mathrm{Id}, \Phi_1(\mathcal{U}) \subset G(x)$ and $\bigcup\{\Phi_t(x) | 0 \le t \le 1\} \subset \mathrm{N}(x;\delta/2)$. By reducing the tube $\mathcal{U}$, we can suppose that $\bigcup\{\Phi_t(\mathcal{W}) | 0 \le t \le 1\} \subset \mathrm{N}(x;\delta)$, where $\mathcal{W}$ is a fine G_x-neighborhood of S in $\mathcal{U}$. Then the desired G_x-contraction $\mathcal{W}$ in x inside $\mathrm{N}(x;\epsilon)$ is defined in two steps: first, G_x-homotopy $F_t \rightleftharpoons$

$\Phi_t \restriction_{\mathcal{W}} : \mathcal{W} \to \mathrm{N}(x;\delta)$ G_x-contracts $\mathcal{W}$ into $\mathrm{N}(x;\delta) \cap G(x)$, and then $\mathrm{N}(x;\delta) \cap G(x)$ is G_x-contracted in x inside $\mathrm{N}(x;\epsilon) \cap G(x)$.

(4) $\Longrightarrow$ (1). Consider an arbitrary tube $\mathcal{V} = G \times_{G_x} S$ around $G(x)$ and fix some G_x-retraction $r : \mathcal{W} \to S$ of a G_x-neighborhood $\mathcal{W} \subset \mathcal{V}$ onto S which exists by the following assertion (this is an easy consequence of [Mu]).

Lemma 1.3. *Let $\mathcal{U} = G \times_{G_x} S$ be any tube around $G(x)$. Then S is a neighborhood G_x-retract of $\mathcal{U}$.*

Proof. Since G_x is G_x-ANE, it follows that G_x is a G_x-retract of some G_x-neighborhood $\mathcal{W}' \subset G$: $r' : \mathcal{W}' \to G_x, r' \circ r' = r'$. Then the desired G_x-retraction $r : \mathcal{W} \to S$ of G_x-neighborhood $\mathcal{W} \rightleftharpoons \mathcal{W}' \times_{G_x} S$ of the slice S is defined by the formula $r([w, s]) = [r'(w), s], w \in \mathcal{W}', s \in S$. □

We now complete the proof of Theorem 1.2. We derive from (4) that for $\epsilon > 0$, such that $\mathrm{N}(x;\epsilon) \subset \mathcal{W}$, there exist a G_x-neighborhood $\mathcal{U} \ni x$ and a G_x-contraction $\Phi_t : \mathcal{U} \to \mathrm{N}(x;\epsilon)$ to the point x. Let the tube $\mathcal{V}_1 = G \times_{G_x} S_1, S_1 \subset S$, be so small that $S_1 \subset \mathcal{U}$. Then the desired G_x-contraction $F_t : S_1 \to S$ is defined by the formula $F_t = r \circ \Phi_t$. □

When the group action is trivial the concept of G-LC agrees with the local contractibility (LC). Next, with respect to simplicity, an example of a G-LC-space is a linear G-space. Since the G-LC property can be easily detected and since it is well-known that G-ANE $\subset$ G-LC, there is an interesting question as to what degree is the local G-contractibility of a space characterized by its G-extensor properties. We observe that if the group action is trivial, the implication G-LC $\Longrightarrow$ G-ANE is true under some dimensional restrictions on X (X is finite-dimensional—the Kuratowski-Dugundji theorem [Hu], X is countable dimensional—Haver's theorem [Ha]). Theorem 1.1 can be considered as a partial extension of the Kuratowski-Dugundji theorem to the equivariant case.

By the Mostow theorem, a separable metric G-space X equivariantly lying in $\mathbb{R}^n$ can be characterized by the property that X has finitely many orbit types and $\dim X < \infty$. Both of these properties are essentially applied in the characterization of equivariant extensors lying in $\mathbb{R}^n$ [Ja]. The present paper is a result of our efforts to understand what happens with this problem when the space $X \in G$-LC is infinite-dimensional and has infinitely many orbit types and the acting group G is not necessarily a Lie group. As a result we have the following theorem.

Theorem 1.4. *Let G be any compact group and suppose that $X \in G$-LC. Then $X \in G$-ANE, provided that X/G is a countable-dimensional space.*

Remarks. (iii) If the acting group is trivial, then Theorem 1.4 coincides with the Haver theorem [Ha].

(iv) Theorem 1.4 is valid, provided that the orbit space X/G satisfies a more general condition of being a C-space (see [AG] and [An, Appendix]).

(v) Beyond the class of compact groups Theorem 1.4 admits the generalization to so-called *proper actions* of locally compact Lie groups in the sense of Palais.

The second consequence is related to the problem of whether a linear metric G-space L belongs to the class G-AE. Below we list known affirmative results.

(vi) $\dim L/G < \infty$ (the proof of this case is easily reduced to the case of a compact Lie group acting on a finite-dimensional space with the orthogonal representation); or

(vii) L is countable-dimensional and $|G| = 1$ (this is a consequence of the Haver theorem); or

(viii) L is locally convex and $L \in G$-ANE(0) [Ag] (in particular, G is a compact Lie group [Mu, p. 489] or L is a Banach G-space [Ab, p. 154]).

On the other hand, there exists a linear metric G-space L which is not G-AE within the class of nonlocally convex spaces [Ca] and the class of countable-dimensional spaces [Ag]. Therefore none of the hypotheses in the corollary below can be omitted.

