EVALUATIONS OF INITIAL IDEALS
AND CASTELNUOVO-MUMFORD REGULARITY

NGÔ VIỆT TRUNG

(Communicated by Wolmer V. Vasconcelos)

Abstract. This paper characterizes the Castelnuovo-Mumford regularity by evaluating the initial ideal with respect to the reverse lexicographic order.

1. Introduction

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$ of arbitrary characteristic. Let $I \subset S$ be an arbitrary homogeneous ideal and

$0 \rightarrow F_p \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow S/I \rightarrow 0$

a graded minimal free resolution of $S/I$. Write $b_i$ for the maximum degree of the generators of $F_i$. The Castelnuovo-Mumford regularity

$\text{reg}(S/I) := \max \{b_i - i | i = 0, \ldots, p \}$

is a measure for the complexity of $I$ in computational problems [EG], [BM], [V]. One can use Buchsberger's syzygy algorithm to compute $\text{reg}(S/I)$. However, such a computation is often very big. Theoretically, if $\text{char}(k) = 0$, $\text{reg}(S/I)$ is equal to the largest degree of the generators of the generic initial ideal of $I$ with respect to the reverse lexicographic order [BS]. But it is difficult to know when an initial ideal is generic. Therefore, it would be of interest to have other methods for the computation of $\text{reg}(S/I)$.

The aim of this paper is to present a simple method for the computation of $\text{reg}(S/I)$ which is based only on evaluations of $\text{in}(I)$, where $\text{in}(I)$ denotes the initial ideal of $I$ with respect to the reverse lexicographic order. We are inspired by a recent paper of Bermejo and Gimenez [BG] which gives such a method for the computation of the Castelnuovo-Mumford regularity of projective curves.

Let $d = \text{dim} S/I$. For $i = 0, \ldots, d$ put $S_i = k[x_1, \ldots, x_{n-i}]$. Let $J_i$ be the ideal of $S_i$ obtained from $\text{in}(I)$ by the evaluation $x_{n-i+1} = \cdots = x_n = 0$. Let $J_i$ denote the ideal of $S_i$ obtained from $J_i$ by the evaluation $x_{n-i} = 1$. These ideals can be easily computed from the generators of $\text{in}(I)$. In fact, if $\text{in}(I) = (f_1, \ldots, f_s)$, where $f_1, \ldots, f_s$ are monomials in $S$, then $J_i$ is generated by the monomials $f_j$ not divided
by any of the variables \(x_{n-i+1}, \ldots, x_n\) and \(\tilde{J}_i\) by those monomials obtained from the latter by setting \(x_{n-i} = 1\). Put
\[
c_i(I) := \sup \{r \mid (\tilde{J}_i/J_i)_r \neq 0\},
\]
with \(c_i(I) = -\infty\) if \(\tilde{J}_i = J_i\) and
\[
r(I) := \sup \{r \mid (S_d/J_d)_r \neq 0\}.
\]

We can express \(\text{reg}(S/I)\) in terms of these numbers as follows. Assume that \(c_i(I) < \infty\) for \(i = 0, \ldots, d-1\). Then
\[
\text{reg}(S/I) = \max\{c_0(I), \ldots, c_{d-1}(I), r(I)\}.
\]
The assumption \(c_i(I) < \infty\) for \(i = 0, \ldots, d-1\) is satisfied for a sufficiently general choice of the variables. If \(I\) is the defining saturated ideal of a projective (not necessarily reduced) curve, this assumption is automatically satisfied if \(k[x_{n-1}, x_n]\) is a Noether normalization of \(S/I\). In this case, \(c_0(I) = -\infty\) and \(\text{reg}(S/I) = \max\{c_1(I), r(I)\}\). From this formula we can easily deduce the results of Bermejo and Gimenez.

Similarly we can compute the partial regularities \(\ell\)-\(\text{reg}(S/I) := \max\{b_i - i \mid i \geq \ell\}\), \(\ell > 0\), which were recently introduced by Bayer, Charalambous and Popescu [BCP] (see also Aramova and Herzog [AH]). These regularities can be defined in terms of local cohomology. Let \(m\) denote the maximal homogeneous ideal of \(S\). Let \(H_m^i(S/I)\) denote the \(i\)th local cohomology module of \(S/I\) with respect to \(m\) and set \(a_i(S/I) = \max\{r \mid H_m^i(S/I)_r \neq 0\}\) with \(a_i(S/I) = -\infty\) if \(H_m^i(S/I) = 0\). For \(t \geq 0\) we define \(\text{reg}_i(S/I) := \max\{a_i(S/I) + i \mid i = 0, \ldots, t\}\). Then \(\text{reg}_i(S/I) = (n-t) - \text{reg}(S/I)\) [T2]. Under the assumption \(c_i(I) < \infty\) for \(i = 0, \ldots, t\) we obtain the following formula:
\[
\text{reg}_i(S/I) = \max\{c_i(I) \mid i = 0, \ldots, t\}.
\]
The numbers \(c_i(I)\) also allow us to determine the place at which \(\text{reg}(S/I)\) is attained in the minimal free resolution of \(S/I\). In fact, \(\text{reg}(S/I) = b_t - t\) if \(c_t(I) = \max\{c_i(I) \mid i = 0, \ldots, d\}\). Moreover, \(r(I)\) can be used to estimate the reduction number of \(S/I\) which is another measure for the complexity of \(I\) [V].

