EVALUATIONS OF INITIAL IDEALS
AND CASTELNUOVO-MUMFORD REGULARITY

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Abstract. This paper characterizes the Castelnuovo-Mumford regularity by evaluating the initial ideal with respect to the reverse lexicographic order.

1. Introduction

Let \( S = k[x_1, \ldots, x_n] \) be a polynomial ring over a field \( k \) of arbitrary characteristic. Let \( I \subset S \) be an arbitrary homogeneous ideal and

\[
0 \rightarrow F_p \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow S/I \rightarrow 0
\]

a graded minimal free resolution of \( S/I \). Write \( b_i \) for the maximum degree of the generators of \( F_i \). The Castelnuovo-Mumford regularity

\[
\text{reg}(S/I) := \max\{b_i - i \mid i = 0, \ldots, p\}
\]

is a measure for the complexity of \( I \) in computational problems [EG], [BM], [V]. One can use Buchsberger’s syzygy algorithm to compute \( \text{reg}(S/I) \). However, such a computation is often very big. Theoretically, if \( \text{char}(k) = 0 \), \( \text{reg}(S/I) \) is equal to the largest degree of the generators of the generic initial ideal of \( I \) with respect to the reverse lexicographic order [BS]. But it is difficult to know when an initial ideal is generic. Therefore, it would be of interest to have other methods for the computation of \( \text{reg}(S/I) \).

The aim of this paper is to present a simple method for the computation of \( \text{reg}(S/I) \) which is based only on evaluations of \( \text{in}(I) \), where \( \text{in}(I) \) denotes the initial ideal of \( I \) with respect to the reverse lexicographic order. We are inspired by a recent paper of Bermejo and Gimenez [BG] which gives such a method for the computation of the Castelnuovo-Mumford regularity of projective curves.

Let \( d = \dim S/I \). For \( i = 0, \ldots, d \) put \( S_i = k[x_1, \ldots, x_{n-i}] \). Let \( J_i \) be the ideal of \( S_i \) obtained from \( \text{in}(I) \) by the evaluation \( x_{n-i+1} = \cdots = x_n = 0 \). Let \( J_i \) denote the ideal of \( S_i \) obtained from \( J_i \) by the evaluation \( x_{n-i} = 1 \). These ideals can be easily computed from the generators of \( \text{in}(I) \). In fact, if \( \text{in}(I) = (f_1, \ldots, f_s) \), where \( f_1, \ldots, f_s \) are monomials in \( S \), then \( J_{i+1} \) is generated by the monomials \( f_j \) not divided by \( x_{n-i} \).
by any of the variables $x_{n-i+1}, \ldots, x_n$ and $\tilde{J}_i$ by those monomials obtained from the latter by setting $x_{n-i} = 1$. Put

$$c_i(I) := \sup\{r \mid (\tilde{J}_i/J_i)_r \neq 0\},$$

with $c_i(I) = -\infty$ if $\tilde{J}_i = J_i$ and

$$r(I) := \sup\{r \mid (S_d/J_d)_r \neq 0\}.$$

We can express $\text{reg}(S/I)$ in terms of these numbers as follows. Assume that $c_i(I) < \infty$ for $i = 0, \ldots, d - 1$. Then

$$\text{reg}(S/I) = \max\{c_0(I), \ldots, c_{d-1}(I), r(I)\}.$$  

The assumption $c_i(I) < \infty$ for $i = 0, \ldots, d - 1$ is satisfied for a sufficiently general choice of the variables. If $I$ is the defining saturated ideal of a projective (not necessarily reduced) curve, this assumption is automatically satisfied if $k[x_{n-1}, x_n]$ is a Noether normalization of $S/I$. In this case, $c_0(I) = -\infty$ and $\text{reg}(S/I) = \max\{c_1(I), r(I)\}$. From this formula we can easily deduce the results of Bermejo and Gimenez. 

Similarly we can compute the partial regularities $\ell\text{-}\text{reg}(S/I) := \max\{b_i - i \mid i \geq \ell\}$, $\ell > 0$, which were recently introduced by Bayer, Charalambous and Popescu [BCP] (see also Aramova and Herzog [AH]). These regularities can be defined in terms of local cohomology. Let $m$ denote the maximal homogeneous ideal of $S$. Let $H_m^n(S/I)$ denote the $n$th local cohomology module of $S/I$ with respect to $m$ and set $a_i(S/I) = \max\{r \mid H_m^n(S/I)_r \neq 0\}$ with $a_i(S/I) = -\infty$ if $H_m^n(S/I) = 0$. For $t \geq 0$ we define $\text{reg}_t(S/I) := \max\{a_i(S/I) + i \mid i = 0, \ldots, t\}$. Then $\text{reg}_t(S/I) = (n - t)\text{-}\text{reg}(S/I)$ [T2]. Under the assumption $c_i(I) < \infty$ for $i = 0, \ldots, t$ we obtain the following formula:

