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EVALUATIONS OF INITIAL IDEALS AND CASTELNUOVO-MUMFORD REGULARITY

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ABSTRACT. This paper characterizes the Castelnuovo-Mumford regularity by evaluating the initial ideal with respect to the reverse lexicographic order.

1. INTRODUCTION

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k of arbitrary characteristic. Let $I \subset S$ be an arbitrary homogeneous ideal and

$$0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow S/I \longrightarrow 0$$

a graded minimal free resolution of S/I. Write b_i for the maximum degree of the generators of F_i. The *Castelnuovo-Mumford regularity*

$$\operatorname{reg}(S/I) := \max\{b_i - i|\ i = 0, \dots, p\}$$

is a measure for the complexity of I in computational problems [EG], [BM], [V]. One can use Buchsberger's syzygy algorithm to compute $\operatorname{reg}(S/I)$. However, such a computation is often very big. Theoretically, if $\operatorname{char}(k) = 0$, $\operatorname{reg}(S/I)$ is equal to the largest degree of the generators of the generic initial ideal of I with respect to the reverse lexicographic order [BS]. But it is difficult to know when an initial ideal is generic. Therefore, it would be of interest to have other methods for the computation of $\operatorname{reg}(S/I)$.

The aim of this paper is to present a simple method for the computation of $\operatorname{reg}(S/I)$ which is based only on evaluations of $\operatorname{in}(I)$, where $\operatorname{in}(I)$ denotes the initial ideal of I with respect to the reverse lexicographic order. We are inspired by a recent paper of Bermejo and Gimenez [BG] which gives such a method for the computation of the Castelnuovo-Mumford regularity of projective curves.

Let $d = \dim S/I$. For $i = 0, \dots, d$ put $S_i = k[x_1, \dots, x_{n-i}]$. Let J_i be the ideal of S_i obtained from $\operatorname{in}(I)$ by the evaluation $x_{n-i+1} = \cdots = x_n = 0$. Let $\tilde{J}_i$ denote the ideal of S_i obtained from J_i by the evaluation $x_{n-i} = 1$. These ideals can be easily computed from the generators of $\operatorname{in}(I)$. In fact, if $\operatorname{in}(I) = (f_1, \dots, f_s)$, where $f_1, \dots, f_s$ are monomials in S, then J_i is generated by the monomials f_j not divided

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by any of the variables $x_{n-i+1}, \dots, x_n$ and $\tilde{J}_i$ by those monomials obtained from the latter by setting $x_{n-i} = 1$. Put

$$c_i(I) := \sup\{r |\ (\tilde{J}_i/J_i)_r \neq 0\},$$

with $c_i(I) = -\infty$ if $\tilde{J}_i = J_i$ and

$$r(I) := \sup\{r |\ (S_d/J_d)_r \neq 0\}.$$

We can express $\operatorname{reg}(S/I)$ in terms of these numbers as follows. Assume that $c_i(I) < \infty$ for $i = 0, \dots, d-1$. Then

$$\operatorname{reg}(S/I) = \max\{c_0(I), \dots, c_{d-1}(I), r(I)\}.$$

The assumption $c_i(I) < \infty$ for $i = 0, \dots, d-1$ is satisfied for a sufficiently general choice of the variables. If I is the defining saturated ideal of a projective (not necessarily reduced) curve, this assumption is automatically satisfied if $k[x_{n-1}, x_n]$ is a Noether normalization of S/I. In this case, $c_0(I) = -\infty$ and $\operatorname{reg}(S/I) = \max\{c_1(I), r(I)\}$. From this formula we can easily deduce the results of Bermejo and Gimenez.

Similarly we can compute the *partial regularities* ℓ–$\operatorname{reg}(S/I) := \max\{b_i - i |\ i \geq \ell\}$, $\ell > 0$, which were recently introduced by Bayer, Charalambous and Popescu [BCP] (see also Aramova and Herzog [AH]). These regularities can be defined in terms of local cohomology. Let $\mathfrak{m}$ denote the maximal homogeneous ideal of S. Let $H^i_{\mathfrak{m}}(S/I)$ denote the ith local cohomology module of S/I with respect to $\mathfrak{m}$ and set $a_i(S/I) = \max\{r |\ H^i_{\mathfrak{m}}(S/I)_r \neq 0\}$ with $a_i(S/I) = -\infty$ if $H^i_{\mathfrak{m}}(S/I) = 0$. For $t \geq 0$ we define $\operatorname{reg}_t(S/I) := \max\{a_i(S/I) + i |\ i = 0, \dots, t\}$. Then $\operatorname{reg}_t(S/I) = (n-t)$–$\operatorname{reg}(S/I)$ [T2]. Under the assumption $c_i(I) < \infty$ for $i = 0, \dots, t$ we obtain the following formula:

$$\operatorname{reg}_t(S/I) = \max\{c_i(I) |\ i = 0, \dots, t\}.$$

The numbers $c_i(I)$ also allow us to determine the place at which $\operatorname{reg}(S/I)$ is attained in the minimal free resolution of S/I. In fact, $\operatorname{reg}(S/I) = b_t - t$ if $c_t(I) = \max\{c_i(I) |\ i = 0, \dots, d\}$. Moreover, $r(I)$ can be used to estimate the reduction number of S/I which is another measure for the complexity of I [V].