It turns out that the numbers \(c_i(I)\) and \(r(I)\) can be described combinatorially in terms of the lattice vectors of the generators of \(\text{in}(I)\) (see Propositions 4.1–4.3 for details). These descriptions together with the above formulae give an effective method for the computation of \(\text{reg}(S/I)\) and \(\text{reg}_i(S/I)\). From this we can derive the estimation
\[
\text{reg}_i(S/I) \leq \max\{\deg g_i - n + i \mid i = 0, \ldots, t\},
\]
where \(g_i\) is the least common multiple of the minimal generators of \(\text{in}(I)\) which are not divided by any of the variables \(x_{n-i+1}, \ldots, x_n\).

This paper is organized as follows. In Section 2 we prepare some facts on the Castelnuovo-Mumford regularity. In Section 3 we prove the above formulae for \(\text{reg}(S/I)\) and \(\text{reg}_i(S/I)\). The combinatorial descriptions of \(c_i(I)\) and \(r(I)\) are given in Section 4. Section 5 deals with the case of projective curves.

### 2. Filter-regular sequence of linear forms

We shall keep the notations of the preceding section. Let \(z = z_1, \ldots, z_{t+1}\) be a sequence of homogeneous elements of \(S\), \(t \geq 0\). We call \(z\) a filter-regular sequence for \(S/I\) if \(z_{i+1} \notin \mathfrak{p}\) for any associated prime \(\mathfrak{p} \neq m\) of \((I, z_1, \ldots, z_i), i = 0, \ldots, t\).
For $i = 0, \ldots, t$ we put
\[ a^t_z(S/I) := \sup \{ r \mid [I, z_1, \ldots, z_i] : z_{i+1}]_r \neq (I, z_1, \ldots, z_i)_r \}, \]
with $a^t_z(S/I) = -\infty$ if $[I, z_1, \ldots, z_i] : z_{i+1} = (I, z_1, \ldots, z_i)$. These invariants can be $\infty$ and they are a measure for how far $z$ is from being a regular sequence in $S/I$. It can be shown that $z$ is a filter-regular sequence for $S/I$ if and only if $a^t_z(S/I) < \infty$ for $i = 0, \ldots, t$. There is the following close relationship between these numbers and the partial regularity of $S/I$.

**Theorem 2.1** ([T1 Proposition 2.2]). Let $z$ be a filter-regular sequence of linear forms for $S/I$. Then
\[ \text{reg}_z(S/I) = \max \{ a^t_z(S/I) \mid i = 0, \ldots, t \}. \]

We will use the following characterization of $a^t_z(S/I)$.

**Lemma 2.2.** $a^t_z(S/I) = \max \{ r \mid \bigcup_{m \geq 1} [I, z_1, \ldots, z_i] : z_{i+1}^m]_r \neq (I, z_1, \ldots, z_i)_r \}$. 

**Proof.** Put $r_0 = \max \{ r \mid \bigcup_{m \geq 1} [I, z_1, \ldots, z_i] : z_{i+1}^m]_r \neq (I, z_1, \ldots, z_i)_r \}$. By definition, $a^t_z(S/I) \leq r_0$. Conversely, if $y$ is an element of $\bigcup_{m \geq 1} [I, z_1, \ldots, z_i] : z_{i+1}^m]_{r_0}$, then
\[ y z_{i+1} \in \bigcup_{m \geq 1} [I, z_1, \ldots, z_i] : z_{i+1}^m]_{r_0+1} = (I, z_1, \ldots, z_i)_{r_0+1}. \]
Hence $y \in [(I, z_1, \ldots, z_i) : z_{i+1}]_{r_0}$. This implies $r_0 \leq a^t_z(S/I)$. So we get $r_0 = a^t_z(S/I)$. \( \square \)

Since $\text{reg}(S/I) = \text{reg}_d(S/I)$, to compute $\text{reg}(S/I)$ we need a filter-regular sequence of linear forms of length $d + 1$. But that can be avoided by the following observation.