$$\text{reg}_t(S/I) = \max\{c_i(I) \mid i = 0, \ldots, t\}.$$  

The numbers $c_i(I)$ also allow us to determine the place at which $\text{reg}(S/I)$ is attained in the minimal free resolution of $S/I$. In fact, $\text{reg}(S/I) = b_t - t$ if $c_t(I) = \max\{c_i(I) \mid i = 0, \ldots, d\}$. Moreover, $r(I)$ can be used to estimate the reduction number of $S/I$ which is another measure for the complexity of $I$ [V].

It turns out that the numbers $c_i(I)$ and $r(I)$ can be described combinatorially in terms of the lattice vectors of the generators of $\text{in}(I)$ (see Propositions 4.1-4.3 for details). These descriptions together with the above formulae give an effective method for the computation of $\text{reg}(S/I)$ and $\text{reg}_t(S/I)$. From this we can derive the estimation

$$\text{reg}_t(S/I) \leq \max\{\deg g_i - n + i \mid i = 0, \ldots, t\},$$

where $g_i$ is the least common multiple of the minimal generators of $\text{in}(I)$ which are not divided by any of the variables $x_{n-i+1}, \ldots, x_n$.

This paper is organized as follows. In Section 2 we prepare some facts on the Castelnuovo-Mumford regularity. In Section 3 we prove the above formulae for $\text{reg}(S/I)$ and $\text{reg}_t(S/I)$. The combinatorial descriptions of $c_i(I)$ and $r(I)$ are given in Section 4. Section 5 deals with the case of projective curves.

2. Filter-regular sequence of linear forms

We shall keep the notations of the preceding section. Let $z = z_1, \ldots, z_{t+1}$ be a sequence of homogeneous elements of $S$, $t \geq 0$. We call $z$ a filter-regular sequence for $S/I$ if $z_{t+1} \not\in p$ for any associated prime $p \neq m$ of $(I, z_1, \ldots, z_i)$, $i = 0, \ldots, t$. 

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This notion was introduced in order to characterize generalized Cohen-Macaulay rings \cite{STC}. Recall that $S/I$ is a generalized Cohen-Macaulay ring if and only if $I$ is equidimensional and $(R/I)_{p}$ is a Cohen-Macaulay ring for every prime ideal $p \neq m$. This condition is satisfied if $I$ is the defining ideal of a projective curve. We call $z$ a homogeneous system of parameters for $S/I$ if $t + 1 = d$ and $(I, z_{1}, \ldots, z_{d})$ is an $m$-primary ideal. It is known that every homogeneous system of parameters for $S/I$ is a filter-regular sequence if $S/I$ is a generalized Cohen-Macaulay ring. In general, a homogeneous system of parameters need not be a filter-regular sequence. However, if $k$ is an infinite field, any ideal which is primary to the maximal graded ideal and which is generated by linear forms can be generated by a homogeneous filter-regular sequence (proof of \cite{T1} Lemma 3.1).

For $i = 0, \ldots, t$ we put

$$a_{z}^{i}(S/I) := \sup \{|r| \mid (I, z_{1}, \ldots, z_{i})_{r} \neq (I, z_{1}, \ldots, z_{i})_{r}\},$$

with $a_{z}^{i}(S/I) = -\infty$ if $(I, z_{1}, \ldots, z_{i})_{r} = (I, z_{1}, \ldots, z_{i})_{r}$. These invariants can be infinite and they are a measure for how far $z$ is from being a regular sequence in $S/I$. It can be shown that $z$ is a filter-regular sequence for $S/I$ if and only if $a_{z}^{i}(S/I) < \infty$ for $i = 0, \ldots, t$ \cite{T1} Lemma 2.1. Note that our definition of $a_{z}^{i}(S/I)$ is one less than that in \cite{T1}. There is the following close relationship between these numbers and the partial regularity of $S/I$.

**Theorem 2.1** (\cite{T1} Proposition 2.2). Let $z$ be a filter-regular sequence of linear forms for $S/I$. Then

$$\text{reg}_{z}(S/I) = \max \{a_{z}^{i}(S/I) \mid i = 0, \ldots, t\}.$$

We will use the following characterization of $a_{z}^{i}(S/I)$.

**Lemma 2.2.** $a_{z}^{i}(S/I) = \max \{|r| \mid \bigcup_{m \geq 1} (I, z_{1}, \ldots, z_{i})_{r} \neq (I, z_{1}, \ldots, z_{i})_{r}\}$. 

