L$^p$ ESTIMATES ON FUNCTIONS OF MARKOV OPERATORS

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Abstract. We prove L$^p$ estimates for functions of Markov operators on a discrete measure space of superpolynomial volume growth.

Let $X$ be a discrete, measurable space endowed with a measure $dx$ and a measurable distance $d(\cdot,\cdot)$. Let us denote by $B(x;r)$ the ball of center $x$ and radius $r$. If $|B(x,r)|$ is the $dx$-measure of $B(x,r)$, we assume that there exist $0 < \alpha' \leq \alpha \leq 1$ and $\kappa, \kappa', c, c' > 0$ such that

$$c'e^{\kappa' r^{\alpha'}} \leq |B(x,r)| \leq ce^{\kappa r^\alpha}, \quad \forall x \in X, r > 0,$$

i.e. $X$ has superpolynomial ($\alpha < 1$) or exponential ($\alpha' = \alpha = 1$) volume growth.

Let us consider a bounded symmetric Markov kernel $P(x,y)$ on $X$ and let us set $P_0(x,y) = \delta_x(y)$, where $\delta_x$ is the Dirac mass at $x$, $P_1(x,y) = P(x,y)$ and $P_n(x,y) = \int P_{n-1}(x,z)P(z,y)dz$ for $n \geq 2$. We assume that there exists constants $c, \beta > 0$ such that

$$P_n(x,y) \leq ce^{-\beta d(x,y)^2/n}$$

for any $x, y \in X$ and $n \in \mathbb{N}$.

Markov chains with transition kernels satisfying an estimate such as (2) were first studied by N.Th. Varopoulos [8]. T.K. Carne [3] proves (2) in the case when $X$ is countable by improving the result of [8]. G. Alexopoulos [1] generalised this in the context of continuous groups.

In the presence of a group structure on $X$, translation invariant, symmetric Markov kernels are obtained by the convolution powers of a probability measure $\mu$ on $X$. In fact, if $\mu$ has a bounded symmetric density $f$ with respect to the left invariant Haar measure $dx$, then the Markov kernel

$$P(x,y) = f(x^{-1}y)$$

is translation invariant and satisfies

$$P_n(x,y) = f^n(x^{-1}y)$$

where $f^n$ is the $n-$convolution power of $f$. 

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Let \(|x|\) be a word distance. Then by [8], [3] and [1], there exist a \(\beta > 0\) such that
\[
f^n(x^{-1}y) \leq c e^{-\beta \frac{|x^{-1}y|^2}{n}},
\]
for any \(x, y \in X\) and \(n \in \mathbb{N}\).

It is worth mentioning that every locally compact group is at most of exponential volume growth. Further, in [5], Grigorchuck proved that there exist discrete finitely generated groups such that
\[
ce^\alpha \leq |B(x, r)| \leq C e^{r^{\alpha'}}
\]
with \(\alpha, \alpha' \in (0, 1)\). In this case, for a class of symmetric and bounded probability densities \(f\), one can prove that
\[
f^n(x^{-1}y) \leq c e^{-\frac{2^{1/2}}{e} - \beta \frac{|x^{-1}y|^2}{n}}, \quad \forall x, y \in X;
\]
see [6], Remark 1, p. 690.

If \(P\) is the Markov operator with kernel \(P(x,y)\), then \(I-P\) is symmetric, positive, bounded on \(L^2\) and admits the spectral decomposition
\[
I-P = \int_0^{\infty} \lambda dE_\lambda.
\]
Also, for any bounded Borel function \(m\) on \(\mathbb{R}\), by the spectral theorem we can define the operator
\[
m(I-P) = \int_0^{\infty} m(\lambda) dE_\lambda
\]
which is bounded on \(L^2\).

Let us consider the following class \(T\) of Borel functions: \(m \in T\) iff its Fourier transform satisfies
\[
|\hat{m}(t)| \leq c e^{-W|t|}, \quad \forall t \in \mathbb{R},
\]
for some \(W > 0\). The class \(T\) is of the type of multipliers introduced in [4] and [7]. In fact, the class \(\mathcal{F}_0(e^{-W|t|}, b), b > 0\) ([7], p. 787), contains functions \(m\) which satisfy
\[
|\hat{m}^{(k)}(t)| \leq c \left(\frac{k}{b}\right)^k e^{-W|t|}
\]
for any \(t\) and \(k \geq 0\).