It turns out that the numbers $c_i(I)$ and $r(I)$ can be described combinatorially in terms of the lattice vectors of the generators of $\operatorname{in}(I)$ (see Propositions 4.1–4.3 for details). These descriptions together with the above formulae give an effective method for the computation of $\operatorname{reg}(S/I)$ and $\operatorname{reg}_t(S/I)$. From this we can derive the estimation

$$\operatorname{reg}_t(S/I) \leq \max\{\deg g_i - n + i |\ i = 0, \dots, t\},$$

where g_i is the least common multiple of the minimal generators of $\operatorname{in}(I)$ which are not divided by any of the variables $x_{n-i+1}, \dots, x_n$.

This paper is organized as follows. In Section 2 we prepare some facts on the Castelnuovo-Mumford regularity. In Section 3 we prove the above formulae for $\operatorname{reg}(S/I)$ and $\operatorname{reg}_t(S/I)$. The combinatorial descriptions of $c_i(I)$ and $r(I)$ are given in Section 4. Section 5 deals with the case of projective curves.

2. Filter-regular sequence of linear forms

We shall keep the notations of the preceding section. Let $\mathbf{z} = z_1, \dots, z_{t+1}$ be a sequence of homogeneous elements of S, $t \geq 0$. We call $\mathbf{z}$ a *filter-regular sequence* for S/I if $z_{i+1} \notin \mathfrak{p}$ for any associated prime $\mathfrak{p} \neq \mathfrak{m}$ of $(I, z_1, \dots, z_i)$, $i = 0, \dots, t$.

This notion was introduced in order to characterize generalized Cohen-Macaulay rings [STC]. Recall that S/I is a generalized Cohen-Macaulay ring if and only if I is equidimensional and $(R/I)_{\mathfrak{p}}$ is a Cohen-Macaulay ring for every prime ideal $\mathfrak{p} \neq \mathfrak{m}$. This condition is satisfied if I is the defining ideal of a projective curve. We call $\mathbf{z}$ a homogeneous system of parameters for S/I if $t+1 = d$ and $(I, z_1, \dots, z_d)$ is an $\mathfrak{m}$-primary ideal. It is known that every homogeneous system of parameters for S/I is a filter-regular sequence if S/I is a generalized Cohen-Macaulay ring. In general, a homogeneous system of parameters need not be a filter-regular sequence. However, if k is an infinite field, any ideal which is primary to the maximal graded ideal and which is generated by linear forms can be generated by a homogeneous filter-regular sequence (proof of [T1, Lemma 3.1]).

For $i = 0, \dots, t$ we put

$$a_{\mathbf{z}}^i(S/I) := \sup\{r|\ [(I, z_1, \dots, z_i) : z_{i+1}]_r \neq (I, z_1, \dots, z_i)_r\},$$

with $a_{\mathbf{z}}^i(S/I) = -\infty$ if $(I, z_1, \dots, z_i) : z_{i+1} = (I, z_1, \dots, z_i)$. These invariants can be ∞ and they are a measure for how far $\mathbf{z}$ is from being a regular sequence in S/I. It can be shown that $\mathbf{z}$ is a filter-regular sequence for S/I if and only if $a_{\mathbf{z}}^i(S/I) < \infty$ for $i = 0, \dots, t$ [T1, Lemma 2.1]. Note that our definition of $a_{\mathbf{z}}^i(S/I)$ is one less than that in [T1]. There is the following close relationship between these numbers and the partial regularity of S/I.

Theorem 2.1 ([T1, Proposition 2.2]). *Let $\mathbf{z}$ be a filter-regular sequence of linear forms for S/I. Then*

$$\operatorname{reg}_t(S/I) = \max\{a_{\mathbf{z}}^i(S/I)|\ i = 0, \dots, t\}.$$

We will use the following characterization of $a_{\mathbf{z}}^i(S/I)$.

Lemma 2.2. $a_{\mathbf{z}}^i(S/I) = \max\{r|\ [\bigcup_{m\geq 1}(I, z_1, \dots, z_i) : z_{i+1}^m]_r \neq (I, z_1, \dots, z_i)_r\}$.

