

## IMMERSIONS OF SURFACES IN ALMOST-COMPLEX 4-MANIFOLDS

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(Communicated by Ronald A. Fintushel)

**ABSTRACT.** In this paper, we investigate the relation between double points and complex points of immersed surfaces in almost-complex 4-manifolds and show how estimates for the minimal genus of embedded surfaces lead to inequalities between the number of double points and the number of complex points of an immersion.

### 1. INTRODUCTION

Suppose that  $X$  is a 4-manifold with an almost complex structure  $J$  and  $F \hookrightarrow X$  is an immersed oriented surface. For a generic immersion, there are two types of distinguished points on this surface. On the one hand, we have the singularities of the embedding which we assume to be ordinary double points (this is always the case for a generic immersion). On the other hand, there is a finite number of complex points, i.e., points at which the almost complex structure  $J$  preserves the tangent space of the surface. At such a point, the orientation induced by  $J$  on the tangent space may coincide with the orientation of the surface—in which case we will call the point a **positive complex point**—or it may not, then it is called a **negative complex point**.

In this paper, we use results of H.F. Lai to derive relations between the number of double points and the number of complex points of an immersion, thus extending the results of [CG], where only the case of embedded surfaces is treated. Our main result concerns immersed surfaces in the neighborhood of almost complex submanifolds. So let us assume that  $F_0 \subset X$  is an almost complex curve, i.e., an embedded oriented surface whose points are all positive complex points. Then clearly the number  $n^-$  of negative complex points is zero, and so is the number  $d_-$  of double points having negative sign. In particular, we have the inequality  $n^- \leq d_-$ . It turns out that a similar inequality is true for immersed surfaces “near”  $F_0$ .

**Theorem 1.** *Let  $X$  be an almost complex 4-manifold and suppose that  $F_0 \subset X$  is an embedded pseudoholomorphic curve with  $F_0 \cdot F_0 > 0$ . Now let  $F$  be an immersed surface contained in a tubular neighborhood of  $F_0$ . Then the following hold:*

1. *If  $F \cdot F_0 > 0$ , then  $n^-(F) \leq d_-(F)$ .*
2. *If  $F \cdot F_0 < 0$ , then  $n^+(F) \leq d_-(F)$ .*

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Received by the editors September 8, 2000 and, in revised form, November 1, 2000.

1991 *Mathematics Subject Classification.* Primary 57M99, 53C15.

The author was supported by the Graduiertenkolleg “Mathematik im Bereich ihrer Wechselwirkung mit der Physik” at the University of Munich.

Here  $n^+$  and  $n^-$  denote the numbers of positive and negative complex points of  $F$  (counted properly; see Section 2 for details), and  $d_{\pm}$  denotes the number of positive respectively negative double points. The proof of this theorem, which relies on Lai's results and facts from the Seiberg–Witten gauge theory, will be given in Section 4. In Section 2, we review the results of Lai we will need and relate them to the existence results for pseudoholomorphic curves proved in [B1]. In Section 3, we generalize one of the genus estimates given by Rokhlin in [Ro] to the case of immersed surfaces and use this to derive an inequality between the numbers of complex points and double points of certain immersions.

Finally, I would like to thank my advisor, Dieter Kotschick, for many helpful discussions and suggestions.

## 2. COMPLEX POINTS OF IMMERSIONS

In this section, we will briefly describe the paper [Lai] of H.F. Lai which contains a formula for the algebraic number of complex points of a surface in an almost complex 4-manifold, and show how this formula is related to the results of [B1]. We will then apply Lai's results to derive some relations between the number of complex points and the number of double points of immersed surfaces in almost complex 4-manifolds.