**Lemma 2.3.** Let $z = z_1, \ldots, z_d$ be a filter-regular sequence for $S/I$, $d = \dim(S/I)$. Then $z$ is a system of parameters for $S/I$.

**Proof.** Let $p$ be an arbitrary associated prime $p$ of $(I, z_1, \ldots, z_i)$ with $\dim S/p = d - i$, $i = 0, \ldots, d - 1$. Then $p \neq m$ because $\dim S/p > 0$. By the definition of a filter-regular sequence, $z_{i+1} \notin p$. Hence $z$ is a homogeneous system of parameters for $S/I$. \( \square \)

If $z$ is a homogeneous system of parameters for $S/I$, then $S/(I, z_i, \ldots, z_d)$ is of finite length. Hence $(S/(I, z_i, \ldots, z_d))_r = 0$ for $r$ large enough. Following [NR] we call
\[ r_z(S/I) := \max \{ r \mid (S/(I, z_1, \ldots, z_d))_r \neq 0 \} \]
the reduction number of $S/I$ with respect to $z$. It is equal to the maximum degree of the generators of $S/I$ as a module over $k[z_1, \ldots, z_d]$. Note that the minimum of $r_z(S/I)$ is called the reduction number of $S/I$.

**Theorem 2.4** ([BS, Theorem 1.10], [11] Corollary 3.3). Let $z$ be a filter-regular sequence of $d$ linear forms for $S/I$. Then

$$\text{reg}(S/I) = \max\{a^0_z(S/I), \ldots, a^{d-1}_z(S/I), r_z(S/I)\}.$$

**Remark.** Theorem 2.4 was proved in [BS] under an additional condition on the maximum degree of the generators of $I$.

3. Evaluations of the Initial Ideal

Let $c_i(I)$, $i = 0, \ldots, d$, and $r(I)$ be the invariants defined in Section 1 by means of evaluations of $\text{in}(I)$, where $\text{in}(I)$ is the initial ideal of $I$ with respect to the reverse lexicographic order. We will use the results of Section 2 to express $\text{reg}_z(S/I)$ and $\text{reg}(S/I)$ in terms of $c_i(I)$ and $r(I)$.

**Lemma 3.1.** For $z = x_{n-t}, \ldots, x_{n-1}$ and $i = 0, \ldots, t$ we have

$$a^i_z(S/I) = c_i(I).$$

**Proof.** By [BS, Lemma (2.2)], $[(I, x_{n-t}, \ldots, x_{n-i+1}) : x_{n-i}]_r = (I, x_{n-t}, \ldots, x_{n-i+1})_r$ if and only if $[(\text{in}(I), x_{n-t}, \ldots, x_{n-i+1}) : x_{n-i}]_r = (\text{in}(I), x_{n-t}, \ldots, x_{n-i+1})_r$ for all $r \geq 0$. Therefore

$$a^i_z(S/I) = a^i_z(S/\text{in}(I)).$$

By Lemma 2.2 we get

$$a^i_z(S/\text{in}(I)) = \sup\{r \mid \bigcup_{m \geq 1} (\text{in}(I), x_{n-t}, \ldots, x_{n-i+1}) : x_{n-i}^m\rceil_r \\ \neq (\text{in}(I), x_{n-t}, \ldots, x_{n-i+1})_r\}. $$

Note that $J_i$ is the ideal of $S_i = k[x_1, \ldots, x_{n-i}]$ obtained from $\text{in}(I)$ by the evaluation $x_{n-i+1} = \cdots = x_n = 0$ and that this evaluation corresponds to the canonical isomorphism $S/(x_{n-i+1}, \ldots, x_n) \cong S_i$. Then we may rewrite the above formula as

$$a^i_z(S/\text{in}(I)) = \sup\{r \mid \bigcup_{m \geq 1} J_i : x_{n-i}^m\rceil_r \neq (J_i)_r\}. $$

Since $J_i$ is a monomial ideal, $\bigcup_{m \geq 1} J_i : x_{n-i}^m$ is generated by the monomials $g$ in the variables $x_1, \ldots, x_{n-i-1}$ for which there exists an integer $m \geq 1$ such that $g x_{n-i}^m \in J_i$. Such a monomial $g$ is determined by the condition $g \in \tilde{J}_i$. Hence

$$a^i_z(S/\text{in}(I)) = \sup\{r \mid (\tilde{J}_i)_r \neq (J_i)_r\} = c_i(I).$$

As a consequence of Lemma 3.1 we can use the invariants $c_i(I)$ to check when $x_{n-t}$ is a regular resp. filter-regular sequence for $S/I$.

**Corollary 3.2.** $x_{n-i}$ is a non-zerodivisor in $S/(I, x_{n-t}, \ldots, x_{n-i+1})$ if and only if $c_i(I) = -\infty$.