Proof. Put $r_0 = \max\{r|\ [\bigcup_{m\geq 1}(I, z_1, \dots, z_i) : z_{i+1}^m]_r \neq (I, z_1, \dots, z_i)_r\}$. By definition, $a_{\mathbf{z}}^i(S/I) \leq r_0$. Conversely, if y is an element of $\bigcup_{m\geq 1}(I, z_1, \dots, z_i) : z_{i+1}^m]_{r_0}$, then

$$y z_{i+1} \in [\bigcup_{m\geq 1}(I, z_1, \dots, z_i) : z_{i+1}^m]_{r_0+1} = (I, z_1, \dots, z_i)_{r_0+1}.$$

Hence $y \in [(I, z_1, \dots, z_i) : z_{i+1}]_{r_0}$. This implies $r_0 \leq a_{\mathbf{z}}^i(S/I)$. So we get $r_0 = a_{\mathbf{z}}^i(S/I)$. ☐

Since $\operatorname{reg}(S/I) = \operatorname{reg}_d(S/I)$, to compute $\operatorname{reg}(S/I)$ we need a filter-regular sequence of linear forms of length $d+1$. But that can be avoided by the following observation.

Lemma 2.3. *Let $\mathbf{z} = z_1, \dots, z_d$ be a filter-regular sequence for S/I, $d = \dim(S/I)$. Then $\mathbf{z}$ is a system of parameters for S/I.*

Proof. Let $\mathfrak{p}$ be an arbitrary associated prime $\mathfrak{p}$ of $(I, z_1, \dots, z_i)$ with $\dim S/\mathfrak{p} = d - i$, $i = 0, \dots, d-1$. Then $\mathfrak{p} \neq \mathfrak{m}$ because $\dim S/\mathfrak{p} > 0$. By the definition of a filter-regular sequence, $z_{i+1} \notin \mathfrak{p}$. Hence $\mathbf{z}$ is a homogeneous system of parameters for S/I. ☐

If $\mathbf{z}$ is a homogeneous system of parameters for S/I, then $S/(I, z_1, \dots, z_d)$ is of finite length. Hence $(S/(I, z_1, \dots, z_d))_r = 0$ for r large enough. Following [NR] we call

$$r_{\mathbf{z}}(S/I) := \max\{r|\ (S/(I, z_1, \dots, z_d))_r \neq 0\}$$

the *reduction number* of S/I with respect to $\mathbf{z}$. It is equal to the maximum degree of the generators of S/I as a module over $k[z_1, \dots, z_d]$ [V]. Note that the minimum of $r_{\mathbf{z}}(S/I)$ is called the reduction number of S/I.

Theorem 2.4 ([BS, Theorem 1.10], [T1, Corollary 3.3])**.** *Let* $\mathbf{z}$ *be a filter-regular sequence of* d *linear forms for* S/I*. Then*

$$\operatorname{reg}(S/I) = \max\{a_{\mathbf{z}}^0(S/I), \dots, a_{\mathbf{z}}^{d-1}(S/I), r_{\mathbf{z}}(S/I)\}.$$

Remark. Theorem 2.4 was proved in [BS] under an additional condition on the maximum degree of the generators of I.

3. Evaluations of the initial ideal

Let $c_i(I)$, $i = 0, \dots, d$, and $r(I)$ be the invariants defined in Section 1 by means of evaluations of $\operatorname{in}(I)$, where $\operatorname{in}(I)$ is the initial ideal of I with respect to the reverse lexicographic order. We will use the results of Section 2 to express $\operatorname{reg}_t(S/I)$ and $\operatorname{reg}(S/I)$ in terms of $c_i(I)$ and $r(I)$.

Lemma 3.1. *For* $\mathbf{z} = x_n, \dots, x_{n-t}$ *and* $i = 0, \dots, t$ *we have*

$$a_{\mathbf{z}}^i(S/I) = c_i(I).$$

Proof. By [BS, Lemma (2.2)], $[(I, x_n, \dots, x_{n-i+1}) : x_{n-i}]_r = (I, x_n, \dots, x_{n-i+1})_r$ if and only if $[(\operatorname{in}(I), x_n, \dots, x_{n-i+1}) : x_{n-i}]_r = (\operatorname{in}(I), x_n, \dots, x_{n-i+1})_r$ for all $r \geq 0$. Therefore

$$a_{\mathbf{z}}^i(S/I) = a_{\mathbf{z}}^i(S/\operatorname{in}(I)).$$

By Lemma 2.2 we get

$$a_{\mathbf{z}}^i(S/\operatorname{in}(I)) = \sup\{r \mid [\bigcup_{m\geq 1} (\operatorname{in}(I), x_n, \dots, x_{n-i+1}) : x_{n-i}^m]_r \neq (\operatorname{in}(I), x_n, \dots, x_{n-i+1})_r\}.$$