We note that if \(m\) is smooth in the zone \(\Omega_W = \{\lambda \in \mathbb{C} : |\text{Im} \lambda| \leq W\}\) and holomorphic on \(\Omega_W\), then it belongs to \(\mathcal{F}_0(e^{-W|t|}, b)\) for some \(b > 0\), iff
\[
|m(\lambda)| \leq c \left(\frac{k}{b}\right)^k (1 + |\lambda|)^{-k/2}
\]
for any \(\lambda \in \overline{\Omega}\) and \(k \geq 0\) ([2], Lemma 5.5).

In [2], G. Alexopoulos proved an analog of the Mikhlin-Hörmander multiplier theorem for random walks on discrete groups of polynomial volume growth. In this article we prove the following analog of the main result of M. Taylor [7].

**Theorem.** Let as assume that \(P_n\) satisfies [2], \(m \in T\) and that either
(i) \(X\) is of superpolynomial volume growth but not exponential, i.e. assumption [1] is valid with \(\alpha', \alpha \in (0, 1)\),
(ii) $X$ is of exponential volume growth and $\beta > \kappa + \delta$, $W \delta > \frac{\xi}{2}$ where $\delta$ is the supremum of $\eta \in (0, e^{-1})$ such that $\eta \leq \max(\beta, e^{-\beta})$.

Then $m(I - P)$ is bounded on $L^p$, $p \geq 1$.

The proof of the Theorem is based on the following lemmas.

For a fixed $y \in X$ we shall denote by $A_p(y)$, $p \in \mathbb{N}$, the shell $\{x : 2^{p/2} \leq d(x, y) \leq 2^{(p+1)/2}\}$. Let us also recall that the Dirac mass $\delta_y$ at $y$ is in $L^2(X)$ if $X$ is discrete.

**Lemma 1.** There is $\eta \in (0, e^{-1})$ with $\eta \leq \max(\beta, e^{-\beta})$ such that for all $p \in \mathbb{N}$, $x \in A_p(y)$ and $\|t\| \leq \eta 2^{p/2}$,

$$|e^{itP}(\delta_y)(x)| \leq ce^{-(\beta - \eta)d(x, y)}. \quad (4)$$

**Proof.** We have that

$$e^{itP}\delta_y(x) = \sum_{n \geq 0} \frac{(it)^n}{n!} P^n \delta_y(x)$$

$$= \sum_{n \leq 2^{p/2}} \frac{(it)^n}{n!} P^n(x, y) + \sum_{n > 2^{p/2}} \frac{(it)^n}{n!} P^n(x, y) = I_1 + I_2.$$ 

It follows from (4) that for $x \in A_p(y)$ and $\|t\| \leq \eta 2^{p/2}$

$$|I_1| \leq c \sum_{n \leq 2^{p/2}} \frac{(\eta 2^{p/2})^n}{n!} e^{-\beta 2^{p/2}}$$

$$\leq ce^{-\beta 2^{p/2}} e^{\eta 2^{p/2}} = ce^{-(\beta - \eta)2^{p/2}}.$$

By (2) and Stirling’s formula $n! \sim \sqrt{n} \left(\frac{n}{e}\right)^n$, we get that

$$|I_2| \leq c \sum_{n > 2^{p/2}} \frac{(\eta 2^{p/2})^n}{n!} \leq c \sum_{n > 2^{p/2}} \left(\frac{\eta 2^{p/2} e}{n}\right)^n \frac{1}{\sqrt{n}}$$

$$\leq c \frac{1}{2^{(p-1)/4}} \sum_{n > 2^{p/2}} (\eta e)^n \leq c \frac{1}{2^{(p-1)/4}} (\eta e)^{2^{p/2}}$$

provided that $\eta e < 1$.

Now,

$$(\eta e)^{2^{p/2}} = e^{2^{p/2} \log(\eta e)} \leq e^{-(\beta - \eta)2^{p/2}}$$

provided that

$$\log(\eta e) \leq -(\beta - \eta)$$

which holds true if

$$\eta e \leq e^\eta e^{-\beta} \text{ or } \eta \leq e^{-\beta}$$

since $\eta < 1$.

Since

$$m(I - P) = \int_{\mathbb{R}} \hat{m}(t)e^{it(I - P)}dt$$
the kernel $K(x, y)$ of the operator $m(I - P)$ is given by

$$K(x, y) = \int_{\mathbb{R}} \hat{m}(t)e^{it(I - P)}\delta_y(x)dt$$

$$= \int_{|t| \leq \eta 2^{p/2}} + \int_{|t| \geq \eta 2^{p/2}} = K_0(x, y) + K_\infty(x, y).$$

We have the following:

**Lemma 2.** Let $\eta$ be as in the Theorem. Then there is $c > 0$ such that

$$\int_X |K_0(x, y)| \, dx \leq c \sum_{p \geq 0} e^{-\eta(W - \epsilon)2^{p/2}} 2^{p/2} |A_p(y)|.$$