Suppose that  $\eta$  is a complex vector bundle of rank  $n$  over an oriented manifold  $X$  and that  $\eta$  splits as a direct sum

$$\eta = \xi \oplus \xi'$$

of a complex bundle  $\xi$  of rank  $(n - 1)$  and a complex line bundle  $\xi'$ . Then the Chern product formula for direct sums implies an obvious relation between the Chern classes of the bundles  $\eta, \xi$  and  $\xi'$ . The question Lai examined in his paper is the following. Suppose that we have a *real* subbundle  $\xi \hookrightarrow \eta$  of real dimension  $k = 2n - 2$ . Then again  $\eta$  splits as above, with the important difference that the splitting is now a splitting as a real bundle of rank  $2n$ . Are there still relations between the Euler classes of the bundles  $\xi$  and  $\xi'$  and the Chern classes of  $\eta$ ?

For this purpose, he uses a certain notion of “complex point” which we will now explain. Consider the bundle  $G_k(\eta)$  whose fibre over a point  $x \in X$  is the Grassmannian of oriented  $k$ -dimensional real subspaces of the fibre  $\eta_x$ . The inclusion  $\xi \hookrightarrow \eta$  defines a section (“Gauss map”)

$$t : X \rightarrow G_k(\eta)$$

in this bundle, given by  $t(x) = \xi_x$ . Let  $G_{n-1}^{\mathbb{C}}(\eta)$  denote the bundle of complex subspaces of dimension  $n - 1$  in the fibres of  $\eta$ . Note that every complex subspace carries a canonical orientation and is therefore an *oriented*  $k$ -dimensional real subspace. Hence we have a canonical inclusion  $G_{n-1}^{\mathbb{C}}(\eta) \rightarrow G_k(\eta)$  whose image will be denoted by  $K_{\eta}^+$  (this is  $K_{\eta}$  in [Lai]). Since the fibres of  $G_{n-1}^{\mathbb{C}}(\eta)$  are complex manifolds and  $X$  is oriented,  $K_{\eta}^+$  carries a natural orientation and therefore defines a homology class in  $H_*(G_k(\eta))$ . In a similar manner, we can fix an orientation on the fibre of  $G_k(\eta)$  (for example given by the Schubert calculus) to obtain an orientation of the total space  $G_k(\eta)$ . If we equip every complex subspace in  $\eta$  with the opposite, non-complex orientation, we obtain a second embedding of  $G_{n-1}^{\mathbb{C}}(\eta)$  into  $G_k(\eta)$  whose image will be denoted by  $K_{\eta}^-$ . Note that if  $\nu : G_k(\eta) \rightarrow G_k(\eta)$  denotes the involution given by reversing the orientation,  $K_{\eta}^- = \nu(K_{\eta}^+)$ . We will

orient  $K_\eta^-$  such that  $\nu$  maps the orientation of  $K_\eta^+$  onto minus the orientation of  $K_\eta^-$ . Lai then proved the following result (Theorem 5.10 in [Lai]).

**Theorem 2** (Lai). *Let  $\eta \rightarrow X$  be a complex vector bundle of rank  $n$  and  $\eta = \xi \oplus \xi'$  a splitting (as a real bundle) into oriented real vector bundles  $\xi$  and  $\xi'$  of ranks  $k = 2n - 2$  and  $2$ . Suppose that the orientations of  $\xi$  and  $\xi'$  are compatible with the complex orientation of  $\eta$  (i.e., an oriented basis of  $\xi$  together with an oriented basis of  $\xi'$  defines an oriented basis for  $\eta$ ), and let  $t : X \rightarrow G_k(\eta)$  denote the “Gauss section” defined by  $\xi$ . Then*

$$e(\xi) + \sum_{r=0}^{n-1} e(\xi')^r \cup c_{n-r-1}(\eta) = 2t^*PD(K_\eta^+),$$

where  $PD$  denotes Poincaré duality.

Note that Lai’s formula also implies a statement about  $t^*PD(K_\eta^-)$ , namely

$$(1) \quad e(\xi) + \sum_{r=0}^{n-1} (-1)^{r+1} e(\xi')^r \cup c_{n-r-1}(\eta) = 2t^*PD(K_\eta^-),$$

which can easily be derived from Theorem 2 by reversing the orientations of  $\bar{\xi}$  and  $\bar{\xi}'$ .