**Proof.** By definition, $a^i_z(S/I) = -\infty$ if and only if $x_{n-i}$ is a non-zerodivisor in $S/(I, x_{n-t}, \ldots, x_{n-i+1})$. Hence the conclusion follows from Lemma 3.1. 

Corollary 3.3. Let \( z = x_n, \ldots, x_{n-t} \). Then \( z \) is a filter-regular sequence for \( S/I \) if and only if \( c_i(I) < \infty \) for \( i = 0, \ldots, t \).

Proof. It is known that \( z \) is a filter-regular sequence for \( S/I \) if and only if \( a_z^d(S/I) < \infty \) for \( i = 0, \ldots, t \) [11, Lemma 2.1]. □

Now we can characterize \( \text{reg}_t(S/I) \) as follows.

Theorem 3.4. Assume that \( c_i(I) < \infty \) for \( i = 0, \ldots, t \). Then

\[
\text{reg}_t(S/I) = \max\{c_i(I) \mid i = 0, \ldots, t\}.
\]

Proof. This follows from Theorem 2.4, Lemma 3.1, Corollary 3.3 and Lemma 3.5. □

We can also give a characterization of \( \text{reg}(S/I) \) which involves \( r(I) \).

Lemma 3.5. Assume that \( c_i(I) < \infty \) for \( i = 0, \ldots, d-1 \). Then

\[
r_z(S/I) = r(I).
\]

Proof. By Corollary 3.3, \( z = x_n, \ldots, x_{n-d+1} \) is a filter-regular sequence for \( S/I \). By Lemma 2.3 and [12, Theorem 4.1], this implies that \( z \) is a homogeneous system of parameters for \( S/\text{in}(I) \) with

\[
r_z(S/I) = r_z(S/\text{in}(I)).
\]

Note that \( S/(x_{n-d+1}, \ldots, x_n) \cong S_d \) and that \( J_d \) is the ideal obtained from \( \text{in}(I) \) by the evaluation \( x_{n-d+1} = \cdots = x_n = 0 \). Then

\[
\begin{align*}
r_z(S/\text{in}(I)) &= \max\{r\mid (S/(\text{in}(I), x_n, \ldots, x_{n-d+1}))_r \neq 0\} \\
&= \max\{r\mid (S_d/J_d)_r \neq 0\} \\
&= r(I).
\end{align*}
\]

□

Theorem 3.6. Assume that \( c_i(I) < \infty \) for \( i = 0, \ldots, d-1 \). Then

\[
\text{reg}(S/I) = \max\{c_0(I), \ldots, c_{d-1}(I), r(I)\}.
\]

Proof. This follows from Theorem 2.4, Lemma 3.1, Corollary 3.3 and Lemma 3.5. □

4. COMBINATORIAL DESCRIPTION

First, we want to show that the condition \( c_i(I) < \infty \) can be easily checked in terms of the lattice vectors of the generators of \( \text{in}(I) \). Let \( \mathcal{B} \) be the (finite) set of monomials which minimally generates \( \text{in}(I) \). We set

\[
E_i := \{v \in \mathbb{N}^{n-i} \mid x^v \in \mathcal{B}\},
\]

where \( x^v = x_1^{v_1} \cdots x_s^{v_s} \) if \( v = (v_1, \ldots, v_s) \). For \( j = 1, \ldots, n-i \) we denote by \( p_j \) the projection from \( \mathbb{N}^{n-i} \) to \( \mathbb{N}^{n-i-1} \) which deletes the \( j \)th coordinate. For two lattice vectors \( a = (\alpha_1, \ldots, \alpha_s) \) and \( b = (\beta_1, \ldots, \beta_s) \) of the same size we say \( a \geq b \) if \( \alpha_j \geq \beta_j \) for \( j = 1, \ldots, s \).

Lemma 4.1. \( c_i(I) < \infty \) if and only if for every element \( a \in p_{n-i}(E_i) \setminus E_{i+1} \) there are elements \( b_j \in E_{i+1} \) such that \( p_j(a) \geq p_j(b_j) \), \( j = 1, \ldots, n-i-1 \).
Proof. Recall that $c_i(I) = \sup \{ r \mid (J_i/J_i)_r \neq 0 \}$. Then $c_i(I) < \infty$ if and only if $J_i/J_i$ is of finite length. By the definition of $J_i$ and $\tilde{J}_i$, the latter condition is equivalent to the existence of a number $r$ such that $x^{r_i}_i \tilde{J}_i \subseteq J_i$ for $j = 1, \ldots, n - i$. It is clear that $J_i$ is generated by the monomials $x^v$ with $v \in E_i$. From this it follows that $J_i$ is generated by $J_i$ and the monomials $x^a$ with $a \in p_{n-i}(E_i) \setminus E_{i+1}$. For such a monomial $x^a$ we can always find a number $r$ such that $x^{r_i}_i x^a \in J_i$. For $j < n - i$, $x^{r_i}_i x^a \in J_i$ if and only if $x^{r_i}_i x^a$ is divided by a generator $x^{b_j}$ of $J_i$. Since $x^{r_i}_i x^a$ does not contain $x_{n-i}, \ldots, x_n$, so does $x^{b_j}$. Hence $b_j \in E_{i+1}$. Setting $x_j = 1$ we see that $x^{r_i}_i x^a$ is divided by $x^{b_j}$ for some number $r$ if and only if $p_j(a) \geq p_j(b_j)$. \[ \Box \]

