Lp ESTIMATES ON FUNCTIONS OF MARKOV OPERATORS

MICHEL MARIAS

(Communicated by Christopher D. Sogge)

Abstract. We prove Lp estimates for functions of Markov operators on a discrete measure space of superpolynomial volume growth.

Let X be a discrete, measurable space endowed with a measure dx and a measurable distance d(.,.). Let us denote by B(x,r) the ball of center x and radius r. If |B(x,r)| is the dx-measure of B(x,r), we assume that there exist 0 < α' ≤ α ≤ 1 and κ, κ', c, c' > 0 such that

\[ c' e^{\kappa' r^{\alpha'}} \leq |B(x,r)| \leq c e^{\kappa r^\alpha}, \quad \forall x \in X, \ r > 0, \]

i.e. X has superpolynomial (α < 1) or exponential (α' = α = 1) volume growth.

Let us consider a bounded symmetric Markov kernel P(x,y) on X and let us set P_0(x,y) = δ_x(y), where δ_x is the Dirac mass at x, P_1(x,y) = P(x,y) and P_n(x,y) = \int P_{n-1}(x,z)P(z,y)dz for n ≥ 2. We assume that there exists constants c, β > 0 such that

\[ P_n(x,y) \leq c e^{-\beta d(x,y)^2/n}, \]

for any x, y ∈ X and n ∈ N.

Markov chains with transition kernels satisfying an estimate such as (2) were first studied by N.Th. Varopoulos [8]. T.K. Carne [3] proves (2) in the case when X is countable by improving the result of [8]. G. Alexopoulos [1] generalised this in the context of continuous groups.

In the presence of a group structure on X, translation invariant, symmetric Markov kernels are obtained by the convolution powers of a probability measure μ on X. In fact, if μ has a bounded symmetric density f with respect to the left invariant Haar measure dx, then the Markov kernel

\[ P(x,y) = f(x^{-1}y) \]

is translation invariant and satisfies

\[ P_n(x,y) = f^n(x^{-1}y) \]

where f^n is the n-convolution power of f.
Let $|x|$ be a word distance. Then by [8], [3] and [1], there exist a $\beta > 0$ such that

$$f^n(x^{-1}y) \leq c e^{-\beta \frac{|x^{-1}y|^2}{n}},$$

for any $x, y \in X$ and $n \in \mathbb{N}$.

It is worth mentioning that every locally compact group is at most of exponential volume growth. Further, in [5], Grigorchuck proved that there exist discrete finitely generated groups such that

$$ce^{\alpha} \leq |B(x, r)| \leq Ce^{\alpha'}$$

with $\alpha, \alpha' \in (0, 1)$. In this case, for a class of symmetric and bounded probability densities $f$, one can prove that

$$f^n(x^{-1}y) \leq c e^{-\frac{\alpha/2}{\alpha} \frac{|x^{-1}y|^2}{n}}, \quad \forall x, y \in X;$$

see [5], Remark 1, p. 690.

If $P$ is the Markov operator with kernel $P(x, y)$, then $I - P$ is symmetric, positive, bounded on $L^2$ and admits the spectral decomposition

$$I - P = \int_0^\infty \lambda dE_\lambda.$$

Also, for any bounded Borel function $m$ on $\mathbb{R}$, by the spectral theorem we can define the operator

$$m(I - P) = \int_0^\infty m(\lambda) dE_\lambda$$

which is bounded on $L^2$.

Let us consider the following class $T$ of Borel functions: $m \in T$ iff its Fourier transform satisfies

$$|\hat{m}(t)| \leq ce^{-W|t|}, \quad \forall t \in \mathbb{R},$$

for some $W > 0$. The class $T$ is of the type of multipliers introduced in [4] and [7]. In fact, the class $F_0(e^{-W|t|}, b), b > 0$ ([7], p. 787), contains functions $m$ which satisfy

$$|\hat{m}(k)(t)| \leq c \left(\frac{k}{b}\right)^k e^{-W|t|}$$

for any $t$ and $k \geq 0$.

We note that if $m$ is smooth in the zone $\Omega_W = \{ \lambda \in \mathbb{C} : |\text{Im} \lambda| \leq W \}$ and holomorphic on $\Omega_W$, then it belongs to $F_0(e^{-W|t|}, b)$ for some $b > 0$, iff

$$|m(\lambda)| \leq c \left(\frac{k}{b}\right)^k (1 + |\lambda|)^{-k/2}$$

for any $\lambda \in \Omega$ and $k \geq 0$ ([2], Lemma 5.5).