**Proof.** Put $r_{0} = \max \{|r| \mid \bigcup_{m \geq 1} (I, z_{1}, \ldots, z_{i})_{r} \neq (I, z_{1}, \ldots, z_{i})_{r}\}$. By definition, $a_{z}^{i}(S/I) \leq r_{0}$. Conversely, if $y$ is an element of $\bigcup_{m \geq 1} (I, z_{1}, \ldots, z_{i})_{r_{0}}$, then

$$y_{z_{i+1}} \in \bigcup_{m \geq 1} (I, z_{1}, \ldots, z_{i})_{r_{0+1}} = (I, z_{1}, \ldots, z_{i})_{r_{0+1}}.$$

Hence $y \in (I, z_{1}, \ldots, z_{i})_{r_{0}}$. This implies $r_{0} \leq a_{z}^{i}(S/I)$. So we get $r_{0} = a_{z}^{i}(S/I)$.

Since $\text{reg}(S/I) = \text{reg}_{d}(S/I)$, to compute $\text{reg}(S/I)$ we need a filter-regular sequence of linear forms of length $d + 1$. But that can be avoided by the following observation.

**Lemma 2.3.** Let $z = z_{1}, \ldots, z_{d}$ be a filter-regular sequence for $S/I$, $d = \dim(S/I)$. Then $z$ is a system of parameters for $S/I$.

**Proof.** Let $p$ be an arbitrary associated prime $p$ of $(I, z_{1}, \ldots, z_{i})$ with $\dim S/p = d - i$, $i = 0, \ldots, d - 1$. Then $p \neq m$ because $\dim S/p > 0$. By the definition of a filter-regular sequence, $z_{i+1} \notin p$. Hence $z$ is a homogeneous system of parameters for $S/I$.

If $z$ is a homogeneous system of parameters for $S/I$, then $S/(I, z_{1}, \ldots, z_{d})$ is of finite length. Hence $(S/(I, z_{1}, \ldots, z_{d}))_{r} = 0$ for $r$ large enough. Following \cite{NR} we call

$$r_{z}(S/I) := \max \{|r| \mid (S/(I, z_{1}, \ldots, z_{d}))_{r} \neq 0\}.$$
the reduction number of $S/I$ with respect to $z$. It is equal to the maximum degree of the generators of $S/I$ as a module over $k[z_1, \ldots, z_d]$ [V]. Note that the minimum of $r_z(S/I)$ is called the reduction number of $S/I$.

**Theorem 2.4** ([BS Theorem 1.10], [I]) Corollary 3.3]. Let $z$ be a filter-regular sequence of $d$ linear forms for $S/I$. Then

\[ \operatorname{reg}(S/I) = \max\{a^0_z(S/I), \ldots, a^{d-1}_z(S/I), r_z(S/I)\}. \]

**Remark.** Theorem 2.4 was proved in [BS] under an additional condition on the maximum degree of the generators of $I$.

3. Evaluations of the Initial Ideal

Let $c_i(I)$, $i = 0, \ldots, d$, and $r(I)$ be the invariants defined in Section 1 by means of evaluations of in$(I)$, where in$(I)$ is the initial ideal of $I$ with respect to the reverse lexicographic order. We will use the results of Section 2 to express reg$_z(S/I)$ and reg$(S/I)$ in terms of $c_i(I)$ and $r(I)$.

**Lemma 3.1.** For $z = x_n, \ldots, x_{n-t}$ and $t = 0, \ldots, t$ we have

\[ a^t_z(S/I) = c_i(I). \]

**Proof.** By [BS Lemma (2.2)] and [(I), x, $x_{n-i+1}: x_{n-i}] = (I, x, \ldots, x_{n-i+1})$] if and only if $[(\text{in}(I), x, \ldots, x_{n-i+1}) : x_{n-i}] = (\text{in}(I), x, \ldots, x_{n-i+1})]$ for all $r \geq 0$. Therefore

\[ a^t_z(S/I) = a^t_z(S/\text{in}(I)). \]

By Lemma 2.2 we get

\[ a^t_z(S/\text{in}(I)) = \sup\{r \mid \bigcup_{m \geq 1} (\text{in}(I), x, \ldots, x_{n-i+1}) : x^m_{n-i}] 
\neq (\text{in}(I), x, \ldots, x_{n-i+1})]. \]

Note that $J_i$ is the ideal of $S_i = k[x_1, \ldots, x_n]$ obtained from in$(I)$ by the evaluation $x_{n-i+1} = \cdots = x_n = 0$ and that this evaluation corresponds to the canonical isomorphism $S/(x_{n-i+1}, \ldots, x_n) \cong S_i$. Then we may rewrite the above formula as

\[ a^t_z(S/\text{in}(I)) = \sup\{r \mid \bigcup_{m \geq 1} J_i : x^m_{n-i}] 
\neq (J_i)\}. \]