If $c_i(I) = \infty$, we should make a random linear transformation of the variables $x_1, \ldots, x_{n-i}$ and test the condition $c_i(I) < \infty$ again. By Lemma 3.1 the linear transformation does not change the invariants $c_j(I)$ for $j < i$. Moreover, instead of \[ \text{in}(I) \] we only need to compute the smaller initial ideal in($I_i$), where $I_i$ denotes the ideal of $S_i$ obtained from $I$ by the evaluation $x_{n-i+1} = \ldots = x_n = 0$. Let $B_i$ be the set of monomials which minimally generates in($I_i$). It is easy to see that $B_i$ is the set of the monomials of $B$ which are not divided by $x_{n-i+1}, \ldots, x_n$. From this it follows that $E_j = \{ v \in \mathbb{N}^{n-j} \mid x^v \in B_i \}$ for $j \leq i$. Thus, we can use this formula to compute $E_j$ and to check the condition $c_j(I) < \infty$ for $j \leq i$. Once we know $c_i(I) < \infty$ we can proceed to compute $c_i(I)$.

In the lattice $\mathbb{N}^{n-i}$ we delete the shadow of $E_i$, that is, the set of elements $a$ for which there is $v \in E_i$ with $v \leq a$. The remaining lattice has the shape of a staircase and we will denote by $F_i$ the set of its corners. It is easy to see that $F_i$ is the set of the elements of the form $a = \max(v_1, \ldots, v_{n-i}) - (1, \ldots, 1)$ with $a \npreceq v$ for any element $v \in E_i$, where $v_1, \ldots, v_{n-i}$ is a family of $n - i$ elements of $E_i$ for which the $j$th coordinate of $v_j$ is greater than the $j$th coordinate of $v_i$ for all $h \neq j, j = 1, \ldots, n - i$, and $\max(v_1, \ldots, v_{n-i})$ denotes the element whose coordinates are the maxima of the corresponding coordinates of $v_1, \ldots, v_{n-i}$. If $a = (a_1, \ldots, a_{n-i})$, we set $|a| := a_1 + \ldots + a_{n-i}$.

**Proposition 4.2.** Assume that $c_i(I) < \infty$. Then $c_i(I) = -\infty$ if $F_i = \emptyset$ and $c_i(I) = \max_{a \in F_i} |a|$ if $F_i \neq \emptyset$.

**Proof.** Let $a$ be an arbitrary element of $F_i$. Then $a = \max(v_1, \ldots, v_{n-i}) - (1, \ldots, 1)$ for some family $v_1, \ldots, v_{n-i}$ of $S_i$. Let $\varepsilon_j = (\varepsilon_{j1}, \ldots, \varepsilon_{jn-i})$, $j = 1, \ldots, n-i$. Then $a = (\varepsilon_{11} - 1, \ldots, \varepsilon_{n-in-i} - 1)$. Since $\varepsilon_{jj} > \varepsilon_{hh}$ for $h \neq j$, we get $a \succeq (\varepsilon_{n-i1}, \ldots, \varepsilon_{n-in-i-1}, 0)$. Therefore, $x^a$ is divided by the monomial obtained from $x^{\varepsilon_{n-i}}$ by setting $x_{n-i} = 1$. Note that $\tilde{J}_i$ is generated by the monomials $x^v$ with $v \in E_i$. Since $v_{n-i} \in E_i$, we have $x^{\varepsilon_{n-i}} \in \tilde{J}_i$, whence $x^a \in \tilde{J}_i$. On the other hand, $x^a \npreceq \tilde{J}_i$ because $a \npreceq v$ for any element $v \in E_i$. Since $|a| = \deg x^a$, this implies $(\tilde{J}_i/\tilde{J}_i)|a| = 0$. Hence $|a| \leq c_i(I)$. So we obtain $\max_{a \in F_i} |a| \leq c_i(I)$ if $F_i \neq \emptyset$.