In [2], G. Alexopoulos proved an analog of the Mikhlin-Hörmander multiplier theorem for random walks on discrete groups of polynomial volume growth. In this article we prove the following analog of the main result of M. Taylor [7].

**Theorem.** Let as assume that $P_n$ satisfies [2], $m \in T$ and that either

(i) $X$ is of superpolynomial volume growth but not exponential, i.e. assumption [1] is valid with $\alpha', \alpha \in (0, 1)$,
(ii) $X$ is of exponential volume growth and $\beta > \kappa + \delta$, $W_\delta > \frac{\delta}{2}$ where $\delta$ is the supremum of $\eta \in (0, e^{-1})$ such that $\eta \leq \max(\beta, e^{-\beta})$.

Then $m(I - P)$ is bounded on $L^p$, $p \geq 1$.

The proof of the Theorem is based on the following lemmas.

For a fixed $y \in X$ we shall denote by $A_p(y)$, $p \in \mathbb{N}$, the shell $\{x : 2^{p/2} \leq d(x, y) \leq 2^{(p+1)/2}\}$. Let us also recall that the Dirac mass $\delta_y$ at $y$ is in $L^2(X)$ if $X$ is discrete.

Lemma 1. There is $\eta \in (0, e^{-1})$ with $\eta \leq \max(\beta, e^{-\beta})$ such that for all $p \in \mathbb{N}$, $x \in A_p(y)$ and $|t| \leq \eta 2^p$, 

$$|e^{itP}(\delta_y)(x)| \leq ce^{-(\beta - \eta)d(x, y)}.$$  

Proof. We have that

$$e^{itP}\delta_y(x) = \sum_{n \geq 0} \frac{(it)^n}{n!} P^n \delta_y(x)$$

$$= \sum_{n \leq 2^p/2} \frac{(it)^n}{n!} P^n(x, y) + \sum_{n > 2^p/2} \frac{(it)^n}{n!} P^n(x, y) = I_1 + I_2.$$

It follows from (4) that for $x \in A_p(y)$ and $|t| \leq \eta 2^p/2$

$$|I_1| \leq c \sum_{n \leq 2^p/2} \frac{(\eta 2^p/2)^n}{n!} e^{-\beta 2^p/2}$$

$$\leq ce^{-\beta 2^p/2} e^{(\eta 2^p/2)} = ce^{-(\beta - \eta)2^p/2}.$$  

By (2) and Stirling’s formula $n! \sim \sqrt{n} \left(\frac{n}{e}\right)^n$, we get that

$$|I_2| \leq c \sum_{n > 2^p/2} \frac{(\eta 2^p/2)^n}{n!} \leq c \sum_{n > 2^p/2} \left(\frac{\eta 2^p/2}{n}\right)^n \frac{1}{\sqrt{n}}$$

$$\leq c \frac{1}{2^{(p-1)/4}} \sum_{n > 2^p/2} (\eta e)^n \leq c \frac{1}{2^{(p-1)/4}} (\eta e)^{2^p/2}$$

provided that $\eta e < 1$.

Now,

$$(\eta e)^{2^p/2} = e^{2^{p/2} \log(\eta e)} \leq e^{-(\beta - \eta)2^p/2}$$

provided that

$$\log(\eta e) \leq -(\beta - \eta)$$

which holds true if

$$\eta e \leq e^\eta e^{-\beta} \text{ or } \eta \leq e^{-\beta}$$

since $\eta < 1$. \hfill \square

Since

$$m(I - P) = \int_{\mathbb{R}} \hat{m}(t)e^{it(I-P)}dt$$
the kernel $K(x, y)$ of the operator $m(I - P)$ is given by
\[ K(x, y) = \int_{\mathbb{R}} \hat{m}(t)e^{it(I-P)}\delta_y(x)dt \]
\[ = \int_{|t|\leq \eta 2^{p/2}} + \int_{|t|\geq \eta 2^{p/2}} = K_0(x, y) + K_\infty(x, y). \]

We have the following:

**Lemma 2.** Let $\eta$ be as in the Theorem. Then there is $c > 0$ such that
\[ \int_X |K_0(x, y)| \, dx \leq c \sum_{p \geq 0} e^{-(\beta - \eta)2^{p/2}} 2^{p/2} |A_p(y)|. \]