Since $J_i$ is a monomial ideal, $\bigcup_{m \geq 1} J_i : x^m_{n-i}$ is generated by the monomials $g$ in the variables $x_1, \ldots, x_{n-i-1}$ for which there exists an integer $m \geq 1$ such that $gx^m_{n-i} \in J_i$. Such a monomial $g$ is determined by the condition $g \in J_i$. Hence

\[ a^t_z(S/\text{in}(I)) = \sup\{r \mid (J_i)_r \neq (J_i)\} = c_i(I). \]

As a consequence of Lemma 3.1 we can use the invariants $c_i(I)$ to check when $x_n, \ldots, x_{n-t}$ is a regular resp. filter-regular sequence for $S/I$.

**Corollary 3.2.** $x_{n-i}$ is a non-zerodivisor in $S/(I, x_n, \ldots, x_{n-i+1})$ if and only if $c_i(I) = -\infty$.

**Proof.** By definition, $a^t_z(S/I) = -\infty$ if and only if $x_{n-i}$ is a non-zerodivisor in $S/(I, x_n, \ldots, x_{n-i+1})$. Hence the conclusion follows from Lemma 3.1. \qed
Corollary 3.3. Let $z = x_n, \ldots, x_{n-t}$. Then $z$ is a filter-regular sequence for $S/I$ if and only if $c_i(I) < \infty$ for $i = 0, \ldots, t$.

Proof. It is known that $z$ is a filter-regular sequence for $S/I$ if and only if $a^*_z(S/I) < \infty$ for $i = 0, \ldots, t$ [11, Lemma 2.1].

Now we can characterize $\reg_t(S/I)$ as follows.

Theorem 3.4. Assume that $c_i(I) < \infty$ for $i = 0, \ldots, t$. Then

$$\reg_t(S/I) = \max\{c_i(I) \mid i = 0, \ldots, t\}.$$ 

Proof. This follows from Theorem 2.1, Lemma 3.1 and Corollary 3.3.

Lemma 3.5. Assume that $c_i(I) < \infty$ for $i = 0, \ldots, d - 1$. Then

$$r_z(S/I) = r(I).$$

Proof. By Corollary 3.3, $z = x_n, \ldots, x_{n-d+1}$ is a filter-regular sequence for $S/I$. By Lemma 2.3 and [12, Theorem 4.1], this implies that $z$ is a homogeneous system of parameters for $S/(\in(I))$ with

$$r_z(S/I) = r_z(S/(\in(I))).$$

Note that $S/(x_{n-d+1}, \ldots, x_n) \cong S_d$ and that $J_d$ is the ideal obtained from $\in(I)$ by the evaluation $x_{n-d+1} = \cdots = x_n = 0$. Then

$$r_z(S/(\in(I))) = \max\{r \mid (S/(\in(I), x, \ldots, x_{n-d+1}))_r \neq 0\} = \max\{r \mid (S_d/J_d)_r \neq 0\} = r(I).$$

Theorem 3.6. Assume that $c_i(I) < \infty$ for $i = 0, \ldots, d - 1$. Then

$$\reg(S/I) = \max\{c_0(I), \ldots, c_{d-1}(I), r(I)\}.$$ 

Proof. This follows from Theorem 2.4, Lemma 3.1, Corollary 3.3 and Lemma 3.5.

4. Combinatorial description

First, we want to show that the condition $c_i(I) < \infty$ can be easily checked in terms of the lattice vectors of the generators of $\in(I)$. Let $B$ be the (finite) set of monomials which minimally generates $\in(I)$. We set

$$E_i := \{v \in \mathbb{N}^{n-i} \mid x^v \in B\},$$

where $x^v = x_1^{v_1} \cdots x_s^{v_s}$ if $v = (\varepsilon_1, \ldots, \varepsilon_s)$. For $j = 1, \ldots, n-i$ we denote by $p_j$ the projection from $\mathbb{N}^{n-i}$ to $\mathbb{N}^{n-i-1}$ which deletes the $j$th coordinate. For two lattice vectors $a = (\alpha_1, \ldots, \alpha_s)$ and $b = (\beta_1, \ldots, \beta_s)$ of the same size we say $a \geq b$ if $\alpha_j \geq \beta_j$ for $j = 1, \ldots, s$.