To prove the converse inequality we assume that $\tilde{J}_i/\tilde{J}_i \neq 0$. Since $c_i(I) < \infty$, there is a monomial $x^b \in \tilde{J}_i \setminus \tilde{J}_i$ such that $\deg x^b = c_i(I)$. Since $x^b \npreceq \tilde{J}_i$, $b \npreceq v$ for any element $v \in E_i$. For $j = 1, \ldots, n-i$ we have $x_j x^b \in \tilde{J}_i$ because $\deg x_j x^b = c_i(I) + 1$. Therefore, $x_j x^b$ is divided by some monomial $x^v$ with $v_j \in E_i$. Let $b = (\beta_1, \ldots, \beta_{n-i})$ and $v_j = (\varepsilon_{j1}, \ldots, \varepsilon_{jn-i})$. Then $\beta_h \geq \varepsilon_{jh}$ for $h \neq j$ and $\beta_j + 1 \geq \varepsilon_{jj}$.
Since $b \not \in v_j$, we must have $\beta_j < \epsilon_{jj}$, hence $\beta_j = \epsilon_{jj} - 1$. It follows that $\epsilon_{jj} = \beta_j + 1 > \epsilon_{hj}$ for all $h \neq j$. Thus, the family $v_1, \ldots, v_{n-i}$ belongs to $S_i$ and $b = \max(v_1, \ldots, v_{n-i}) - (1, \ldots, 1)$. So we have proved that $b \in F_i$. Hence $c_i(I) = \deg x^b = |b| \leq \max_{a \in F_i}|a|.

The above argument also shows that $F_i \neq \emptyset$ if $\hat{I}_i \neq I_i$. So $c_i(I) = -\infty$ if $F_i = \emptyset$.

By Corollary 3.3 if $c_i(I) < \infty$ for $i = 0, \ldots, d - 1$, then $z = x_1, \ldots, x_{n-d+1}$ is a filter-regular sequence for $S/I$. By Lemma 4.2 and Lemma 5.3 that implies $r(I) = r_d(S/I) < \infty$. In this case, we have the following description of $r(I)$.

**Proposition 4.3.** Assume that $r(I) < \infty$. Then $r(I) = \max_{a \in F_d}|a|$.

**Proof.** This can be proved similarly to the proof of Lemma 4.2.

Combining the above results with Theorem 3.4 and Theorem 4.6 we get a simple method to compute $\text{reg}_a(S/I)$ and $\text{reg}(S/I)$. We will illustrate the above method by an example at the end of the next section. Moreover, we get the following estimation for $\text{reg}_a(S/I)$.

**Corollary 4.4.** Let $x_1, \ldots, x_{n-i}$ be a filter-regular sequence for $S/I$. Let $c_i$ denote the least common multiple of the minimal generators of $\text{in}(I)$ which are not divided by any of the variables $x_{n-i+1}, \ldots, x_n$. Then

$$\text{reg}_a(S/I) \leq \max\{\deg g_i - n + i | i = 0, \ldots, t\}.$$  

**Proof.** By Corollary 3.3, the assumption implies that $c_i(I) < \infty$ for $i = 0, \ldots, t$. Thus, combining Theorem 3.4 and Lemma 4.2 we get

$$\text{reg}_a(S/I) \leq \max\{|a| | a \in F_i, i = 0, \ldots, t\}.$$  

It is easily seen from the definition of $F_i$ that $\max_{a \in F_i}|a| \leq \deg g_i - n + i, i = 0, \ldots, t$, hence the conclusion.

**Remark.** Bruns and Herzog [BH, Theorem 3.1(a)], resp. Hoa and Trung [HT, Theorem 3.1], proved that for any monomial ideal $I$, $\text{reg}(S/I) \leq \deg f - 1$, resp. $\deg f - \text{ht} I$, where $f$ is the least common multiple of the minimal generators of $I$. Note that the mentioned result of Bruns and Herzog is valid for multigraded modules.

5. **The case of projective curves**

Let $I_C \subset k[x_1, \ldots, x_n]$ be the defining saturated ideal of a (not necessarily reduced) projective curve $C \subset \mathbb{P}^{n-1}, n \geq 3$. We will assume that $k[x_{n-1}, x_n] \hookrightarrow S/I_C$ is a Noether normalization of $S/I_C$. In this case, Theorem 3.6 can be reformulated as follows.