Also, for any $\epsilon > 0$, there is $c > 0$ such that

$$\int_X |K_\infty(x, y)| \, dx \leq c \sum_{p \geq 0} e^{-\eta(W - \epsilon)2^{p/2}} |A_p(y)|^{1/2}.$$

**Proof.** It follows from (4) and (3) that for $x \in A_p(y)$

$$|K_0(x, y)| \leq c \int_{|t| \leq \eta 2^{p/2}} |\hat{m}(t)| e^{-(\beta - \eta)d(x, y)} \, dt$$

$$\leq c e^{-(\beta - \eta)2^{p/2}} \eta 2^{p/2}.$$

Thus, by using (11)

$$\int_X |K_0(x, y)| \, dx = \sum_{p \geq 0} \int_{A_p(y)} |K_0(x, y)| \, dx$$

$$\leq c \sum_{p \geq 0} e^{-(\beta - \eta)2^{p/2}} \eta 2^{p/2} \int_{A_p(y)} dx$$

$$\leq c \sum_{p \geq 0} e^{-(\beta - \eta)2^{p/2}} 2^{p/2} |A_p(y)|.$$

On the other hand, by the Cauchy-Schwarz inequality, (11), (2), the decay of $\hat{m}(t)$ for $t$ large and the fact that the Dirac mass $\delta_y \in L^2(X)$, we get that

$$\int_{A_p(y)} |K_\infty(x, y)| \, dx \leq \int_{A_p(y)} \left( \int_{|t| \geq \eta 2^{p/2}} |\hat{m}(t)| e^{it(I - P)}\delta_y(x) \, dt \right) \, dx$$

$$\leq c \int_{|t| \geq \eta 2^{p/2}} |\hat{m}(t)| dt \int_{A_p(y)} \left| e^{itP}\delta_y(x) \right| \, dx$$

$$\leq c \int_{|t| \geq \eta 2^{p/2}} |\hat{m}(t)| |A_p(y)|^{1/2} \left\| e^{itP} \right\|_2 \|\delta_y\|_2 dt$$

$$\leq c |A_p(y)|^{1/2} \int_{|t| \geq \eta 2^{p/2}} |\hat{m}(t)| dt$$

$$\leq c |A_p(y)|^{1/2} e^{-\eta(W - \epsilon)2^{p/2}} \int_{|t| \geq \eta 2^{p/2}} e^{-\epsilon|t|} \, dt$$

$$\leq c |A_p(y)|^{1/2} e^{-\eta(W - \epsilon)2^{p/2}}.$$
Therefore,
\[
\int_X |K_\infty(x, y)| \, dx = \sum_{p \geq 0} \int A_p(y) |K_\infty(x, y)| \, dx \\
\leq c \sum_{p \geq 0} c |A_p(y)|^{1/2} e^{-W_\varepsilon 2^p/2}.
\]

Proof of the Theorem. (i) Let us assume that (1) holds for \( \alpha', \alpha \in (0, 1) \). Then by (5) and (6) we get that
\[
\int_X |K(x, y)| \, dx \\
\leq c \sum_{p \geq 0} e^{-(\beta - \eta)2^p/2} 2^p/2 |A_p(y)| + c \sum_{p \geq 0} e^{-\eta(W_\varepsilon)2^p/2} |A_p(y)|^{1/2} \\
\leq c \sum_{p \geq 0} e^{-(\beta - \eta)2^p/2} 2^p/2 e^{\kappa(2^p/2)\alpha} + \sum_{p \geq 0} e^{-\eta(W_\varepsilon)2^p/2} e^{\frac{\kappa}{2}(2^p/2)\alpha} < \infty.
\]
This implies that \( m(I - P) \) is bounded on \( L^\infty(X) \). As already mentioned, by the spectral theorem \( m(I - P) \) is bounded on \( L^2(X) \) and by interpolation and duality we obtain the boundedness of \( m(I - P) \) on \( L^p(X) \), \( p \geq 1 \).

(ii) Similarly, if \( \|B(x, r)\| \leq c e^{\kappa r} \), then
\[
\int_X |K(x, y)| \, dx \\
\leq c \sum_{p \geq 0} e^{-(\beta - \eta)2^p/2} 2^p/2 e^{\kappa2^p/2} + \sum_{p \geq 0} e^{-\eta(W_\varepsilon)2^p/2} e^{\frac{\kappa}{2}2^p/2} < \infty,
\]
provided that \( \beta > \kappa + \eta \) and \( \eta W > \kappa/2 \) and the boundedness of \( m(I - P) \) on \( L^\infty(X) \) follows.

REFERENCES


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