Corollary 1.5. *A metric linear G-space L is a G-*AE *space if $L \in G$-*ANE(0) *and L/G is countable dimensional.*

A few comments about the proofs are in order. The class $\mathfrak{C}$ of all G-spaces satisfying the hypotheses of Theorem 1.4 is closed under the multiplication by $J = [0, 1): \ \mathfrak{C} \supset \mathfrak{C} \times J$. In this situation the problem of an *exact extension* of a G-map $f\colon A \to X$ is solved by reducing to a less difficult problem of an *approximate extension* of a partial G-map.

Definition 1.6. A G-space X is called an approximate absolute neighborhood G-extensor ($X \in G$-A-ANE), provided that for every open cover $\omega \in \operatorname{cov} X$ and every partial G-map $Z \hookleftarrow A \xrightarrow{f} X$ there exists a G-map $\widetilde{f}\colon U \to X$, called an *$\omega$-extension of f*, defined on a G-neighborhood U of A such that $\operatorname{dist}(f, \widetilde{f}\restriction_A) \prec \omega$.

The following theorem plays the key role in our exposition:

Theorem 1.7. *Suppose that a class of metric G-spaces $\mathfrak{P}$ is closed under the multiplication by J. Then $\mathfrak{P} \subset G$-*ANE *if and only if $\mathfrak{P} \subset G$-*A-ANE.

Implicitly this criterion with no group action being considered can be found in [Do], [Ma], [To]. However, its fundamental role in the theory of absolute extensors becomes clear only now since:

(α) it is substantially easier to construct an approximate extension of a G-map than to get an exact one;

(β) if the class $\mathfrak{P}$ of G-spaces is closed under the multiplication by J, then the problems of detection of G-ANE and approximation G-ANE are equivalent within the class $\mathfrak{P}$; and

(γ) the concept of an approximate G-ANE fits well with different concepts and constructions in the theory of extensors.

Since the class $\mathfrak{C}$ of G-spaces is closed under the multiplication by J, it suffices by virtue of Theorem 1.2 to establish that $X \in G$-A-ANE, for every $X \in \mathfrak{C}$. This enables one to apply the concept of fiberwise trivial maps in the sense of Ancel [An], due to its close connection with G-A-ANE.

Proposition 1.8. *If for every metric G-space Z and for every partial G-map $Z \hookleftarrow A \xrightarrow{\varphi} X$, the projection $\pi_\varphi\colon G_\varphi \to A$ of the graph $G_\varphi \subset Z \times X$ of φ onto A is fiberwise G-trivial inside the projection $p \rightleftharpoons \operatorname{pr}_Z\colon Z \times X \to Z$, then X is G-*A-ANE.

Proposition 1.8 is an easy consequence of Proposition 4.4.

2. Preliminaries

Hereafter we shall assume that all spaces (resp. maps) are metric (resp. continuous), unless otherwise stated. We shall denote the set of all open covers of X by

$\operatorname{cov} X$. The *ε-neighborhood of* $A \subset X, \varepsilon > 0$, is the set

$$\mathrm{N}(A;\varepsilon) \rightleftharpoons \{x \in X \,|\, \operatorname{dist}(x, A) < \varepsilon\}.$$

The *body* of a system of open sets ω is the set

$$\cup\omega \rightleftharpoons \bigcup\{U | U \in \omega\}.$$

As always, $\omega \succ \omega_1$ denotes that the cover ω is a refinement of ω_1. It is well-known that every cover $\sigma \in \operatorname{cov} X$ of a metric space X admits a *star refinement* (the Stone theorem). Below we give a well-known and useful criterion for the star-refinability of covers:

Proposition 2.1. *The cover $\sigma = \{S_\lambda | \lambda \in \Lambda\}$ is a star refinement of $\omega = \{W_\beta | \beta \in B\}$ if and only if for every λ there exists $\beta = \beta(\lambda)$ such that $\bigcap_{\lambda\in\Lambda'} S_\lambda \neq \emptyset, \Lambda' \subset \Lambda$, implies that:*

(1) $\bigcup_{\lambda\in\Lambda'} S_\lambda \subset \bigcap_{\lambda\in\Lambda'} W_{\beta_\lambda}$.

If $f, g \colon X \to Y$ are any maps, $A \subset Y$ and $\omega \in \operatorname{cov} Y$, then the condition of ω-closeness of f and g will be denoted by $\operatorname{dist}(f, g) \prec \omega$ and $f \restriction_A$ will denote the restriction of f onto A.

Hereafter, we shall consider only compact groups. An *action* of G on a space X is a homeomorphism $T : G \to \operatorname{Aut} X$ of the group G into the group $\operatorname{Aut} X$ of all autohomeomorphisms of X such that the map $G \times X \to X$, given by $(g, x) \mapsto T(g)(x) = gx$, is continuous. A space X with a fixed action of G is called a *G-space*, and a map $f \colon X \to Y$ of G-spaces preserving an action of G (i.e. $f(g\cdot x) = g\cdot f(x)$) is called a *$G$-map.* For a metric G-space Z, every G-map $\varphi \colon A \to X$, defined on a closed invariant subset $A \subset Z$ of Z, is called a *partial G-map.*

For a point $x \in X$, the *isotropy subgroup* of x, or the *stabilizer* of x, is defined as $G_x = \{g \in G | gx = x\}$, and the *orbit* of x as $G(x) = \{gx | g \in G\}$. The space of all orbits is denoted by X/G and the natural map $\pi : X \to X/G$, given by $\pi(x) = G(x)$, is called the *orbit projection.* The orbit space X/G is equipped with the quotient topology, induced by π.