As an application of his formula, Lai considers a complex manifold  $X$  and an immersed surface  $F \subset X$ . He then proves an equation involving the Euler classes of the normal bundle of  $F$ , the Euler class of its tangent bundle, the Chern class of  $X$  and the algebraic number of complex points of  $F$ . In [CG], Chkhenkeli and Garrity observed that Lai’s arguments still hold if we consider almost complex manifolds instead of complex manifolds, since he only deals with vector bundles but does not make use of the fact that the complex structure on  $TX$  is integrable. However, there seems to be some confusion about the signs and the question of how to count complex points in [CG], so we work out this point in greater detail.

Let  $X$  be a 4-manifold which carries an almost complex structure  $J : TX \rightarrow TX$ . We orient  $X$  using the orientation given by  $J$ . Assume that  $\iota : F \hookrightarrow X$  is an immersion of a connected and oriented surface  $F$  into  $X$ . If, as above,  $G_2(TX)$  denotes the bundle of Grassmannians of 2-dimensional oriented real subspaces, the immersion defines a Gauss map  $t_F : F \rightarrow G_2(\iota^*TX)$ .

Now let us consider the submanifolds  $K_{\iota^*TX}^\pm$  in  $G_2(\iota^*TX)$ , which will be abbreviated by  $K^\pm$  in the sequel. The points  $x \in F$  with  $t_F(x) \in K^+$  are exactly the points where  $J$  respects the tangent space of  $F$  (i.e.,  $J$  commutes with  $d\iota$ ) and the orientation induced by  $J$  on  $T_xF$  equals the orientation of  $F$ . Similarly,  $t_F(x) \in K^-$  means that  $J$  preserves  $T_xF$ , but induces the opposite orientation on  $T_xF$ . We will call the points of the first kind **positive complex points** and the points of the second kind **negative complex points**.

For a generic immersion,  $t_F$  will be transversal to  $K^\pm$ , and since  $K^\pm$  has codimension 2 in  $G_2(\iota^*TX)$ , we have well-defined intersection numbers  $n^\pm$  between  $t_F(F)$  and  $K^\pm$ , given by the relation

$$(2) \quad n^\pm = \langle t_F^*PD([K^\pm]), [F] \rangle.$$

We will refer to these numbers as the algebraic sums of positive respectively negative complex points. Now suppose that  $\iota$  is an embedding. Then we have

$$\iota^*TX = N \oplus TF,$$

where  $N$  denotes the normal bundle, and  $\langle e(N), [F] \rangle = F \cdot F$  equals the self-intersection number of  $F$ . Using Lai's result, applied to  $\iota^*TX$ , we therefore obtain the following equation:

$$(3) \quad g(F) + n^+ = 1 + \frac{1}{2}(F \cdot F + \langle \iota^*c_1(X), [F] \rangle) = 1 + \frac{1}{2}(F \cdot F - K \cdot F).$$

Using the equation for  $\iota_F^*K^-$  derived above, we also obtain

$$(4) \quad g(F) + n^- = 1 + \frac{1}{2}(F \cdot F + K \cdot F).$$

A nice example is a holomorphic curve  $F$  in a complex surface  $X$ . Then the image of the Gauss map does not meet  $K^-$  at all, but is entirely contained in  $K^+$ . Therefore,  $n^- = 0$  and equation (4) is just the adjunction equality. So it seems that the “ $F \cdot C$ ” used by Chkhenkeli and Garrity should be our  $n^-$  to obtain the correct results.