Lemma 4.1. $c_i(I) < \infty$ if and only if for every element $a \in p_{n-i}(E_i) \setminus E_{i+1}$ there are elements $b_j \in E_{i+1}$ such that $p_j(a) \geq p_j(b_j)$, $j = 1, \ldots, n-i-1$.
Proof. Recall that \( c_i(I) = \sup \{ r \mid (J_i/J_r)_r \neq 0 \} \). Then \( c_i(I) < \infty \) if and only if \( J_i/J_r \) is of finite length. By the definition of \( J_i \) and \( J_r \), the latter condition is equivalent to the existence of a number \( r \) such that \( x_i^{n_i}J_i \subseteq J_r \) for \( j = 1, \ldots, n - i \).

It is clear that \( J_i \) is generated by the monomials \( x^v \) with \( v \in E_i \). From this it follows that \( J_i \) is generated by \( J_i \) and the monomials \( x^a \) with \( a \in p_{n-i}(E_i) \setminus E_{i+1} \). For such a monomial \( x^a \) we can always find a number \( r \) such that \( x_i^{n_i}x^a \in J_r \). For \( j < n - i, x_jx^a \in J_i \) if and only if \( x_jx^a \) is divided by a generator \( x^{b_j} \) of \( J_i \). Since \( x_jx^a \) does not contain \( x_{n-i}, \ldots, x_n \), so does \( x^{b_j} \). Hence \( b_j \in E_{i+1} \). Setting \( x_j = 1 \) we see that \( x_jx^a \) is divided by \( x^{b_j} \) for some number \( r \) if and only if \( p_j(a) \geq p_j(b_j) \). \( \Box \)

If \( c_i(I) = \infty \), we should make a random linear transformation of the variables \( x_1, \ldots, x_{n-i} \) and test the condition \( c_i(I) < \infty \) again. By Lemma 5.1 the linear transformation does not change the invariants \( c_j(I) \) for \( j < i \). Moreover, instead of \( \text{in}(I) \) we only need to compute the smaller initial ideal \( \text{in}(I_i) \), where \( I_i \) denotes the ideal of \( S_i \) obtained from \( I \) by the evaluation \( x_{n-i+1} = \cdots = x_n = 0 \). Let \( B_i \) be the set of monomials which minimally generates \( \text{in}(I_i) \). It is easy to see that \( B_i \) is the set of the monomials of \( B \) which are not divided by \( x_{n-i+1}, \ldots, x_n \). From this it follows that \( E_j = \{ v \in \mathbb{N}^{n-i} \mid x^v \in B_j \} \) for \( j \leq i \). Thus, we can use this formula to compute \( E_j \) and to check the condition \( c_j(I) < \infty \) for \( j \leq i \). Once we know \( c_i(I) < \infty \) we can proceed to compute \( c_i(I) \).

In the lattice \( \mathbb{N}^{n-i} \) we delete the shadow of \( E_i \), that is, the set of elements \( a \) for which there is \( v \in E_i \) with \( v \leq a \). The remaining lattice has the shape of a staircase and we will denote by \( F_i \) the set of its corners. It is easy to see that \( F_i \) is the set of the elements of the form \( a = \max(v_1, \ldots, v_{n-i}) - (1, \ldots, 1) \) with \( a \not\leq v \) for any element \( v \in E_i \), where \( v_1, \ldots, v_{n-i} \) is a family of \( n - i \) elements of \( E_i \) for which the \( j \)th coordinate of \( v_j \) is greater than the \( j \)th coordinate of \( v_h \) for all \( h \neq j, j = 1, \ldots, n - i \), and \( \max(v_1, \ldots, v_{n-i}) \) denotes the element whose coordinates are the maxima of the corresponding coordinates of \( v_1, \ldots, v_{n-i} \). If \( a = (a_1, \ldots, a_{n-i}) \), we set

\[
|a| := a_1 + \cdots + a_{n-i}.
\]

**Proposition 4.2.** Assume that \( c_i(I) < \infty \). Then \( c_i(I) = -\infty \) if \( F_i = \emptyset \) and \( c_i(I) = \max_{a \in F_i} |a| \) if \( F_i \neq \emptyset \).

Proof. Let \( a \) be an arbitrary element of \( F_i \). Then \( a = \max(v_1, \ldots, v_{n-i}) - (1, \ldots, 1) \) for some family \( v_1, \ldots, v_{n-i} \) of \( S_i \). Let \( v_j = (\varepsilon_{j1}, \ldots, \varepsilon_{jn-i}) \), \( j = 1, \ldots, n - i \). Then \( a = (\varepsilon_{11} - 1, \ldots, \varepsilon_{n-i,n-i} - 1) \). Since \( \varepsilon_{jj} > \varepsilon_{kj} \) for \( h \neq j \), we get \( a \geq (\varepsilon_{n-i,1}, \ldots, \varepsilon_{n-i,n-i} - 1, 0) \). Therefore, \( x^a \) is divided by the monomial obtained from \( x^{v_{n-i}} \) by setting \( x_{n-i} = 1 \). Note that \( J_i \) is generated by the monomials \( x^v \) with \( v \in E_i \). Since \( v_{n-i} \in E_i \), we have \( x^{v_{n-i}} \in J_i \), whence \( x^a \in J_i \). On the other hand, \( x^a \not\in J_i \) because \( a \not\geq v \) for any element \( v \in E_i \). Since \( |a| = \deg x^a \), this implies \( (J_i/J_i)_{|a|} \neq 0 \). Hence \( |a| \leq c_i(I) \). So we obtain \( \max_{a \in F_i} |a| \leq c_i(I) \) if \( F_i \neq \emptyset \).