Note that J_i is the ideal of $S_i = k[x_1, \dots, x_{n-i}]$ obtained from $\operatorname{in}(I)$ by the evaluation $x_{n-i+1} = \cdots = x_n = 0$ and that this evaluation corresponds to the canonical isomorphism $S/(x_{n-i+1}, \dots, x_n) \cong S_i$. Then we may rewrite the above formula as

$$a_{\mathbf{z}}^i(S/\operatorname{in}(I)) = \sup\{r \mid [\bigcup_{m\geq 1} J_i : x_{n-i}^m]_r \neq (J_i)_r\}.$$

Since J_i is a monomial ideal, $\bigcup_{m\geq 1} J_i : x_{n-i}^m$ is generated by the monomials g in the variables $x_1, \dots, x_{n-i-1}$ for which there exists an integer $m \geq 1$ such that $g x_{n-i}^m \in J_i$. Such a monomial g is determined by the condition $g \in \tilde{J}_i$. Hence

$$a_{\mathbf{z}}^i(S/\operatorname{in}(I)) = \sup\{r \mid (\tilde{J}_i)_r \neq (J_i)_r\} = c_i(I).$$

□

As a consequence of Lemma 3.1 we can use the invariants $c_i(I)$ to check when $x_n, \dots, x_{n-t}$ is a regular resp. filter-regular sequence for S/I.

Corollary 3.2. x_{n-i} *is a non-zerodivisor in* $S/(I, x_n, \dots, x_{n-i+1})$ *if and only if* $c_i(I) = -\infty$*.*

Proof. By definition, $a_{\mathbf{z}}^i(S/I) = -\infty$ if and only if x_{n-i} is a non-zerodivisor in $S/(I, x_n, \dots, x_{n-i+1})$. Hence the conclusion follows from Lemma 3.1. □

Corollary 3.3. *Let* $\mathbf{z} = x_n, \dots, x_{n-t}$. *Then* $\mathbf{z}$ *is a filter-regular sequence for* S/I *if and only if* $c_i(I) < \infty$ *for* $i = 0, \dots, t$.

Proof. It is known that $\mathbf{z}$ is a filter-regular sequence for S/I if and only if $a^i_{\mathbf{z}}(S/I) < \infty$ for $i = 0, \dots, t$ [T1, Lemma 2.1]. □

Now we can characterize $\operatorname{reg}_t(S/I)$ as follows.

Theorem 3.4. *Assume that* $c_i(I) < \infty$ *for* $i = 0, \dots, t$. *Then*

$$\operatorname{reg}_t(S/I) = \max\{c_i(I) \mid i = 0, \dots, t\}.$$

Proof. This follows from Theorem 2.1, Lemma 3.1 and Corollary 3.3. □

We can also give a characterization of $\operatorname{reg}(S/I)$ which involves $r(I)$.

Lemma 3.5. *Assume that* $c_i(I) < \infty$ *for* $i = 0, \dots, d-1$. *Then*

$$r_{\mathbf{z}}(S/I) = r(I).$$

Proof. By Corollary 3.3, $\mathbf{z} = x_n, \dots, x_{n-d+1}$ is a filter-regular sequence for S/I. By Lemma 2.3 and [T2, Theorem 4.1], this implies that $\mathbf{z}$ is a homogeneous system of parameters for $S/\operatorname{in}(I)$ with

$$r_{\mathbf{z}}(S/I) = r_{\mathbf{z}}(S/\operatorname{in}(I)).$$

Note that $S/(x_{n-d+1}, \dots, x_n) \cong S_d$ and that J_d is the ideal obtained from $\operatorname{in}(I)$ by the evaluation $x_{n-d+1} = \cdots = x_n = 0$. Then

$$\begin{aligned} r_{\mathbf{z}}(S/\operatorname{in}(I)) &= \max\{r \mid (S/(\operatorname{in}(I), x_n, \dots, x_{n-d+1}))_r \neq 0\} \\ &= \max\{r \mid (S_d/J_d)_r \neq 0\} \\ &= r(I). \end{aligned}$$

□

Theorem 3.6. *Assume that* $c_i(I) < \infty$ *for* $i = 0, \dots, d-1$. *Then*

$$\operatorname{reg}(S/I) = \max\{c_0(I), \dots, c_{d-1}(I), r(I)\}.$$

Proof. This follows from Theorem 2.4, Lemma 3.1, Corollary 3.3 and Lemma 3.5. □

4. Combinatorial description

First, we want to show that the condition $c_i(I) < \infty$ can be easily checked in terms of the lattice vectors of the generators of $\operatorname{in}(I)$. Let $\mathcal{B}$ be the (finite) set of monomials which minimally generates $\operatorname{in}(I)$. We set

$$E_i := \{v \in \mathbb{N}^{n-i} \mid x^v \in \mathcal{B}\},$$

where $x^v = x_1^{\varepsilon_1} \cdots x_s^{\varepsilon_s}$ if $v = (\varepsilon_1, \dots, \varepsilon_s)$. For $j = 1, \dots, n-i$ we denote by p_j the projection from $\mathbb{N}^{n-i}$ to $\mathbb{N}^{n-i-1}$ which deletes the jth coordinate. For two lattice vectors $a = (\alpha_1, \dots, \alpha_s)$ and $b = (\beta_1, \dots, \beta_s)$ of the same size we say $a \geq b$ if $\alpha_j \geq \beta_j$ for $j = 1, \dots, s$.