Using equation (4), we are now able to give another formulation of the condition for the existence of an almost complex structure adapted to a surface as given in Lemma 1 of [B1], namely:

**Corollary 1.** *Let  $(X, J)$  be an almost complex 4-manifold and assume we are given an embedded surface  $F \subset X$ . Then the following conditions are equivalent:*

1. *There is a (generic)  $J'$  homotopic to  $J$  such that the algebraic number of negative complex points of  $F$  with respect to  $J'$  is zero.*
2. *There is a  $J''$  homotopic to  $J$  such that  $F$  is pseudoholomorphic with respect to  $J''$ .*

*Proof.* First suppose that the algebraic number  $n^-$  of negative complex points with respect to  $J'$  is zero. Then, according to equation (4), the adjunction equality is fulfilled and, by Lemma 1 in [B1], we can find an almost complex structure  $J''$  homotopic to  $J'$  and hence to  $J$  such that  $F$  is pseudoholomorphic with respect to  $J''$ . If, conversely,  $F$  is pseudoholomorphic with respect to  $J''$ , the adjunction equality holds for  $J''$ , and if we choose a generic  $J'$  homotopic to  $J$  (and  $J''$ ), equation (4) implies that  $n^- = 0$ .  $\square$

At this point, the author would like to point out that the proof of Lemma 2 in [B1] contains a minor gap, the arguments given there do not work in the special case  $b^+ = 2 = b^-$ . However, the assertion of the Lemma is true and a slight modification of the published version of the proof also works in this special case. A corrected version can be found in [B2].

Now let us relate Lai's results to the double points of immersed surfaces. In the sequel, we will always assume that an immersion of a surface is proper in the sense that the only singularities are ordinary double points. Recall that there is a natural way to attach a sign to a double point  $p$  of an oriented surface, depending on whether the orientations of the two branches of the surface meeting in  $p$  fit together to give the orientation of  $X$  at this point (then the sign should be +1) or not (sign -1). If the sign of a double point is +1, we will call this point a **positive double point**, otherwise it will be called a **negative double point**. We will need the following relation between the double points of an immersed surface, its normal Euler number and its self-intersection number.

**Lemma 1.** *Assume that  $X$  is a closed and oriented 4-manifold and  $\iota : F \hookrightarrow X$  is an immersion of a connected and oriented surface having  $d_+$  positive and  $d_-$  negative double points. Let  $N \rightarrow F$  denote the normal bundle of the immersion. Then*

$$e(N) = F \cdot F - 2d_+ + 2d_-.$$

*Proof.* For the sake of simplicity, let us assume that there is only one double point  $p$ . First consider the case that  $p$  is positive. Let  $\iota^{-1}(p) = \{x_1, x_2\}$  and choose an orientation preserving chart  $h : U \rightarrow \mathbb{R}^4$  around  $p$  such that:

1.  $h(U \cap \iota(F)) = h(U) \cap (\mathbb{R}^2 \times 0 \cup 0 \times \mathbb{R}^2)$ .
2. For small disks  $D_i$  in  $F$  around  $x_i$ ,  $h(\iota(D_1)) \subset \mathbb{R}^2 \times 0, h(\iota(D_2)) \subset 0 \times \mathbb{R}^2$  and the restrictions  $(h \circ \iota)|_{D_i}$  map the orientation of  $F$  to the canonical orientations of the planes  $\mathbb{R}^2 \times 0$  and  $0 \times \mathbb{R}^2$ .

Then a trivialization for the normal bundle  $N = \iota^*TX/TF$  restricted to  $D_1$  is given by  $dh \circ d\iota$ , followed by the projection onto the second plane  $0 \times \mathbb{R}^2$ , and with respect to this chart, a section of  $N|_{D_1}$  is given by the affine plane  $\mathbb{R}^2 \times 0 + \epsilon$  for a small  $\epsilon > 0$ . If we choose a tubular neighborhood  $\tau : N \hookrightarrow X$  which coincides with the map given by  $dh$  and  $h^{-1}$  around the  $x_i$ , the image of this section will be contained in  $h^{-1}(\mathbb{R}^2 \times 0 + \epsilon)$  and will intersect  $F$  in one positive point. A similar section can be constructed over  $D_2$ , and combined with a generic section of  $N$  outside of the  $D_i$ , we obtain an immersion  $\iota'$  of  $F$  which will intersect  $\iota(F)$  in  $2 + e(N)$  points, counted with signs.