To prove the converse inequality we assume that \( J_i/J_i \neq 0 \). Since \( c_i(I) < \infty \), there is a monomial \( x^b \in J_i \setminus J_i \) such that \( \deg x^b = c_i(I) \). Since \( x^b \not\in J_i \), \( b \not\geq v \) for any element \( v \in E_i \). For \( j = 1, \ldots, n - i \) we have \( x_jx^b \in J_i \) because \( \deg x_jx^b = c_i(I) + 1 \). Therefore, \( x_jx^b \) is divided by some monomial \( x^v \) with \( v_j \in E_i \). Let \( b = (\beta_1, \ldots, \beta_{n-i}) \) and \( v_j = (\varepsilon_{j1}, \ldots, \varepsilon_{jn-i}) \). Then \( \beta_h \geq \varepsilon_{jh} \) for \( h \neq j \) and \( \beta_j + 1 \geq \varepsilon_{jj} \).
Since $b \not\leq v_j$, we must have $\beta_j < \varepsilon_{jj}$, hence $\beta_j = \varepsilon_{jj} - 1$. It follows that $\varepsilon_{jj} = \beta_j + 1 > \varepsilon_{bj}$ for all $h \neq j$. Thus, the family $v_1, \ldots, v_{n-i}$ belongs to $S_{i}$ and $b = \max\{v_1, \ldots, v_{n-i}\} - (1, \ldots, 1)$. So we have proved that $b \in F_i$. Hence $c_i(I) = \deg x^b = |b| \leq \max_{a \in F_i} |a|$.

The above argument also shows that $F_i \neq \emptyset$ if $\tilde{J}_i \neq J_i$. So $c_i(I) = -\infty$ if $F_i = \emptyset$.

By Corollary 3.3, if $c_i(I) < \infty$ for $i = 0, \ldots, d - 1$, then $z = x_n, \ldots, x_{n-d+1}$ is a filter-regular sequence for $S/I$. By Lemma 2.3 and Lemma 5.3, that implies $r(I) = r_d(S/I) < \infty$. In this case, we have the following description of $r(I)$.

**Proposition 4.3.** Assume that $r(I) < \infty$. Then $r(I) = \max_{a \in F_d} |a|$.

**Proof.** This can be proved similarly to the proof of Lemma 4.2.

Combining the above results with Theorem 3.4 and Theorem 6.6, we get a simple method to compute $\text{reg}_I(S/I)$ and $\text{reg}(S/I)$. We will illustrate the above method by an example at the end of the next section. Moreover, we get the following estimation for $\text{reg}_I(S/I)$.

**Corollary 4.4.** Let $x_n, \ldots, x_{n-t}$ be a filter-regular sequence for $S/I$. Let $g_i$ denote the least common multiple of the minimal generators of $\text{in}(I)$ which are not divided by any of the variables $x_{n-i+1}, \ldots, x_n$. Then

$$\text{reg}_I(S/I) \leq \max\{\deg g_i - n + i \mid i = 0, \ldots, t\}.$$

**Proof.** By Corollary 3.3, the assumption implies that $c_i(I) < \infty$ for $i = 0, \ldots, t$. Thus, combining Theorem 3.4 and Lemma 4.2, we get

$$\text{reg}_I(S/I) \leq \max\{|a| \mid a \in F_i, \ i = 0, \ldots, t\}.$$

It is easily seen from the definition of $F_i$ that $\max_{a \in F_i} |a| \leq \deg g_i - n + i, \ i = 0, \ldots, t$, hence the conclusion.

**Remark.** Bruns and Herzog [BH, Theorem 3.1(a)], resp. Hoa and Trung [HT, Theorem 3.1], proved that for any monomial ideal $I$, $\text{reg}(S/I) \leq \deg f - 1$, resp. $\deg f - \text{ht} I$, where $f$ is the least common multiple of the minimal generators of $I$. Note that the mentioned result of Bruns and Herzog is valid for multigraded modules.