Definition 2.2. A G-space X is said to be locally G-contractible, written as $X \in G$-LC, if for every point x and every $\epsilon > 0$ there exist a point $x_0 \in \mathrm{N}(x;\varepsilon)$, a G_{x_0}-neighborhood $V, V \subset \mathrm{N}(x;\varepsilon)$, of x and a G_{x_0}-homotopy $F_t \colon V \to \mathrm{N}(x;\varepsilon)$ such that

(2) $G_x \subset G_{x_0}$ and $G(x_0) \in G$-ANE; and
(3) $F_0 = \mathrm{Id}, F_1(V) = x_0$ (i.e. G_{x_0}-neighborhood V of x is G_{x_0}-contractible to the point x_0 inside the neighborhood $N(x;\epsilon)$).

Remark. For compact Lie groups G, each orbit $G(x_0)$ is G-ANE. Hence, by Theorem 1.2, we can conclude that the definition of local G-contractibility stated above agrees with the one given in Chapter 1.

A space D is said to be *countable-dimensional* if $D = \bigcup_{i=1}^{\infty} D_i$ and D_i is zero-dimensional, for every i. The following result follows easily from Lemma 3.3 of [An] and the openness of the orbit projection $\pi \colon A \to A/G$.

Proposition 2.3. *Let the orbit space A/G of a closed G-set A of the metric G-space Z be countable-dimensional. Then for every sequence $\{\omega_i\}_{i\geq 1}$ of collections of open G-subsets in Z, covering A, there exists a sequence $\{\sigma_i\}_{i\geq 1}$ of collections of open G-subsets in Z such that:*

(4) $\sigma \coloneqq \bigcup\limits_{i=1}^{\infty} \sigma_i$ *covers A;*
(5) *The collection σ_i is disjoint; and*
(6) $\sigma_i \succ \omega_i$, *for every i.*

3. An approximate criterion for extendability of G-maps

Let A be a closed subset of a metric space X. An open cover ω of $Z \setminus A$ is said to be *adjacent to A* if for every point $a \in A$ and its every neighborhood $U = U(a)$ in X there exists a neighborhood $V(a) \subset U$ such that the star of $V(a)$ with respect to the cover ω lies in U. Such covers are usually called *canonical covers* [Bo], [Hu], but we prefer the term 'adjacent' because of its stronger expressiveness.

In the product $X \times J$ of a metric G-space X and $J = [0,1)$ we consider an open cover ω, adjacent to the top level. The next theorem allows one to reduce the problem of the exact extension of partial maps to a less difficult problem of approximate extensions:

Theorem 3.1 (Approximate criterion for extendability)**.** *Let $\omega \in \operatorname{cov} X \times [0,1)$ be a cover which is adjacent to the top level $X \times \{1\}$. Then a partial G-map $Z \supseteq A \xrightarrow{f} X$ has a global [neighborhood] equivariant extension if and only if the partial G-map $Z \times J \supseteq A \times J \xrightarrow{f \times \mathrm{Id}} X \times J$ has an equivariant ω-extension onto $Z \times J$ [onto a neighborhood of $A \times J$].*

It follows immediately from Theorem 3.1 that $X \times J \in G$-A-ANE implies $X \in G$-ANE. This observation completes the proof of Theorem 1.8.

Proof of Theorem 3.1. The nontrivial part of the proof is to establish the sufficiency. Thus, assuming the approximate extension property of $X \times J$ above, one needs to construct a G-extension of a given partial G-map $Z \supseteq A \xrightarrow{f} X$. By the hypothesis, the partial G-map $Z \times J \supseteq A \times J \xrightarrow{g} X \times J$, where $g = f \times \mathrm{Id}_J$, admits a G-map $\widetilde{g}\colon Z \times J \to X \times J$ $\big[$resp. G-map $\widetilde{g}\colon U \to X \times J\big]$, such that $(g, \widetilde{g}\restriction_{A\times J}) \prec \omega$.

The following fact can be derived easily from the hypothesis that ω is adjacent to the top level and the continuity of g.

Claim 3.2. *The G-map $d\colon A \times [0,1] \to X \times [0,1]$, given by the formulae $d\restriction_{A\times J} = \widetilde{g}\restriction_{A\times J}$ and $d\restriction_{A\times\{1\}} = f \times \{1\}$, is continuous.*

We state the next straightforward fact without proof:

Claim 3.3. *Suppose that a G-map $\alpha\colon F \cup W \to T$ is defined on the union of a closed G-subset F and an open G-subset W of a G-space S, such that $\alpha\restriction_F$ and $\alpha\restriction_W$ are continuous. Then there exists a closed G-subset $F_1, F \cup W \supseteq F_1 \supseteq F$, of S such that $\alpha\restriction_{F_1}$ is continuous and $F \cap W \subseteq \mathrm{Int}(F_1)$.*

Apply Claim 3.3 for $S \coloneqq Z \times I, T \coloneqq X \times I, F \coloneqq A \times I \bigcup Z \times \{0\}, W \coloneqq Z \times J$ $\big[$resp. $F \coloneqq A \times I, W \coloneqq U\big]$ and $\alpha \coloneqq d \cup \widetilde{g}$. We get a closed G-subset H of

$Z \times I$, such that

$$A \times I \cup Z \times J \supseteq H \supseteq A \times I \cup Z \times \{0\} \ \big[\text{ resp. } A \times I \cup U \supseteq H \supseteq A \times I \ \big],$$

$$A \times [0,1) \cup Z \times \{0\} \subseteq \operatorname{Int} H \ \big[\text{resp. } A \times [0,1) \subseteq \operatorname{Int} H\big]$$

and $\alpha \restriction_H$ is a continuous G-map.