Lemma 4.1. $c_i(I) < \infty$ *if and only if for every element* $a \in p_{n-i}(E_i) \setminus E_{i+1}$ *there are elements* $b_j \in E_{i+1}$ *such that* $p_j(a) \geq p_j(b_j)$, $j = 1, \dots, n-i-1$.

Proof. Recall that $c_i(I) = \sup\{r|\ (\tilde{J}_i/J_i)_r \neq 0\}$. Then $c_i(I) < \infty$ if and only if $\tilde{J}_i/J_i$ is of finite length. By the definition of J_i and $\tilde{J}_i$, the latter condition is equivalent to the existence of a number r such that $x_j^r\tilde{J}_i \subseteq J_i$ for $j = 1, \dots, n-i$. It is clear that J_i is generated by the monomials x^v with $v \in E_i$. From this it follows that $\tilde{J}_i$ is generated by J_i and the monomials x^a with $a \in p_{n-i}(E_i) \setminus E_{i+1}$. For such a monomial x^a we can always find a number r such that $x_{n-i}^r x^a \in J_i$. For $j < n-i$, $x_j^r x^a \in J_i$ if and only if $x_j^r x^a$ is divided by a generator x^{b_j} of J_i. Since $x_j^r x^a$ does not contain $x_{n-i}, ..., x_n$, so does x^{b_j}. Hence $b_j \in E_{i+1}$. Setting $x_j = 1$ we see that $x_j^r x^a$ is divided by x^{b_j} for some number r if and only if $p_j(a) \geq p_j(b_j)$. □

If $c_i(I) = \infty$, we should make a random linear transformation of the variables $x_1, \dots, x_{n-i}$ and test the condition $c_i(I) < \infty$ again. By Lemma 3.1 the linear transformation does not change the invariants $c_j(I)$ for $j < i$. Moreover, instead of $\mathrm{in}(I)$ we only need to compute the smaller initial ideal $\mathrm{in}(I_i)$, where I_i denotes the ideal of S_i obtained from I by the evaluation $x_{n-i+1} = \cdots = x_n = 0$. Let $\mathcal{B}_i$ be the set of monomials which minimally generates $\mathrm{in}(I_i)$. It is easy to see that $\mathcal{B}_i$ is the set of the monomials of $\mathcal{B}$ which are not divided by $x_{n-i+1}, \dots, x_n$. From this it follows that $E_j = \{v \in \mathbb{N}^{n-j}|\ x^v \in \mathcal{B}_i\}$ for $j \leq i$. Thus, we can use this formula to compute E_j and to check the condition $c_j(I) < \infty$ for $j \leq i$. Once we know $c_i(I) < \infty$ we can proceed to compute $c_i(I)$.

In the lattice $\mathbb{N}^{n-i}$ we delete the shadow of E_i, that is, the set of elements a for which there is $v \in E_i$ with $v \leq a$. The remaining lattice has the shape of a staircase and we will denote by F_i the set of its corners. It is easy to see that F_i is the set of the elements of the form $a = \max(v_1, \dots, v_{n-i}) - (1, \dots, 1)$ with $a \not\geq v$ for any element $v \in E_i$, where $v_1, \dots, v_{n-i}$ is a family of $n-i$ elements of E_i for which the jth coordinate of v_j is greater than the jth coordinate of v_h for all $h \neq j$, $j = 1, \dots, n-i$, and $\max(v_1, \dots, v_{n-i})$ denotes the element whose coordinates are the maxima of the corresponding coordinates of $v_1, \dots, v_{n-i}$. If $a = (\alpha_1, \dots, \alpha_{n-i})$, we set

$$|a| := \alpha_1 + \ldots + \alpha_{n-i}.$$

Proposition 4.2. *Assume that $c_i(I) < \infty$. Then $c_i(I) = -\infty$ if $F_i = \emptyset$ and $c_i(I) = \max_{a \in F_i} |a|$ if $F_i \neq -\emptyset$.*