## 5. The case of projective curves

Let $I_C \subset k[x_1, \ldots, x_n]$ be the defining saturated ideal of a (not necessarily reduced) projective curve $C \subset \mathbb{P}^{n-1}, n \geq 3$. We will assume that $k[x_{n-1}, x_n] \hookrightarrow S/I_C$ is a Noether normalization of $S/I_C$. In this case, Theorem 3.6 can be reformulated as follows.

**Proposition 5.1.** $\text{reg}(S/I_C) = \max\{c_1(I_C), r(I_C)\}$.

**Proof.** By the above assumption $S/I_C$ is a generalized Cohen-Macaulay ring of positive depth and $x_n, x_{n-1}$ is a homogeneous system of parameters for $S/I_C$. Therefore, $x_n, x_{n-1}$ is a filter-regular sequence for $S/I_C$. In particular, $x_n$ is a non-zerodivisor in $S/I_C$. By Lemma 5.2, $c_0(I_C) = -\infty$. Hence the conclusion follows from Theorem 3.6.


Since $S/I_C$ has positive depth, the graded minimal free resolution of $S/I_C$ ends at most at the $(n - 1)$th place:

$$0 \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow S/I_C \rightarrow 0.$$ 

From Theorem 3.4 we obtain the following information on the shifts of $F_{n-1}$. Note that $F_{n-1} = 0$ if $S/I_C$ is a Cohen-Macaulay ring or, in other words, if $C$ is an arithmetically Cohen-Macaulay curve.

**Proposition 5.2.** If $C$ is not an arithmetically Cohen-Macaulay curve, $c_1(I_C) + n - 1$ is the maximum degree of the generators of $F_{n-1}$.

**Proof.** Let $b_{n-1}$ be the maximum degree of the generators of $F_{n-1}$. As we have seen in the introduction, $b_{n-1} = n + 1 = (n - 1) - \text{reg}(S/I_C)$. By Theorem 3.4 \( \text{reg}_1(S/I_C) = \max\{c_0(I_C), c_1(I_C)\} = c_1(I_C) \) because $c_0(I_C) = -\infty$. So we obtain $b_{n-1} = c_1(I_C) + n - 1$.

Now we shall see that Proposition 5.1 contains all main results of Bermejo and Gimenez in [BG]. It should be noted that they did not use strong results such as Theorem 2.4. We follow the notations of [BG].

Let $E := \{a \in \mathbb{N}^{n-2} \mid x^a \in \text{in}(I_C)\}$ and denote by $H(E)$ the smallest integer $r$ such that $a \in E$ if $|a| = r$.

**Corollary 5.3 ([BG] Theorem 2.4).** Assume that $C$ is an arithmetically Cohen-Macaulay curve. Then $\text{reg}(S/I_C) = H(E) - 1$.

**Proof.** Since $x_n, x_{n-1}$ is a regular sequence in $S/I_C$, we have $c_1(I_C) = -\infty$ by Corollary 3.2. By Proposition 5.1 it implies $\text{reg}(S/I_C) = r(I_C)$. But $r(I_C) = \sup\{r \mid (S_2/J_2)_v \neq 0\} = H(E) - 1$ because $J_2$ is generated by the monomials $x^a$, $a \in E$.

Let $I_0$ be the ideal in $S$ generated by the polynomials obtained from $I_C$ by the evaluation $x_{n-1} = x_n = 0$. Then $S/I_0$ is a two-dimensional Cohen-Macaulay ring. Let $\tilde{I}$ denote the ideal in $S$ generated by the monomials obtained from $\text{in}(I_C)$ by the evaluation $x_{n-1} = x_n = 1$. Let

$$F := \{a \in \mathbb{N}^{n-2} \mid a \in \tilde{I} \setminus \text{in}(I_0)\}.$$ 

For every vector $a \in F$ let

$$E_a := \{(\mu, \nu) \in \mathbb{N}^2 \mid x^a x_{n-1}^\mu x_n^\nu \in \text{in}(I_C)\}.$$ 

Let $\mathcal{R} := \bigcup_{a \in F} \{a \times [\mathbb{N}^2 \setminus E_a]\}$ and denote by $H(\mathcal{R})$ the smallest integer $r$ such that the number of the elements $b \in \mathcal{R}$ with $|b| = s$ becomes a constant for $s \geq r$.