Claim 3.4. *There exists a sequence of open G-neighborhoods $V_1 \supseteq V_2 \supseteq \cdots$, $V_i \supseteq \operatorname{Cl} V_{i+1}$, $\bigcap V_i = A$, and a monotone sequence of numbers $0 = r_0 < r_1 < r_2 < \ldots, \lim r_i = 1$, such that $V_i \times [0, r_i] \subseteq H$.*

Let $\xi_i \colon Z \to [r_{i-2}, r_{i-1}], i \geq 2$, be a continuous real-valued function, constant on the orbits, and such that $\xi_i \restriction_{\operatorname{Bd} V_{i-1}} \equiv r_{i-2}$ and $\xi_i \restriction_{\operatorname{Bd} V_i} \equiv r_{i-1}$. Then the G-map $\xi \colon Z \to [0,1]$, defined by $\xi \restriction_{Z \setminus V_1} = 0$, $\xi \restriction_{V_{i-1} \setminus V_i} = \xi_i$ for $i \geq 2$ and $\xi \restriction_A \equiv \operatorname{Id}$, is continuous and $\big(v, \xi(v)\big) \in H$, for each $v \in Z$ $\big[$resp. $\big(v, \xi(v)\big) \in H$, for each $v \in V_1\big]$. The desired extension $\widehat{f}$ of the partial map f is now given by the formula $\widehat{f}(v) = \alpha\big(v, \xi(v)\big)$, where $v \in Z$ $\big[$resp. $v \in V_1\big]$. □

4. Fiberwise G-trivial maps

Consider the projection $p \rightleftharpoons \operatorname{pr}_M \colon M \times N \to M$ of the product of metric G-spaces onto the first factor M. Let the image $p(X)$ of a G-subset $X \subset M \times N$ (not necessarily closed) be contained in a G-subspace $Y \subset M$. Then we shall denote the restriction of p onto X by $\pi \colon X \to Y$. We do not require that a G-map π be surjective.

We introduce some new concepts. A G-embedding $A \hookrightarrow B$ of G-subsets A and B of $M \times N$ is said to be *fiberwise G-contractible inside the projection* p, if there exist a G-map $g \colon p(A) \to B$, $p \circ g = \operatorname{Id}_{p(A)}$, and a G-homotopy $H_t \colon A \to B$ such that the following conditions (1) – (3) are satisfied:

(1) $p \circ H_t = p$, for every $t \in I$ (i.e. H_t is the fiberwise G-homotopy);
(2) $H_0 = \operatorname{Id}_A$; and
(3) $H_1 = g \circ p$ (the condition of factorizability of H_1 via the projection p).

Hereafter Ω shall denote the collection of all G-neighborhoods $\mathcal{U}$ of X such that:

(4) $Y \subset p(\mathcal{U})$; and
(5) for every point $y \in Y$ there exists a neighborhood $O \rightleftharpoons O(y)$ of y such that $O \times \operatorname{pr}_N(O^\bullet \bigcap X) \subset \mathcal{U}$. (We denote the product $C \times N = p^{-1}(C)$ by $C^\bullet$, where $C \subset M$.)

Let $\mathcal{U}, \mathcal{V} \in \Omega$ and $\mathcal{V} \subset \mathcal{U}$. We will say that a G-map $\pi = p \restriction_X \colon X \to Y$ is *fiberwise G-$\mathcal{UV}$-contractible inside the projection* p, if there exists a G-neighborhood $W \subset p(\mathcal{V})$ of the G-set Y such that the embedding $W^\bullet \cap \mathcal{V} \hookrightarrow W^\bullet \cap \mathcal{U}$ is fiberwise contractible inside the projection p.

We shall say that a map $\pi \colon X \to Y$ is *fiberwise G-trivial inside the projection* p, if for every G-neighborhood $\mathcal{U} \in \Omega$ there exists a small G-neighborhood $\mathcal{V} \in \Omega$, $\mathcal{V} \subset \mathcal{U}$, for which the G-map $\pi \colon X \to Y$ is fiberwise G-$\mathcal{UV}$-contractible inside the projection p; *locally fiberwise G-trivial inside the projection* p, if for every G-neighborhood $\mathcal{U} \in \Omega$ there exist a G-neighborhood $\mathcal{V} \in \Omega, \mathcal{V} \subset \mathcal{U}$, and a collection $\sigma = \{O(y) | y \in Y\}$ of open (in M) G-sets, which covers Y such that $p(\mathcal{V}) \supset \bigcup \sigma$ and $O(y)^\bullet \cap \mathcal{V} \hookrightarrow O(y)^\bullet \cap \mathcal{U}$ is fiberwise G-contractible inside p, for every $y \in Y$.

Proposition 4.1. *Suppose that there is a sequence of G-neighborhoods $\mathcal{V}_1 \supset \mathcal{V}_2 \supset \mathcal{V}_3 \supset \dots$ from Ω and that a G-space Y is contained in the union $\bigcup_{i=1}^{\infty} W_i$ of open G-subsets of M. Suppose that for every $i \geq 1$, the following conditions are satisfied:*

(6) *$p(\mathcal{V}_{i+1}) \supset W_i$; and*

(7) *the embedding $W_i^{\bullet} \cap \mathcal{V}_{i+1} \hookrightarrow W_i^{\bullet} \cap \mathcal{V}_i$ is fiberwise contractible inside p.*

Then $\mathcal{V} \rightleftharpoons \bigcup_{i=1}^{\infty} W_i^{\bullet} \cap \mathcal{V}_{i+1} \in \Omega$ and the projection π is fiberwise G-$\mathcal{V}_1\mathcal{V}$-contractible inside p.