If the double point  $p$  is negative, the sections constructed over the  $D_i$  will contribute with sign  $-1$  to the intersection number of  $\iota'(F)$  and  $\iota(F)$ , and we obtain  $F \cdot F = e(N) - 2$ . This proves our assertion in the case that there is only one double point, the proof in the general case is similar. □

Now let us combine Lai's work (namely equation (1)) with Lemma 1 to derive a relation between the homology class of an immersed surface, its genus, the number of complex points and the number of double points. We then obtain the following:

**Proposition 1.** *Let  $(X, J)$  be an almost complex 4-manifold and  $F \hookrightarrow X$  a generic immersion of an oriented surface  $F$  having  $d_+$  positive and  $d_-$  negative double points. Let  $n^-$  denote the algebraic number of negative complex points (as defined by equation (2)). Then*

$$g(F) + n^- - d_- + d_+ = 1 + \frac{1}{2}(F \cdot F + K \cdot F).$$

*Proof.* We have a decomposition  $\iota^*TX = TF \oplus N$ , to which we can apply equation (1) to obtain

$$(2 - 2g) - c_1(J) \cdot F + e(N) = 2n^-.$$

By Lemma 1,  $e(N) = F \cdot F - 2d^+ + 2d^-$ . Substituting this into the last equation leads to the desired result. □

*Remark 1.* Assume that  $F$  is an immersed surface with  $d_{\pm}(F)$  double points and  $n^{\pm}(F)$  complex points. Let  $\bar{F}$  denote the same surface with the reversed orientation. Then clearly  $d_{\pm}(\bar{F}) = d_{\pm}(F)$ , and hence Proposition 1 yields the relations  $n^+(\bar{F}) = n^-(F)$  and  $n^-(\bar{F}) = n^+(F)$ .

## 3. IMMERSED SURFACES AND GENUS ESTIMATES

Proposition 1 can be used to derive estimates for the number of negative complex points of immersed surfaces (note that these estimates always include estimates on the algebraic number of positive complex points, since Theorem 2 and equation (1) together imply  $n^- - n^+ = K \cdot F$ ). As a first example, we will now prove a bound for the number of double points of an immersion by using branched covers as in [Ro].

As the author was informed when this paper was finished, a strengthened version of this proposition also follows from a more general result due to P.M. Gilmer (see Theorem 4.1 in [Gi]).

**Proposition 2.** *Let  $X$  be a closed oriented and simply-connected 4-manifold and  $F \subset X$  be an immersed surface of genus  $g$  having  $d_+$  positive and  $d_-$  negative double points. Assume that the homology class  $[F]$  of  $F$  is divisible by a prime power  $m$ . Then*

$$d_+ + g \geq \frac{m+1}{6m} F \cdot F - b_2^+(X).$$

*Proof.* As demonstrated in [FS], we can find an immersed surface in  $X \# \overline{\mathbb{C}P^2}$  having  $d_+$  positive double points,  $d_- - 1$  negative double points and representing the class  $([F], 0) \in H_2(X \# \overline{\mathbb{C}P^2}; \mathbb{Z}) = H_2(X; \mathbb{Z}) \oplus \mathbb{Z}$ . In fact, pick two generic lines in  $\overline{\mathbb{C}P^2}$  and reverse the orientation of one of them to obtain two spheres  $S_1, S_1$  which intersect in one point with intersection number  $+1$ . Remove a small ball  $B$  around this point and a similar ball  $B'$  around one negative double point of  $F$ . We then can glue  $X \setminus B'$  and  $\overline{\mathbb{C}P^2} \setminus B$  along their boundaries in such a way that the boundary links  $F \cap B'$  and  $(S_1 \cup S_2) \cap B$  get identified. This yields a new immersed surface in  $X \# \overline{\mathbb{C}P^2}$  as desired. By iterating the construction, one can construct an immersed surface in  $X' = X \# d_- \overline{\mathbb{C}P^2}$  having  $d_+$  positive self-intersection points and representing the class  $([F], 0, \dots, 0)$ .