**Corollary 5.4 ([BG] Theorem 2.7)).** $\text{reg}(S/I_C) = \max\{\text{reg}(S/I_0), H(\mathcal{R})\}$.

**Proof.** As in the proof of Corollary 5.3 we have $\text{reg}(S/I_0) = r(I_0)$. But $r(I_0) = r(I_C)$ because $\text{in}(I_0)$ is the ideal generated by the monomials obtained from $\text{in}(I_C)$ by the evaluation $x_{n-1} = x_n = 0$. Thus,

$$\text{reg}(S/I_0) = r(I_C).$$ 

It has been observed in [BG] that the number of the elements $b \in \mathcal{R}$ with $|b| = s$ is the difference $H_{S/I_C}(s) - H_{S/\tilde{I}}(s) = H_{S/\text{in}(I_C)}(s) - H_{S/\text{in}(I_C)}(s) - H_{S/\text{in}(I_C)}(s)$, where $H_E(s)$ denotes the Hilbert function of a graded $S$-module $E$. Since $x_n$ is a non-zero divisor in $S/\text{in}(I_C)$, $H(\mathcal{R}) + 1$ is the smallest integer $r$ such that $H_{(I,x_n)/(\text{in}(I_C),x_n)}(s)$
= 0 for s ≥ r. On the other hand, since \( \text{in}(IC) \) is generated by monomials which do not contain \( x_r \), and since \( J_1 \) is the ideal in \( k[x_1, \ldots, x_{n-1}] \) obtained from \( \text{in}(IC) \) by the evaluation \( x_r = 0 \), we have \( \text{in}(IC) = J_1S \) and \( \overline{I} = J_1S \), whence \( (\overline{I}, x_r)/(\text{in}(IC), x_r) \cong \overline{J}_1/J_1 \). Note that \( c_1(IC) = \max\{r \mid (\overline{J}_1/J_1)_r \neq 0 \} \) with \( c_1(IC) = -\infty \) if \( \overline{J}_1 = J_1 \). Then
\[
H(\mathcal{R}) = \max\{0, c_1(IC)\}.
\]
Thus, applying Proposition 5.1 we obtain \( \text{reg}(S/IC) = \max\{\text{reg}(S/I_0), H(\mathcal{R})\} \). □

**Example.** Let \( C \subset \mathbb{P}^1 \) be the monomial curve \((t^\alpha s^\beta : t^\beta s^\alpha : s^{\alpha+\beta} : t^{\alpha+\beta}), \alpha > \beta \geq 0, \text{g.c.d.} (\alpha, \beta) = 1\). It is known that the defining ideal \( IC \subset k[x_1, x_2, x_3, x_4] \) is generated by the quadric \( x_1^2 - x_3x_4 \) and the forms \( x_1^\beta x_3^{\alpha-\beta} - x_1^{\beta+2} x_3^{\alpha-\beta-2} - \cdots, x_1^{\beta} \). Using the notations of Section 3 we have
\[
E_1 = \{(1, 1, 0), (0, \alpha, 0), (\beta + 1, 0, \alpha - \beta - 1), (\beta + 2, 0, \alpha - \beta - 2), \ldots, (\alpha, 0, 0)\},
\]
\[
E_2 = \{(1, 1), (0, \alpha), (\alpha, 0)\}.
\]
From this it follows that
\[
F_1 = \{(\beta + 1, 0, \alpha - \beta - 2), (\beta + 2, 0, \alpha - \beta - 3), \ldots, (\alpha - 1, 0, 0)\},
\]
\[
F_2 = \{(0, \alpha - 1), (\alpha - 1, 0)\}.
\]
By Proposition 4.2, \( c_1(IC) = \alpha - 1 \) if \( \alpha - \beta \geq 2 \) (\( c_1(IC) = -\infty \) if \( \alpha - \beta = 1 \)) and \( r(IC) = \alpha - 1 \) by Proposition 4.3. Applying Proposition 5.1 we obtain \( \text{reg}(S/IC) = \alpha - 1 \).

The direct computation of the invariant \( H(\mathcal{R}) \) is more complicated than that of \( c_1(IC) \). First, we should interpret \( F \) as the set of the elements of the form \( \alpha \in \mathbb{N}^2 \) such that \( \alpha \geq \beta \) for some elements \( \beta \in p(E_1) \) but \( \alpha \not\geq c \) for any element \( c \in E_2 \). Then we get
\[
F = \{(\beta + 1, 0), (\beta + 2, 0), \ldots, (\alpha - 1, 0)\}.
\]
For all \( \varepsilon = \beta + 1, \ldots, \alpha - 1 \) we verify that \( E_{(\varepsilon)} = (\alpha - \varepsilon, 0) + \mathbb{N}^2 \). It follows that
\[
\mathcal{R} = \{(\varepsilon, 0, \mu, \nu) \in \mathbb{N}^4 \mid \varepsilon = \beta + 1, \ldots, \alpha - 1; \mu \leq \alpha - \varepsilon - 1\}.
\]
If \( \alpha - \beta = 1 \), we have \( \mathcal{R} = \emptyset \), hence \( H(\mathcal{R}) = 0 \). If \( \alpha - \beta \geq 2 \), we can check that \( H(\mathcal{R}) = \alpha - 1 \).

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**References**


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