This proposition is an equivariant version of Lemma 3.6 of [An, p. 11] and its proof is analogous, and is based on the openness and perfectness of the orbit projection.

We now present a condition on a G-space X, which guarantees the local fiberwise G-triviality of the graph projection.

Proposition 4.2. *Let $Z \hookleftarrow A \xrightarrow{\varphi} X$ be a partial G-map onto a G-LC-space X and $G_\varphi \rightleftharpoons \{(a, \varphi(a)) | a \in A\} \subset Z \times X$ be graph of φ. Then the projection $\pi\colon G_\varphi \to A$ of the graph G_φ onto A is locally fiberwise G-trivial inside the projection $p\colon Z \times X \to Z$.*

Proof. It is easy to check that the collection Ω of all G-neighborhoods of G_φ in $Z\times X$ satisfies properties (4)–(5) for $p\colon Z \times X \to Z$. Next we fix a closed G-neighborhood $\mathcal{U}$ of G_φ. We should find a G-neighborhood $\mathcal{V} \in \Omega, G_\varphi \subset \mathcal{V} \subset \mathcal{U} \subset Z \times X$, such that:

(a) For every point $a_0 \in A$ there exists a G-neighborhood $O = O(a_0) \subset Z$, for which $O \subset p(\mathcal{V})$ and the embedding $O^{\bullet} \cap \mathcal{V} \hookrightarrow O^{\bullet} \cap \mathcal{U}$ is fiberwise G-contractible (here $O^{\bullet} \rightleftharpoons O \times X$).

We shall assume that some invariant metrics are given on X and Z [Pa]. We shall investigate a multivalued map

$$\Phi\colon A \rightsquigarrow \mathbb{R}^+ \ , \ \Phi(a) = \{r > 0 | \ \mathrm{N}(G(\bar{a}); r) \subset \mathcal{U}\} \subset \mathbb{R}^+, \text{ where } \bar{a} \rightleftharpoons (a, \varphi(a)) \in G_\varphi.$$

It is easy to see that Φ is lower semi-continuous, convex-and-closed-valued, and is constant on the orbits.

Since the group G acts trivially on J, we can apply the Dowker theorem [RS] to the induced multivalued map $\widetilde{\Phi}\colon A/G \rightsquigarrow \mathbb{R}^+$ and obtain a selection $r\colon A \to (0, \infty)$ of Φ which is constant on each orbit.

Next, we need an auxiliary fact, based on the hypothesis that $X \in G$-LC.

Lemma 4.3. *For every $a \in A$, there exists a G-neighborhood $\mathcal{O}$ of $\bar{a} = (a, \varphi(a))$ such that $\mathcal{O} \subset \mathrm{N}(G(\bar{a}); r(a))$ and the natural embedding $\mathcal{O} \hookrightarrow \mathrm{N}(G(\bar{a}); r(a))$ is fiberwise G-contractible.*

Proof. For $x = \varphi(a) \in X$ and $\varepsilon = r(a) > 0$, there exist $x_0 \in \mathrm{N}(x; \varepsilon)$, a G_{x_0}-neighborhood $V, V \subset \mathrm{N}(x; \varepsilon)$, of x and a G_{x_0}-homotopy $F_t\colon V \to \mathrm{N}(x; \varepsilon)$ such that the conditions (2)–(3) of Definition 2.2 are fulfilled. Since $G(x_0) \in G$-ANE, the natural G-map $f\colon G(a) \to G(x_0), f(g \cdot a) = g \cdot x_0$, can be extended to a G-map $\widehat{f}\colon U(a) \to G(x_0)$ defined on a G-neighborhood $U(a)$ of $G(a)$.

It is not difficult to check that $S \rightleftharpoons \widehat{f}^{-1}(x_0)$ is a G_{x_0}-set of Z, $U(a)$ coincides with $G \times_{G_{x_0}} S$ and $\mathcal{O}(\bar{a}) \rightleftharpoons G \cdot (S \times V)$ is a G-neighborhood of $G(\bar{a})$ in $Z \times X$. The reader can check that the formula $\Phi_t(g \cdot (s, v)) = g \cdot (s, F_t v), s \in S, v \in V, g \in G$, correctly defines a G-homotopy $\Phi_t\colon \mathcal{O}(\bar{a}) \to Z\times X$, which is a fiberwise G-contraction. Reducing the G-neighborhood $U(a)$ if necessary, we can choose $\mathcal{O}(\bar{a})$ as the desired G-neighborhood $\mathcal{O}$. □

We now continue with the proof of Proposition 4.2. Consider yet another multivalued map

$$\Psi\colon A \rightsquigarrow \mathbb{R}^+,\ \Psi(a)=\{t>0|\ r(a)\geq t,\ \text{and the natural embedding}$$
$$\mathrm{N}(G(\bar{a});t)\hookrightarrow \mathrm{N}(G(\bar{a});r(a)) \text{ is fiberwise } G\text{-contractible}\}\subset\mathbb{R}^+,$$

which is also easily seen to be lower semi-continuous, convex-and-closed-valued, and is constant on the orbits. We again invoke the Dowker theorem and find a continuous G-selection $t\colon A\to(0,\infty)$ of Ψ such that $t(a)\leq r(a)$.