Cutting out small disks and gluing in handles at the remaining double points leads to a surface  $\Sigma \subset X'$  having genus  $d_+ + g$  and self-intersection number  $F \cdot F$ . Since the homology class  $[F]$  was divisible by  $m$ , the same is true for  $[\Sigma]$ . Now let  $Y \rightarrow X'$  denote the branched cover of order  $m$  with branch locus  $\Sigma$ . As in [Ro], one can use this cover to obtain estimates for the genus of  $\Sigma$ . We have

$$\begin{aligned} b_2(Y) &= mb_2(X') + 2(m-1)(d_+ + g) \\ &= mb_2(X) + md_- + 2(m-1)(d_+ + g) \end{aligned}$$

and

$$\tau(Y) = m\tau(X) - md_- - \frac{m^2 - 1}{3m} F \cdot F.$$

Now the real cohomology  $H^2(X; \mathbb{R})$  appears in  $H^2(Y; \mathbb{R})$  as the subspace invariant under the action of  $\mathbb{Z}_m$ , and the splitting of  $H^2(Y; \mathbb{R})$  into the eigenspaces of the action is orthogonal with respect to the intersection form (see [Ro]), hence we have the inequality

$$|\tau(Y) - \tau(X)| \leq b_2(Y) - b_2(X),$$

in particular,

$$-\tau(Y) + \tau(X) \leq b_2(Y) - b_2(X).$$

If we substitute the values for  $b_2$  and  $\tau$  from the above equation, we see that the terms containing  $d_-$  cancel out, and this leads to the desired result.  $\square$

Note that reversing the orientation of  $X$  gives an estimate for the number of negative double points. Furthermore, the estimate of Rohklin (inequality 6.3. in [Ro]) appears in the result as the special case  $d_+ = d_- = 0$ .

**Example 1.** There is a simple example where the estimate in Proposition 2 is sharp. Consider the class  $a = (3, 3) \in H_2(\mathbb{C}P^2 \# \mathbb{C}P^2; \mathbb{Z})$ . Suppose there is an immersed sphere  $S$  representing  $a$  which has  $d_+$  positive and  $d_-$  negative double points. Proposition 2 then shows that  $d_+ \geq 2$ . On the other hand, the class  $3[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2; \mathbb{Z})$  can be realized by a sphere with one positive self-intersection point. Taking two copies of this immersed sphere shows that the class  $(3, 3)$  can be represented by a sphere having two positive double points. Hence the bound from Proposition 2 is sharp in this case.

**Example 2.** The result of Fintushel and Stern in [FS] can be summarized by the statement that for a sphere immersed in a rational surface, the minimal number of positive double points is at least the minimal genus of its homology class. The estimate in Proposition 2 goes in the same direction, since its right-hand side equals the right-hand side in [Ro], 6.3 (up to the absolute value). There is a simple example showing that in general, one cannot estimate the number of positive double points by the minimal genus (the same is true for the number of negative double points).

First note that the minimal genus of the class  $a = (3, 1) \in H_2(\mathbb{C}P^2 \# \mathbb{C}P^2; \mathbb{Z})$  is one. In fact, a torus representing  $a$  is obtained by tubing together algebraic representatives in both factors, and by [KeM],  $a$  cannot be represented by a sphere. In the same paper, Kervaire and Milnor show how to represent the class  $(3, 0)$  by a sphere. From a configuration of three lines in  $\mathbb{C}P^2$ , we obtain a sphere with one positive double point representing  $3[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2; \mathbb{Z})$ . Take two lines in  $\mathbb{C}P^2$  and reverse the orientation of one of them to produce two spheres that intersect each other in one negative intersection point. Cutting out balls around the intersection points and identifying the boundaries leads to a sphere  $S \in \mathbb{C}P^2 \# \mathbb{C}P^2$  that represents  $(3, 0)$ .