Proof. Let a be an arbitrary element of F_i. Then $a = \max(v_1, \dots, v_{n-i}) - (1, \dots, 1)$ for some family $v_1, \dots, v_{n-i}$ of S_i. Let $v_j = (\varepsilon_{j1}, \dots, \varepsilon_{jn-i})$, $j = 1, \dots, n-i$. Then $a = (\varepsilon_{11} - 1, \dots, \varepsilon_{n-in-i} - 1)$. Since $\varepsilon_{jj} > \varepsilon_{hj}$ for $h \neq j$, we get $a \geq (\varepsilon_{n-i1}, \dots, \varepsilon_{n-in-i-1}, 0)$. Therefore, x^a is divided by the monomial obtained from $x^{v_{n-i}}$ by setting $x_{n-i} = 1$. Note that J_i is generated by the monomials x^v with $x_v \in E_i$. Since $v_{n-i} \in E_i$, we have $x^{v_{n-i}} \in J_i$, whence $x^a \in \tilde{J}_i$. On the other hand, $x^a \notin J_i$ because $a \not\geq v$ for any element $v \in E_i$. Since $|a| = \deg x^a$, this implies $(\tilde{J}_i/J_i)_{|a|} \neq 0$. Hence $|a| \leq c_i(I)$. So we obtain $\max_{a \in F_i} |a| \leq c_i(I)$ if $F_i \neq \emptyset$.

To prove the converse inequality we assume that $\tilde{J}_i/J_i \neq 0$. Since $c_i(I) < \infty$, there is a monomial $x^b \in \tilde{J}_i \setminus J_i$ such that $\deg x^b = c_i(I)$. Since $x^b \notin J_i$, $b \not\geq v$ for any element $v \in E_i$. For $j = 1, \dots, n-i$ we have $x_j x^b \in J_i$ because $\deg x_j x^b = c_i(I) + 1$. Therefore, $x_j x^b$ is divided by some monomial x^{v_j} with $v_j \in E_i$. Let $b = (\beta_1, \dots, \beta_{n-i})$ and $v_j = (\varepsilon_{j1}, \dots, \varepsilon_{jn-i})$. Then $\beta_h \geq \varepsilon_{jh}$ for $h \neq j$ and $\beta_j + 1 \geq \varepsilon_{jj}$.

Since $b \not\geq v_j$, we must have $\beta_j < \varepsilon_{jj}$, hence $\beta_j = \varepsilon_{jj} - 1$. It follows that $\varepsilon_{jj} = \beta_j + 1 > \varepsilon_{hj}$ for all $h \neq j$. Thus, the family $v_1, \ldots, v_{n-i}$ belongs to $\mathcal{S}_i$ and $b = \max(v_1, \ldots, v_{n-i}) - (1, \ldots, 1)$. So we have proved that $b \in F_i$. Hence $c_i(I) = \deg x^b = |b| \leq \max_{a \in F_i} |a|$.

The above argument also shows that $F_i \neq \emptyset$ if $\tilde{J}_i \neq J_i$. So $c_i(I) = -\infty$ if $F_i = \emptyset$. □

By Corollary 3.3, if $c_i(I) < \infty$ for $i = 0, \ldots, d-1$, then $\mathbf{z} = x_n, \ldots, x_{n-d+1}$ is a filter-regular sequence for S/I. By Lemma 2.3 and Lemma 3.5, that implies $r(I) = r_{\mathbf{z}}(S/I) < \infty$. In this case, we have the following description of $r(I)$.

Proposition 4.3. *Assume that $r(I) < \infty$. Then $r(I) = \max_{a \in F_d} |a|$.*

Proof. This can be proved similarly to the proof of Lemma 4.2. □

Combining the above results with Theorem 3.4 and Theorem 3.6 we get a simple method to compute $\operatorname{reg}_t(S/I)$ and $\operatorname{reg}(S/I)$. We will illustrate the above method by an example at the end of the next section. Moreover, we get the following estimation for $\operatorname{reg}_t(S/I)$.

Corollary 4.4. *Let $x_n, \ldots, x_{n-t}$ be a filter-regular sequence for S/I. Let g_i denote the least common multiple of the minimal generators of* $\operatorname{in}(I)$ *which are not divided by any of the variables $x_{n-i+1}, \ldots, x_n$. Then*

$$\operatorname{reg}_t(S/I) \leq \max\{\deg g_i - n + i |\ i = 0, \ldots, t\}.$$

Proof. By Corollary 3.3, the assumption implies that $c_i(I) < \infty$ for $i = 0, ..., t$. Thus, combining Theorem 3.4 and Lemma 4.2 we get

$$\operatorname{reg}_t(S/I) \leq \max\{|a|\,|\ a \in F_i,\ i = 0, \ldots, t\}.$$

It is easily seen from the definition of F_i that $\max_{a \in F_i} |a| \leq \deg g_i - n + i$, $i = 0, \ldots, t$, hence the conclusion. □

Remark. Bruns and Herzog [BH, Theorem 3.1(a)], resp. Hoa and Trung [HT, Theorem 3.1], proved that for any monomial ideal I, $\operatorname{reg}(S/I) \leq \deg f - 1$, resp. $\deg f - \operatorname{ht} I$, where f is the least common multiple of the minimal generators of I. Note that the mentioned result of Bruns and Herzog is valid for multigraded modules.