By virtue of the continuity of φ and t, it is possible to choose a G-neighborhood $W(a_0)$ of $a_0\in A$ such that:

(b) For every $a\in W(a_0)\cap A$, $\mathrm{N}(G(\bar{a});t(a)/2)\subset \mathrm{N}(G(\bar{a}_0);t(a_0))$, where $\bar{a}_0 = (a_0,\varphi(a_0))$.

The family of G-neighborhoods $\sigma \coloneqq \{W(a)\subset Z|a\in A\}$ covers A. As Z/G is paracompact, there exists a family $\sigma' = \{W'(a)\subset Z|a\in A\}$ of G-neighborhoods, covering A, which is star refinement of σ. By Proposition 2.1, it follows that:

(c) $\bigcap\limits_{i=0}^{n} W'(a_i)\neq\emptyset$ implies $\bigcup\limits_{i=0}^{n} W'(a_i)\subset \bigcap\limits_{i=0}^{n} W(z_{a_i})\subset W(z_{a_0})$.

We associate to every point $a\in A$, a point $z_a\in A$ such that $\mathrm{St}_{\sigma'}(a)\coloneqq \bigcup\limits_{a\in W'(a')} W'(a')\subset W(z_a)$. Finally, the desired G-neighborhood $\mathcal{V}$ is defined by

$$\mathcal{V}\coloneqq\bigcup_{a\in A}\mathrm{N}(G(\bar{a});t(a)/2)\cap(W'(a))^{\bullet}.$$

It is clear that $G_\varphi\subset\mathcal{V}\subset\mathcal{U}$. To verify (a) we should prove the following properties of the G-neighborhood $O\coloneqq W'(a_0)$ of $a_0\in A$:

(d) $O^{\bullet}\cap\mathcal{V}\subset \mathrm{N}(G(\bar{z}_{a_0});t(z_{a_0}))$; and

(e) $O^{\bullet}\cap \mathrm{N}(G(\bar{z}_{a_0});r(z_{a_0}))\subset\mathcal{U}$, where $\bar{z}_{a_0}=(z_{a_0},\varphi(z_{a_0}))\in Z\times X$.

Property (e) is evident as $\bigcup\limits_{a\in A}\mathrm{N}(G(\bar{a});r(a))\subset\mathcal{U}$. To prove (d), pick a point $z\in O$. It follows from the explicit formulae for $\mathcal{V}$ and $O^{\bullet}$ that:

(f) $G(z)^{\bullet}\cap\mathcal{V}\subset \bigcup\limits_{a\in\Lambda}\mathrm{N}(G(\bar{a});t(a)/2)\cap(W'(a))^{\bullet}$, where $\Lambda=\{a\in A|z\in W'(a)\}$.

Since $z\in\bigcap\limits_{a\in\Lambda}W'(a)$ and $a_0\in\Lambda$ (as $z\in O=W'(a_0)$), we have, by (c), that $\bigcup\limits_{a\in\Lambda}W'(a)\subset W(z_{a_0})$. After invoking (b) the last inclusion implies

$$\bigcup_{a\in\Lambda}\mathrm{N}(G(\bar{a});t(a)/2)\subset \mathrm{N}(G(\bar{z}_{a_0});t(z_{a_0})).$$

Since $t\colon A\to(0,\infty)$ is a G-selection of Ψ, $\mathrm{N}(G(\bar{z}_{a_0});t(z_{a_0}))$ is fiberwise G-contractible inside $\mathrm{N}(G(\bar{z}_{a_0});r(z_{a_0}))$. Therefore $O^{\bullet}\cap\mathcal{V}\hookrightarrow O^{\bullet}\cap\mathcal{U}$ is a fiberwise G-contractible embedding and this completes the proof of Proposition 4.2. □

In conclusion we shall strengthen Proposition 1.8, which establishes the connection between fiberwise G-trivial maps and G-A-ANE.

Proposition 4.4. *Let X be a closed subset of G-ANE-space Z. If the projection $\pi_\varphi\colon G_\varphi \to X$ of the graph $G_\varphi \subset Z \times X$ of the partial G-map $Z \hookleftarrow X \xrightarrow{\varphi\,\mathrm{Id}_X} X$ is fiberwise G-trivial inside the projection $p \rightleftharpoons \mathrm{pr}_Z\colon Z \times X \to Z$, then $X \in G$-A-ANE.*

Proof. We shall prove that for every cover $\omega = \{W_\gamma | \gamma \in \Gamma\} \in \operatorname{cov} X$ there exists a G-map $\widetilde{\varphi}\colon W \to X$, defined on a G-neighborhood W of X in Z, such that $\operatorname{dist}(\varphi, \widetilde{\varphi}\restriction_X) \prec \omega$. When this is done, then $X \in G$-A-ANE, due to the following simple fact:

Lemma 4.5. *Let Z be a G-ANE space and suppose that a partial G-map $Z \hookleftarrow X \xrightarrow{\varphi\,\mathrm{Id}_X} X$ has a neighborhood ω-G-extension, for every $\omega \in \operatorname{cov} X$. Then $X \in G$-A-ANE.*

Consider a cover $\omega' = \{W'_\beta\} \in \operatorname{cov} X$ which is star refinement of ω. It follows by Proposition 2.1 that there exists a map of the index sets $\beta \longmapsto \gamma = \gamma(\beta)$, such that the property 2.1(1) holds. In addition, consider an open system $\sigma = \{S_\lambda\}$ in Z, covering X and such that $\{\varphi(S_\lambda \cap X)\} \succ \omega'$. Let $\varphi(S_\lambda \cap X) \subset W'_{\beta=\beta(\lambda)} \subset W_{\gamma=\gamma(\beta)}$.