From the construction one can deduce that  $S$  will intersect a generic line  $\gamma$  in the second factor twice in one negative and one positive intersection point. Tubing together at the positive point leads to an immersed sphere representing the class  $a$  with  $d_+ = 0, d_- = 1$ . This provides an example of an immersed sphere for which the number of positive self-intersection points is smaller than the minimal genus in this homology class. Gluing at the other point gives a representative of the same class with  $d_+ = 1, d_- = 0$ , thus showing that also the number of negative intersection points can be smaller than the minimal genus. Reversing the orientation of the line finally leads to immersed spheres representing the class  $(3, -1)$ .

As already indicated above, we can now combine the estimate given in Proposition 2 and the relation between complex points and double points of an immersion to obtain the following result.

**Corollary 2.** *Suppose that  $X$  is a simply-connected almost complex 4-manifold with canonical class  $K$  and  $F \hookrightarrow X$  an immersion of a surface of genus  $g$  having  $d_+$  positive and  $d_-$  negative double points. Assume that the homology class of  $F$  is divisible by a prime power  $m$ . As usual, let  $n^-$  denote the algebraic number of*

negative complex points. Then

$$d_- - n^- \geq \frac{1 - 2m}{6m} F \cdot F - \frac{1}{2} K \cdot F - b_2^+(X) - 1.$$

*Proof.* This follows immediately from Proposition 2 and Proposition 1. □

**Example 3.** Consider the homology class  $3[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2; \mathbb{Z})$ . Suppose that  $S$  is an immersed sphere representing this class. Then  $S$  cannot be an embedding, because the minimal genus in this homology class is 1 (see [KeM]), hence there must be at least one double point (as always, we assume that the immersion is generic with double points as the only singularities). In fact, Proposition 2 implies that at least one of the double points must have sign +1 (this follows also from [FS]), i.e.,  $d_+ > 0$ . By Corollary 2, we have  $n^- \leq d_-$ .

Since the estimate of Rokhlin is sharp for this homology class, we expect that the inequality  $n^- \leq d_-$  is also sharp, which turns out to be true. For an example where equality occurs, consider the algebraic curve of degree 3 given by the equation  $x^3 + y^3 = 3xyz$ . This curve has one node at the point  $[0 : 0 : 1]$ . From our point of view, it defines an immersion of a sphere with one positive double point which represents the homology class  $3[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2; \mathbb{Z})$  and is pseudoholomorphic with respect to the canonical almost complex structure on  $\mathbb{C}P^2$ . Hence we have  $n^- = d_- = 0$ .

#### 4. PROOF OF THEOREM 1

It is sufficient to prove the first assertion, the second then follows by reversing the orientation of  $F$ , using Remark 1. Let  $T$  denote a tubular neighborhood of  $F_0$  in which  $F$  is contained. Since  $H_2(T; \mathbb{Z}) = H_2(F_0; \mathbb{Z}) = \mathbb{Z}$ , there is a unique integer  $k$  such that  $k[F_0] = [F] \in H_2(T; \mathbb{Z})$ . Clearly this relation also holds in  $H_2(X; \mathbb{Z})$  and therefore  $k > 0$  since  $F_0 \cdot F = kF_0 \cdot F_0 > 0$  and  $F_0 \cdot F_0 > 0$  by assumption. First we will prove the estimate

$$(5) \quad d_+(F) + g(F) \geq 1 + k(g(F_0) - 1) + \binom{k}{2} F_0 \cdot F_0.$$

Note that this is a version of the so-called ‘‘local Thom conjecture’’ for immersed surfaces. In the special case of embedded surfaces, this conjecture has been proved by Lawson (see [La]).