5. The case of projective curves

Let $I_C \subset k[x_1, \ldots, x_n]$ be the defining saturated ideal of a (not necessarily reduced) projective curve $C \subset \mathbb{P}^{n-1}$, $n \geq 3$. We will assume that $k[x_{n-1}, x_n] \hookrightarrow S/I_C$ is a Noether normalization of S/I_C. In this case, Theorem 3.6 can be reformulated as follows.

Proposition 5.1. $\operatorname{reg}(S/I_C) = \max\{c_1(I_C), r(I_C)\}$.

Proof. By the above assumption S/I_C is a generalized Cohen-Macaulay ring of positive depth and x_n, x_{n-1} is a homogeneous system of parameters for S/I_C. Therefore, x_n, x_{n-1} is a filter-regular sequence for S/I_C. In particular, x_n is a non-zerodivisor in S/I_C. By Lemma 3.2, $c_0(I_C) = -\infty$. Hence the conclusion follows from Theorem 3.6. □

Since S/I_C has positive depth, the graded minimal free resolution of S/I_C ends at most at the $(n-1)$th place:

$$0 \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow S/I_C \longrightarrow 0.$$

From Theorem 3.4 we obtain the following information on the shifts of F_{n-1}. Note that $F_{n-1} = 0$ if S/I_C is a Cohen-Macaulay ring or, in other words, if C is an arithmetically Cohen-Macaulay curve.

Proposition 5.2. *If C is not an arithmetically Cohen-Macaulay curve, $c_1(I_C) + n - 1$ is the maximum degree of the generators of F_{n-1}.*

Proof. Let b_{n-1} be the maximum degree of the generators of F_{n-1}. As we have seen in the introduction, $b_{n-1} - n + 1 = (n-1)$–$\operatorname{reg}(S/I_C) = \operatorname{reg}_1(S/I_C)$. By Theorem 3.4, $\operatorname{reg}_1(S/I_C) = \max\{c_0(I_C), c_1(I_C)\} = c_1(I_C)$ because $c_0(I_C) = -\infty$. So we obtain $b_{n-1} = c_1(I_C) + n - 1$. □

Now we shall see that Proposition 5.1 contains all main results of Bermejo and Gimenez in [BG]. It should be noted that they did not use strong results such as Theorem 2.4. We follow the notations of [BG].

Let $E := \{a \in \mathbb{N}^{n-2} \mid x^a \in \operatorname{in}(I_C)\}$ and denote by $H(E)$ the smallest integer r such that $a \in E$ if $|a| = r$.

Corollary 5.3 ([BG, Theorem 2.4]). *Assume that C is an arithmetically Cohen-Macaulay curve. Then* $\operatorname{reg}(S/I_C) = H(E) - 1$.

Proof. Since x_n, x_{n-1} is a regular sequence in S/I_C, we have $c_1(I_C) = -\infty$ by Corollary 3.2. By Proposition 5.1 this implies $\operatorname{reg}(S/I_C) = r(I_C)$. But

$$r(I_C) = \sup\{r \mid (S_2/J_2)_r \neq 0\} = H(E) - 1$$

because J_2 is generated by the monomials x^a, $a \in E$. □

Let I_0 be the ideal in S generated by the polynomials obtained from I_C by the evaluation $x_{n-1} = x_n = 0$. Then S/I_0 is a two-dimensional Cohen-Macaulay ring. Let $\tilde{I}$ denote the ideal in S generated by the monomials obtained from $\operatorname{in}(I_C)$ by the evaluation $x_{n-1} = x_n = 1$. Let

$$F := \{a \in \mathbb{N}^{n-2} \mid x^a \in \tilde{I} \setminus \operatorname{in}(I_0)\}.$$

For every vector $a \in F$ let

$$E_a := \{(\mu, \nu) \in \mathbb{N}^2 \mid x^a x_{n-1}^{\mu} x_n^{\nu} \in \operatorname{in}(I_C)\}.$$

Let $\Re := \bigcup_{a \in F}\{a \times [\mathbb{N}^2 \setminus E_a]\}$ and denote by $H(\Re)$ the smallest integer r such that the number of the elements $b \in \Re$ with $|b| = s$ becomes a constant for $s \geq r$.

Corollary 5.4 ([BG, Theorem 2.7]). $\operatorname{reg}(S/I_C) = \max\{\operatorname{reg}(S/I_0), H(\Re)\}$.