Since the map $\pi_\varphi\colon G_\varphi \to X$ is fiberwise G-trivial inside $p\colon Z \times X \to Z$, it follows that for the neighborhood $\mathcal{U} \rightleftharpoons \bigcup\limits_\lambda S_\lambda \times W'_{\beta(\lambda)}$ of the graph G_φ, there exist a G-neighborhood $W, Z \supset W \supset X$, and a G-neighborhood $\mathcal{V}, G_\varphi \subset \mathcal{V} \subset \mathcal{U}$, such that $p(\mathcal{V}) \supset W$ and $W^\bullet \cap \mathcal{V}$ is fiberwise G-contractible inside $W^\bullet \cap \mathcal{U}$.

Consequently, there exists a G-map $\widetilde{\varphi}\colon W \to X$ such that $(x, \widetilde{\varphi}(x)) \in \mathcal{U}$, for every $x \in X$. We will show that $\widetilde{\varphi}$ is in fact the desired map. For $x_0 \in X$ let Λ' be the set $\{\lambda | x_0 \in S_\lambda\} \neq \varnothing$.

It is easy to see that $x_0^\bullet \cap \mathcal{U} = \bigcup\limits_{\lambda\in\Lambda'} x_0 \times W'_{\beta(\lambda)} = x_0 \times \big(\bigcup\limits_{\lambda\in\Lambda'} W'_{\beta(\lambda)}\big)$ and hence, by virtue of Proposition 2.1, $x_0^\bullet \cap \mathcal{U} \subset x_0 \times \bigcap\limits_{\lambda\in\Lambda'} W_{\gamma(\beta(\lambda))}$. Since $(x_0, \widetilde{\varphi}(x_0)) \in x_0^\bullet \cap \mathcal{U}$, it follows that $\widetilde{\varphi}(x_0) \in \bigcap\limits_{\lambda\in\Lambda'} W_{\gamma(\beta(\lambda))}$. Hence for every $\lambda \in \Lambda'$, $\varphi(x_0)$ and $\widetilde{\varphi}(x_0)$ are contained in $W_{\gamma(\beta(\lambda))} \in \omega$. □

5. Proof of Theorem 1.4

Consider the situation from the beginning of Chapter 4. This in particular means that $X \subset M \times N$ and $Y \subset N$ are G-subsets of metric G-spaces and $p(X) \subset Y$ where $p \rightleftharpoons \mathrm{pr}_M\colon M \times N \to M$ is the projection onto the first factor M. Let Ω be a collection of all G-neighborhoods of X satisfying conditions (4)–(5) from Chapter 4.

Theorem 5.1. *Let $\pi\colon X \to Y$ be a restriction of the projection $p \rightleftharpoons \mathrm{pr}_M\colon M \times N \to M$ onto X. Let the orbit space Y/G of the G-space Y be countable-dimensional and the projection $\pi\colon X \to Y$ a locally fiberwise G-trivial inside p. Then the projection π is fiberwise G-trivial inside p.*

Proof. We fix a neighborhood $\mathcal{V}_1 \in \Omega$ of the G-space X in $M \times N$. Since π is locally fiberwise G-trivial inside p, there exist a family $\sigma_1 = \{O_1(y) | y \in Y\}$ of open G-sets in M, which covers Y, and a G-neighborhood $\mathcal{V}_2 \subset \mathcal{V}_1, \mathcal{V}_2 \in \Omega$, such that $p(\mathcal{V}_2) \supset \bigcup \sigma_1$ and the embedding $(O_1(y))^\bullet \cap \mathcal{V}_2 \hookrightarrow (O_1(y))^\bullet \cap \mathcal{V}_1$ is fiberwise G-contractible inside p, for every $y \in Y$.

Analogously, for $j \geq 3$, one can construct G-neighborhoods $\mathcal{V}_j \in \Omega, \mathcal{V}_j \subset \mathcal{V}_{j-1}$, and a collection $\sigma_{j-1} = \{O_{j-1}(y) | y \in Y\}$ of open G-sets in M, covering Y, such that:

1) $\sigma_{j-1} \succ \sigma_{j-2}$; and

2) $p(\mathcal{V}_j) \supset \bigcup \sigma_{j-1}$ and the embedding $(O_{j-1}(y))^\bullet \cap \mathcal{V}_j \hookrightarrow (O_{j-1}(y))^\bullet \cap \mathcal{V}_{j-1}$ is fiberwise G-contractible inside p, for every $y \in Y$.

Since Y/G is countable-dimensional, Proposition 2.3 implies that

$$Y \subset \bigcup_{j=1}^{\infty} \bigcup_{\lambda \in \Lambda_j} D_j(\lambda),$$

where $\{D_j(\lambda) | \lambda \in \Lambda_j\}$ is a collection of disjoint open G-sets in M which refines σ_j. We denote by $W_j = \bigcup_{\lambda \in \Lambda_j} D_j(\lambda)$, for $j \geq 1$.