For the proof of (5), we will not make use of the almost complex structure on  $X$  and of the fact that  $F_0$  is a pseudoholomorphic curve. By general position, we can assume that  $F$  and  $F_0$  intersect transversely, in particular, no double point of  $F$  is lying on  $F_0$ . Pick a complex structure on  $F_0$  and a holomorphic line bundle  $L$  over  $F_0$  having degree  $\deg(L) = \langle c_1(L), [F_0] \rangle = F_0 \cdot F_0$ . If we choose a metric on  $L$ , we can identify  $T$  with the unit disk bundle of  $L$ . Let  $E = L \oplus \mathbb{C}$  and let  $Y = \mathbb{P}E$  be the total space of the projective bundle associated to  $E$ . Then  $Y$  is an algebraic surface, and  $b_2^+(Y) = 1 + 2p_g(Y) = 1$  (see for instance [BPV], IV.2.6).

We have an embedding  $L \hookrightarrow Y$ , given by  $l \mapsto [(l, 1)]$ , and the image of  $F_0$  under this embedding—that again will be denoted by  $F_0$ —is an algebraic curve in  $Y$  having self–intersection number  $F_0 \cdot F_0 = \deg(L)$ . If  $K$  denotes the canonical class of  $Y$ , this implies  $K \cdot F_0 = 2g(F_0) - 2 - F_0 \cdot F_0$ .

Now let  $Y' = Y \#_{d_-(F)} \overline{\mathbb{C}P^2}$ , where the blow–up is performed at the positive self–intersection points of  $F$ . Using the construction of [FS] as in the proof of

Proposition 2, we can construct an embedded surface in  $Y'$  representing the homology class  $[F] = k[F_0]$  having  $d_+$  positive double points (here we think of  $H_2(Y; \mathbb{Z})$  as a subgroup of  $H_2(Y'; \mathbb{Z})$ ). Replacing the remaining double points by handles leads to an embedded surface  $F'$  of genus  $g(F) + d_+$  with  $[F'] = [F]$ . Note that  $Y'$  is again an algebraic surface with canonical class  $K' = K - \sum_i E_i$ , where  $E_i$  denotes the exceptional curve in  $\overline{\mathbb{C}P^2}$ .

Choose a Kähler metric  $g'$  on  $Y'$  and let  $\omega'$  denote its fundamental form. Since  $F_0 \subset Y'$  is holomorphic, we have  $[\omega'] \cdot [F_0] > 0$ . Now the Kähler metric  $g'$  defines a symplectic structure with symplectic form  $\omega'$ , and  $[F'][\omega'] = k[F_0][\omega'] > 0$ ,  $F' \cdot F' = k^2 F_0 \cdot F_0 > 0$ , hence we can apply Theorem E in [LL] to conclude that

$$g(F') \geq 1 + \frac{1}{2}(K' \cdot F' + F' \cdot F').$$

Substituting the values for  $g(F')$  and  $[F']$  leads to

$$d_+ + g(F) \geq 1 + \frac{1}{2}(k(2g(F_0) - 2 - F_0 \cdot F_0) + k^2 F_0 \cdot F_0),$$

and inequality (5) follows.

Let us now proceed with the proof of Theorem 1. The inequality (5) is an estimate for the genus of  $F$  and the number of double points in terms of its homology class. In a second step, we can now use Proposition 1 to obtain an inequality between the number of complex points and the number of double points. In fact, Proposition 1 and the inequality (5) together imply

$$d_-(F) - n^-(F) \geq k g(F_0) - \frac{1}{2} k K \cdot F_0 - k - \frac{1}{2} k F_0 \cdot F_0.$$

The right-hand side of this estimate can still be simplified using the fact that  $F_0$  was assumed to be pseudoholomorphic and thus the adjunction equality holds. Therefore, we finally obtain

$$d_-(F) - n^-(F) \geq 0,$$

and this is the desired estimate.

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