Proof. As in the proof of Corollary 5.3 we have $\operatorname{reg}(S/I_0) = r(I_0)$. But $r(I_0) = r(I_C)$ because $\operatorname{in}(I_0)$ is the ideal generated by the monomials obtained from $\operatorname{in}(I_C)$ by the evaluation $x_{n-1} = x_n = 0$. Thus,

$$\operatorname{reg}(S/I_0) = r(I_C).$$

It has been observed in [BG] that the number of the elements $b \in \Re$ with $|b| = s$ is the difference $H_{S/I_C}(s) - H_{S/\tilde{I}}(s) = H_{S/\operatorname{in}(I_C)}(s) - H_{S/\tilde{I}}(s) = H_{\tilde{I}/\operatorname{in}(I_C)}(s)$, where $H_E(s)$ denotes the Hilbert function of a graded S-module E. Since x_n is a non-zerodivisor in $S/\operatorname{in}(I_C)$, $H(\Re)+1$ is the least integer r such that $H_{(\tilde{I},x_n)/(\operatorname{in}(I_C),x_n)}(s)$

$= 0$ for $s \geq r$. On the other hand, since $\mathrm{in}(I_C)$ is generated by monomials which do not contain x_n and since J_1 is the ideal in $k[x_1, \dots, x_{n-1}]$ obtained from $\mathrm{in}(I_C)$ by the evaluation $x_n = 0$, we have $\mathrm{in}(I_C) = J_1 S$ and $\tilde{I} = \tilde{J}_1 S$, whence $(\tilde{I}, x_n)/(\mathrm{in}(I_C), x_n) \cong \tilde{J}_1/J_1$. Note that $c_1(I_C) = \max\{r |\ (\tilde{J}_1/J_1)_r \neq 0\}$ with $c_1(I_C) = -\infty$ if $\tilde{J}_1 = J_1$. Then

$$H(\Re) = \max\{0, c_1(I_C)\}.$$

Thus, applying Proposition 5.1 we obtain $\mathrm{reg}(S/I_C) = \max\{\mathrm{reg}(S/I_0), H(\Re)\}$. $\square$

Example. Let $C \subset \mathbb{P}^3$ be the monomial curve $(t^\alpha s^\beta : t^\beta s^\alpha : s^{\alpha+\beta} : t^{\alpha+\beta})$, $\alpha > \beta > 0$, g.c.d. $(\alpha, \beta) = 1$. It is known that the defining ideal $I_C \subset k[x_1, x_2, x_3, x_4]$ is generated by the quadric $x_1x_2 - x_3x_4$ and the forms $x_1^{\beta+r} x_3^{\alpha-\beta-r} - x_2^{\alpha-r} x_4^r$, $r = 0, \dots, \alpha-\beta$, and that this is a Gröbner basis of I_C for the reverse lexicographic order with $x_1 > x_2 > x_3 > x_4$ [CM, Théorèm 3.9]. Therefore,

$$\mathrm{in}(I_C) = (x_1x_2, x_2^\alpha, x_1^{\beta+1}x_3^{\alpha-\beta-1}, x_1^{\beta+2}x_3^{\alpha-\beta-2}, \dots, x_1^\alpha).$$

Using the notations of Section 3 we have

$$\begin{aligned} E_1 &= \{(1,1,0), (0,\alpha,0), (\beta+1, 0, \alpha-\beta-1), (\beta+2, 0, \alpha-\beta-2), \dots, (\alpha, 0, 0)\}, \\ E_2 &= \{(1,1), (0,\alpha), (\alpha, 0)\}. \end{aligned}$$

From this it follows that

$$\begin{aligned} F_1 &= \{(\beta+1, 0, \alpha-\beta-2), (\beta+2, 0, \alpha-\beta-3), \dots, (\alpha-1, 0, 0)\}, \\ F_2 &= \{(0, \alpha-1), (\alpha-1, 0)\}. \end{aligned}$$

By Proposition 4.2, $c_1(I_C) = \alpha - 1$ if $\alpha - \beta \geq 2$ ($c_1(I_C) = -\infty$ if $\alpha - \beta = 1$) and $r(I_C) = \alpha - 1$ by Proposition 4.3. Applying Proposition 5.1 we obtain $\mathrm{reg}(S/I_C) = \alpha - 1$.

The direct computation of the invariant $H(\Re)$ is more complicated than that of $c_1(I_C)$. First, we should interpret F as the set of the elements of the form $a \in \mathbb{N}^2$ such that $a \geq b$ for some elements $b \in p(E_1)$ but $a \not\geq c$ for any element $c \in E_2$. Then we get

$$F = \{(\beta+1, 0), (\beta+2, 0), \dots, (\alpha-1, 0)\}.$$

For all $\varepsilon = \beta+1, \dots, \alpha-1$ we verify that $E_{(\varepsilon,0)} = (\alpha - \varepsilon, 0) + \mathbb{N}^2$. It follows that

$$\Re = \{(\varepsilon, 0, \mu, \nu) \in \mathbb{N}^4 |\ \varepsilon = \beta+1, \dots, \alpha-1;\ \mu \leq \alpha - \varepsilon - 1\}.$$

If $\alpha - \beta = 1$, we have $\Re = \emptyset$, hence $H(\Re) = 0$. If $\alpha - \beta \geq 2$, we can check that $H(\Re) = \alpha - 1$.

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