**Proposition 5.1.** $\text{reg}(S/I_C) = \max\{c_1(I_C), r(I_C)\}$.

**Proof.** By the above assumption $S/I_C$ is a generalized Cohen-Macaulay ring of positive depth and $x_n, x_{n-1}$ is a homogeneous system of parameters for $S/I_C$. Therefore, $x_n, x_{n-1}$ is a filter-regular sequence for $S/I_C$. In particular, $x_n$ is a non-zerodivisor in $S/I_C$. By Lemma 5.2 $c_0(I_C) = -\infty$. Hence the conclusion follows from Theorem 3.6.
Since $S/I_C$ has positive depth, the graded minimal free resolution of $S/I_C$ ends at most at the $(n - 1)$th place:

$$0 \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow S/I_C \rightarrow 0.$$  

From Theorem 3.4 we obtain the following information on the shifts of $F_{n-1}$. Note that $F_{n-1} = 0$ if $S/I_C$ is a Cohen-Macaulay ring or, in other words, if $C$ is an arithmetically Cohen-Macaulay curve.

**Proposition 5.2.** If $C$ is not an arithmetically Cohen-Macaulay curve, $c_1(I_C) + n - 1$ is the maximum degree of the generators of $F_{n-1}$.

**Proof.** Let $b_{n-1}$ be the maximum degree of the generators of $F_{n-1}$. As we have seen in the introduction, $b_{n-1} - n + 1 = (n - 1) - \text{reg}(S/I_C) = \text{reg}_1(S/I_C)$. By Theorem 3.4, $\text{reg}_1(S/I_C) = \max\{c_0(I_C), c_1(I_C)\} = c_1(I_C)$ because $c_0(I_C) = -\infty$. So we obtain $b_{n-1} = c_1(I_C) + n - 1$.

Now we shall see that Proposition 5.1 contains all main results of Bermejo and Gimenez in [BG]. It should be noted that they did not use strong results such as Theorem 2.4. We follow the notations of [BG].

Let $E := \{a \in \mathbb{N}^{n-2} \mid x^a \in \text{in}(I_C)\}$ and denote by $H(E)$ the smallest integer $r$ such that $a \in E$ if $|a| = r$.

**Corollary 5.3 ([BG Theorem 2.4]).** Assume that $C$ is an arithmetically Cohen-Macaulay curve. Then $\text{reg}(S/I_C) = H(E) - 1$.

**Proof.** Since $x_n, x_{n-1}$ is a regular sequence in $S/I_C$, we have $c_1(I_C) = -\infty$ by Corollary 3.2. By Proposition 5.1, this implies $\text{reg}(S/I_C) = r(I_C)$. But

$$r(I_C) = \sup\{r \mid (S_2/J_2)_r \neq 0\} = H(E) - 1$$

because $J_2$ is generated by the monomials $x^a$, $a \in E$.

Let $I_0$ be the ideal in $S$ generated by the polynomials obtained from $I_C$ by the evaluation $x_{n-1} = x_n = 0$. Then $S/I_0$ is a two-dimensional Cohen-Macaulay ring. Let $\tilde{I}$ denote the ideal in $S$ generated by the monomials obtained from $\text{in}(I_C)$ by the evaluation $x_{n-1} = x_n = 1$. Let

$$F := \{a \in \mathbb{N}^{n-2} \mid x^a \in \tilde{I} \setminus \text{in}(I_0)\}.$$  

For every vector $a \in F$ let

$$E_a := \{(\mu, \nu) \in \mathbb{N}^2 \mid x^a x_n^{\mu} x_n^{\nu} \in \text{in}(I_C)\}.$$  

Let $\mathcal{R} := \bigcup_{a \in F} \{a \times [\mathbb{N}^2 \setminus E_a]\}$ and denote by $H(\mathcal{R})$ the smallest integer $r$ such that the number of the elements $b \in \mathcal{R}$ with $|b| = s$ becomes a constant for $s \geq r$.

**Corollary 5.4 ([BG Theorem 2.7]).** $\text{reg}(S/I_C) = \max\{\text{reg}(S/I_0), H(\mathcal{R})\}$.

**Proof.** As in the proof of Corollary 5.3, we have $\text{reg}(S/I_0) = r(I_0)$. But $r(I_0) = r(I_C)$ because $\text{in}(I_0)$ is the ideal generated by the monomials obtained from $\text{in}(I_C)$ by the evaluation $x_{n-1} = x_n = 0$. Thus,

$$\text{reg}(S/I_0) = r(I_C).$$  

It has been observed in [BG] that the number of the elements $b \in \mathcal{R}$ with $|b| = s$ is the difference $H_{S/I_C}(s) - H_{S/I}(s) = H_{S/\text{in}(I_C)}(s) - H_{F/\text{in}(I_C)}(s)$, where $H_E(s)$ denotes the Hilbert function of a graded $S$-module $E$. Since $x_n$ is a non-zerodivisor in $S/\text{in}(I_C)$, $H(\mathcal{R}) + 1$ is the least integer $r$ such that $H_{(\tilde{I},x_n)/(\text{in}(I_C),x_n)}(s)$
\[= 0 \text{ for } s \geq r. \] On the other hand, since \(\text{in}(I_C)\) is generated by monomials which do not contain \(x_s\), and since \(J_1\) is the ideal in \(k[x_1, \ldots, x_{n-1}]\) obtained from \(\text{in}(I_C)\) by the evaluation \(x_s = 0\), we have \(\text{in}(I_C) = J_1 S\) and \(I = J_1 S\), whence \((\bar{I}, x_s)/(\text{in}(I_C), x_s) \cong \bar{J}_1/\bar{J}_1 \). Note that \(c_1(I_C) = \max\{r | (\bar{J}_1/\bar{J}_1)_r \neq 0\}\) with \(c_1(I_C) = -\infty\) if \(\bar{J}_1 = \bar{J}_1\). Then