Also, for any $\varepsilon > 0$, there is $c > 0$ such that
\[ \int_X |K_\infty(x, y)| \, dx \leq c \sum_{p \geq 0} e^{-\eta(W-\varepsilon)2^{p/2}} |A_p(y)|^{1/2}. \]

**Proof.** It follows from (4) and (3) that for $x \in A_p(y)$
\[ |K_0(x, y)| \leq c \int_{|t|\leq \eta 2^{p/2}} |\hat{m}(t)| e^{-(\beta - \eta)d(x, y)} \, dt \leq ce^{-(\beta - \eta)2^{p/2}} \eta 2^{p/2}. \]

Thus, by using (4)
\[ \int_X |K_0(x, y)| \, dx = \sum_{p \geq 0} \int_{A_p(y)} |K_0(x, y)| \, dx \leq c \sum_{p \geq 0} e^{-(\beta - \eta)2^{p/2}} \eta 2^{p/2} \int_{A_p(y)} \, dx \leq c \sum_{p \geq 0} e^{-(\beta - \eta)2^{p/2}} 2^{p/2} |A_p(y)|. \]

On the other hand, by the Cauchy-Schwarz inequality, (4), (2), the decay of $\hat{m}(t)$ for $t$ large and the fact that the Dirac mass $\delta_y \in L^2(X)$, we get that
\[
\int_{A_p(y)} |K_\infty(x, y)| \, dx \leq \int_{A_p(y)} \left( \int_{|t|\geq \eta 2^{p/2}} |\hat{m}(t)e^{it(I-P)}\delta_y(x)| \, dt \right) \, dx
\leq c \int_{|t|\geq \eta 2^{p/2}} |\hat{m}(t)| \, dt \int_{A_p(y)} \left| e^{itP} \delta_y(x) \right| \, dx
\leq c \int_{|t|\geq \eta 2^{p/2}} |\hat{m}(t)| |A_p(y)|^{1/2} \left\| e^{itP} \right\|_2 \left\| \delta_y \right\|_2 \, dt
\leq c |A_p(y)|^{1/2} \int_{|t|\geq \eta 2^{p/2}} |\hat{m}(t)| \, dt
\leq c |A_p(y)|^{1/2} e^{-\eta(W-\varepsilon)2^{p/2}} \int_{|t|\geq \eta 2^{p/2}} e^{-\varepsilon|t|} \, dt
\leq c |A_p(y)|^{1/2} e^{-\eta(W-\varepsilon)2^{p/2}}. \]
Therefore,
\[
\int_X |K_\infty(x, y)| \, dx = \sum_{p \geq 0} \int A_p(y) \, dx
\leq c \sum_{p \geq 0} c |A_p(y)|^{1/2} e^{-\eta(W-\epsilon)2p/2}.
\]

**Proof of the Theorem.** (i) Let us assume that (1) holds for \( \alpha', \alpha \in (0, 1) \). Then by (5) and (6) we get that
\[
\int_X |K(x, y)| \, dx
\leq c \sum_{p \geq 0} e^{-2p/2} |A_p(y)| + c \sum_{p \geq 0} e^{-\eta(W-e)2p/2} |A_p(y)|^{1/2}
\leq c \sum_{p \geq 0} e^{-2p/2} e^{\kappa(2p/2)^\alpha} + \sum_{p \geq 0} e^{-\eta(W-e)2p/2} e^{\frac{\kappa}{2}(2p/2)^\alpha} < \infty.
\]
This implies that \( m(I - P) \) is bounded on \( L^\infty(X) \). As already mentioned, by the spectral theorem \( m(I - P) \) is bounded on \( L^2(X) \) and by interpolation and duality we obtain the boundedness of \( m(I - P) \) on \( L^p(X) \), \( p \geq 1 \).

(ii) Similarly, if \( |B(x, r)| \leq ce^{\kappa r} \), then
\[
\int_X |K(x, y)| \, dx
\leq c \sum_{p \geq 0} e^{-2p/2} e^{\kappa(2p/2)^\beta/2} + \sum_{p \geq 0} e^{-\eta(W-e)2p/2} e^{\frac{\kappa}{2}(2p/2)^\beta/2} < \infty,
\]
provided that \( \beta > \kappa + \eta \) and \( \eta W > \kappa/2 \) and the boundedness of \( m(I - P) \) on \( L^\infty(X) \) follows.

**References**


Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki 54006, Greece

E-mail address: marias@ccf.auth.gr