It is evident that $p\mathcal{V}_{j+1} \supset W_j$, for every $j \geq 1$ and the embedding $W_j^\bullet \cap \mathcal{V}_{j+1} \hookrightarrow W_j^\bullet \cap \mathcal{V}_j$ is fiberwise G-contractible inside p. Therefore conditions (6) and (7) from from Chapter 4 are satisfied. By virtue of Proposition 4.1, $\mathcal{V} \rightleftharpoons \bigcup_{j=1}^{\infty} W_j^\bullet \cap \mathcal{V}_{j+1} \in \Omega$ and π is fiberwise G-$\mathcal{V}_1\mathcal{V}$-contractible inside p. This completes the proof. □

Proof of Theorem 1.4. Since the class of spaces, which satisfy the hypotheses of Theorem 1.4, is closed under the multiplication by $[0, 1)$, it suffices, by Theorem 3.1, to prove that $X \in G$-LC implies $X \in G$-A-ANE.

Take a closed embedding of X into a linear normed space L [Bo]. By Z we denote a linear normed space $\mathrm{C}(G, L)$ of all continuous maps $f\colon G \to L$ endowed by the action of G: $(g \cdot f)(h) = f(g^{-1} \cdot h)$, where $f \in \mathrm{C}(G, L), g, h \in G$. By [Mx, Theorem 3], $Z \in G$-ANE. Consider a closed G-embedding $\varphi\colon X \hookrightarrow Z$ of X into Z, given by the formula $\varphi(x)(g) = g^{-1} \cdot x$, where $x \in X$ and $g \in G$, and define $G_\varphi \subset Z \times X, \pi_\varphi\colon G_\varphi \to X$ and $p\colon Z \times X \to Z$ as in Proposition 4.4.

Since $X \in G$-LC, it follows by Proposition 4.2 that π_φ is a locally fiberwise G-trivial projection inside p. Since X has a countable-dimensional orbit space, all hypotheses of Theorem 5.1 are satisfied and consequently π_φ is a fiberwise G-trivial projection inside p. Finally it follows by Proposition 4.4 that $X \in G$-A-ANE. □

Acknowledgements

The first author was supported in part by the INTAS grant No. 96-0712. The second author was supported in part by the Ministry of Science and Technology of the Republic of Slovenia grant No. J1–0885–0101–98. We thank the referee for very close reading of the manuscript and several comments and suggestions.

References

[Ab] H. Abels, *A universal proper G-space*, Math. Z. **159** (1978), 143-158. MR **58:**18504

[AG] D. F. Addis and J. H. Gresham, *A class of infinite-dimensional spaces I, Dimension theory and Aleksandroff's problem*, Fund. Math. **101** (1978), 195–205. MR **80b:**54041

[Ag] S. M. Ageev, *An equivariant Dugundji theorem*, Uspehi Mat. Nauk **45**:5 (1990), 116-117 (in Russian); Engl. transl. in: Russ. Math. Surv. **45**:5 (1990), 219–220. MR **92h:**54026

[An] F. D. Ancel, *The role of countable dimensionality in the theory of cell-like relations*, Trans. Amer. Math. Soc. **287** (1985), 1–40. MR **86b:**54012

[Bo] K. Borsuk, *Theory of Retracts*, Monogr. Matem. **44**, PWN, Warsaw, 1967. MR **35:**7306

[Br] G. E. Bredon, *Introduction to Compact Transformation Groups*, Pure and Appl. Math. **46**, Academic Press, New York–London, 1972. MR **54:**1265

[Ca] R. Cauty, *Un espace metrique lineaire qui n'est pas un retracte absolu*, Fund. Math. **146** (1994), 85–99. MR **95j:**54022

[Do] C. H. Dowker, *Homotopy extension theorems*, Proc. London Math. Soc. **6** (1956), 100–116. MR **17:**518f

[Ha] W. E. Haver, *Locally contractible spaces that are absolute neighborhood retracts*, Proc. Amer. Math. Soc. **40** (1973), 280–284. MR **48:**9645

[Hu] S.-T. Hu, *Theory of Retracts*, Wayne State Univ. Press, Detroit, 1965. MR **31:**6202

[Ja] J. Jaworowski, *Extensions of G-maps and Euclidean G-retracts*, Math. Z. **146** (1976), 143–148. MR **52:**15351

[Ma] S. Mardešić, *Approximate polyhedra, resolutions of maps and shape fibrations*, Fund. Math. **114** (1981), 53–78. MR **83d:**55016

[Mx] C. N. Maxwell, *Homeomorphisms of topological transformation groups into function spaces*, Duke Math. J. **33** (1966), 567–574. MR **34:**3563

[Mu] T. Murayama, *On G-ANE's and their G-homotopy types*, Osaka J. Math. **20** (1983), 479–512. MR **85f:**57022

[Pa] R. S. Palais, *The classification of G-spaces*, Memoirs Amer. Math. Soc. **36** (1960). MR **31:**1664

[RS] D. Repovš and P. V. Semenov, *Continuous Selections of Multivalued Mappings*, Mathematics and Its Appl. **455**, Kluwer, Dordrecht, 1998. MR **2000a:**54002

[To] H. Toruńczyk, *Concerning locally homotopy negligible sets and characterization of l_2-manifolds*, Fund. Math. **101** (1978), 93–110. MR **80g:**57019

[Vi] G. Villalobos, *Inversion of the theorem on the existence of slices*, Vestnik MSU (1) (2000), 26-29 (in Russian).

Department of Mathematics, Brest State University, Brest, Belarus 224011
E-mail address: ageev@highmath.brsu.brest.by

Institute of Mathematics, Physics and Mechanics, University of Ljubljana, P.O. Box 2964, Ljubljana, Slovenia 1001
E-mail address: dusan.repovs@uni-lj.si