\[
H(\mathcal{R}) = \max\{0, c_1(I_C)\}.
\]

Thus, applying Proposition [5.1] we obtain \(\text{reg}(S/I_C) = \max\{\text{reg}(S/I_0), H(\mathcal{R})\}\). \(\square\)

**Example.** Let \(C \subset \mathbb{P}^3\) be the monomial curve \((t^{\alpha} s^\beta : t^\beta s^\alpha : s^{\alpha+\beta} : t^{\alpha+\beta})\), \(\alpha > \beta > 0\), g.c.d. \((\alpha, \beta) = 1\). It is known that the defining ideal \(I_C \subset k[x_1, x_2, x_3, x_4]\) is generated by the quadric \(x_1 x_2 - x_3 x_4\) and the forms \(x_1^{\beta+r} x_3^{\alpha-\beta-r} - x_2^{\alpha-\beta} r^r\), \(r = 0, \ldots, \alpha - \beta\), and that this is a Gröbner basis of \(I_C\) for the reverse lexicographic order with \(x_1 > x_2 > x_3 > x_4\) [CM, Théorème 3.9]. Therefore,

\[
\text{in}(I_C) = (x_1 x_2, x_2^\alpha, x_1^{\beta+1} x_3^{\alpha-\beta-1}, x_1^{\beta+2} x_3^{\alpha-\beta-2}, \ldots, x_1^\alpha).
\]

Using the notations of Section 3 we have

\[
E_1 = \{(1, 1, 0), (0, \alpha, 0), (\beta + 1, 0, \alpha - \beta - 1), (\beta + 2, 0, \alpha - \beta - 2), \ldots, (\alpha, 0, 0)\},
\]

\[
E_2 = \{(1, 1), (0, \alpha), (\alpha, 0)\}.
\]

From this it follows that

\[
F_1 = \{ (\beta + 1, 0, \alpha - \beta - 2), (\beta + 2, 0, \alpha - \beta - 3), \ldots, (\alpha - 1, 0, 0) \},
\]

\[
F_2 = \{ (0, \alpha - 1), (\alpha - 1, 0) \}.
\]

By Proposition [4.2] \(c_1(I_C) = \alpha - 1\) if \(\alpha - \beta \geq 2\) \((c_1(I_C) = -\infty\) if \(\alpha - \beta = 1\)\) and \(r(I_C) = \alpha - 1\) by Proposition [4.3]. Applying Proposition [5.1] we obtain \(\text{reg}(S/I_C) = \alpha - 1\).

The direct computation of the invariant \(H(\mathcal{R})\) is more complicated than that of \(c_1(I_C)\). First, we should interpret \(F\) as the set of the elements of the form \(\alpha \in \mathbb{N}^2\) such that \(\alpha \geq b\) for some elements \(b \in p(E_1)\) but \(a \not\geq c\) for any element \(c \in E_2\). Then we get

\[
F = \{ (\beta + 1, 0), (\beta + 2, 0), \ldots, (\alpha - 1, 0) \}.
\]

For all \(\varepsilon = \beta + 1, \ldots, \alpha - 1\) we verify that \(E_{(\varepsilon, 0)} = (\alpha - \varepsilon, 0) + \mathbb{N}^2\). It follows that

\[
\mathcal{R} = \{ (\varepsilon, 0, \mu, \nu) \in \mathbb{N}^4 | \varepsilon = \beta + 1, \ldots, \alpha - 1; \mu \leq \alpha - \varepsilon - 1 \}.
\]

If \(\alpha - \beta = 1\), we have \(\mathcal{R} = \emptyset\), hence \(H(\mathcal{R}) = 0\). If \(\alpha - \beta \geq 2\), we can check that \(H(\mathcal{R}) = \alpha - 1\).

**Acknowledgement**

The author would like to thank M. Morales for raising his interest in the paper of Bermejo and Gimenez [BG] and L.T. Hoa for useful suggestions.

**References**


Institute of Mathematics, Box 631, Bô Hô, Hanoi, Vietnam
E-mail address: nvtrung@hn.vnn.vn