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SIMPLE ALGEBRAS OF WEYL TYPE, II

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Abstract. Over a field $\mathbb{F}$ of any characteristic, for a commutative associative algebra A, and for a commutative subalgebra D of $\mathrm{Der}_{\mathbb{F}}(A)$, the vector space $A[D]$ which consists of polynomials of elements in D with coefficients in A and which is regarded as operators on A forms naturally an associative algebra. It is proved that, as an associative algebra, $A[D]$ is simple if and only if A is D-simple. Suppose A is D-simple. Then, (a) $A[D]$ is a free left A-module; (b) as a Lie algebra, the subquotient $[A[D], A[D]]/(Z(A[D]) \cap [A[D], A[D]])$ is simple (except for one case), where $Z(A[D])$ is the center of $A[D]$. The structure of this subquotient is explicitly described. This extends the results obtained by Su and Zhao.

1. Introduction

There are many known classes of infinite dimensional simple Lie algebras and simple associative algebras. Generalizations of the infinite dimensional simple Lie algebras of Cartan type over a field of characteristic zero have been obtained by Kawamoto [K], Osborn [O], Dokovic and Zhao [DZ1, DZ2, DZ3], Osborn and Zhao [OZ1, OZ2], Zhao [Z3] and Xu [X]. Passman [P] and Jordan [J1] studied the Lie algebras $AD = A \otimes D$ of generalized Witt type constructed from a commutative associative algebra A with an identity element and its commutative derivation subalgebra D over a field $\mathbb{F}$ of arbitrary characteristic. Passman proved that AD is simple if and only if A is D-simple (that is, A does not have nontrivial D-stable ideals) and $A^D D$ acts faithfully on A except for the case $\mathrm{char}\,\mathbb{F} = 2, \dim_{\mathbb{F}_1} \mathbb{F}_1 D = 1$ and $D(A) \neq A$, where $\mathbb{F}_1 = A^D = \{x \in A | D(x) = 0\}$.

In [SZ1], for a commutative associative algebra A with an identity element over $\mathbb{F}$ and for the polynomial algebra $\mathbb{F}[D]$ of its commutative derivation subalgebra D, Su and Zhao constructed the associative algebras $A \otimes \mathbb{F}[D]$ of Weyl type, and proved that $A \otimes \mathbb{F}[D]$ is simple if and only if A is D-simple and $A^D \otimes \mathbb{F}[D]$ acts faithfully on A, both as a Lie algebra (modulo its center) and as an associative algebra. The simple algebras $A \otimes \mathbb{F}[D]$ (or modulo its center as Lie algebras) are always infinite dimensional. But as operators on A, some nonzero elements in $A \otimes \mathbb{F}[D]$ are possibly zero operators on A. In this case, $A \otimes \mathbb{F}[D]$ certainly cannot be a simple Lie algebra (modulo its center) or a simple associative algebra. If we directly consider such

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operators on A, we can avoid this case so that we still get simple algebras. This is the objective of this paper.

Throughout this paper, $\mathbb{F}$ is a field of arbitrary characteristic.

In this paper, for a commutative associative algebra A with an identity element over $\mathbb{F}$ and for a commutative derivation subalgebra $D \subset \mathrm{Der}(A)$, we study the associative algebra $A[D]$ which consists of polynomials of elements in D with coefficients in A and which is regarded as operators on A. These associative algebras can also be realized as iterated skew polynomial rings if $\dim D < \infty$ (see [J2]). As an associative algebra, it is proved that $A[D]$ is simple if and only if A is D-simple (see Theorem 2.3). Suppose A is D-simple, then (a) $A[D]$ is a free left A-module (see Theorem 3.5); (b) as a Lie algebra, the subquotient $[A[D], A[D]]/(Z(A[D]) \cap [A[D], A[D]])$ is simple (except for one case), where $Z(A[D])$ is the center of $A[D]$. The structure of this subquotient is explicitly described (see Theorems 3.8, 3.9). The Lie algebras obtained by Su and Zhao in [SZ1] are some infinite dimensional Lie algebra studied in this paper. In particular we can get a lot of finite dimensional simple modular Lie algebras. At the end of this paper, we give some finite or infinite dimensional examples which cannot be covered by [SZ1].

2. Simple associative algebras of Weyl type

Let A be a commutative associative algebra with an identity element 1 over $\mathbb{F}$, and let D be a nonzero $\mathbb{F}$-vector space spanned by some commuting $\mathbb{F}$-derivations of A. Let

$$\{\partial_i | i \in \dot{I}\}, \quad \text{where } \dot{I} \text{ is some indexing set,} \tag{2.1}$$

be an $\mathbb{F}$-basis of D. Denote

$$\begin{aligned} \dot{J} = \{\alpha = (\alpha_i | i \in \dot{I}) | \alpha_i \in \mathbb{Z}_+, \forall i \in \dot{I} \text{ and } \alpha_i = 0 \\ \text{for all but a finite number of } i \in \dot{I}\}, \end{aligned} \tag{2.2}$$

where $\alpha = (\alpha_i | i \in \dot{I})$, simply denoted by $\alpha = (\alpha_i)$, is a collection of nonnegative integers indexed by $\dot{I}$. Define

$$|\alpha| = \sum_{i \in \dot{I}} \alpha_i, \tag{2.3}$$

which is called the *level* of α. For any $\alpha \in \dot{J}, x \in A$, we define the operator on A

$$x\partial^\alpha = x \prod_{i \in \dot{I}} \partial_i^{\alpha_i}, \tag{2.4}$$

by $(x\partial_1\partial_2 \cdots \partial_n)(y) = x(\partial_1(\partial_2 \cdots (\partial_n(y)) \cdots))$ for any $y \in A$. Let $A[D]$ be the $\mathbb{F}$-vector space spanned by all the following operators on A:

$$x\partial^\alpha, \quad \forall x \in A, \alpha \in \dot{J}. \tag{2.5}$$

If we define the product of $A[D]$ as the composition of operators on A, then $A[D]$ becomes an associative algebra. It is not difficult to verify that

$$(u\partial^\alpha) \cdot (v\partial^\beta) = u \sum_{\gamma \in \dot{J}(\alpha)} \binom{\alpha}{\gamma} \partial^\gamma(v) \partial^{\alpha+\beta-\gamma}, \tag{2.6}$$

for all $u, v \in A$, $\alpha, \beta \in \dot{J}$, where

$$\dot{J}(\alpha) = \{\gamma \in \dot{J} | \gamma_i \leq \alpha_i,\ \forall i \in \dot{I}\},$$
$$\binom{\alpha}{\gamma} = \prod_{i \in \dot{I}} \binom{\alpha_i}{\gamma_i}, \quad \gamma \in \dot{J}(\alpha). \tag{2.7}$$

We call the associative algebra $A[D]$ an *associative algebra of Weyl type*. This is different from that in [SZ1].

Remark. It is clear that A is a left $A[D]$-module and $A[D]$ is a left A-module.

For any

$$x = \sum_{\alpha \in J} u_\alpha \partial^\alpha \in A[D], \tag{2.8}$$

the expression in (2.8) is sometimes not unique. An expression of x in (2.8) is called *principal* if the integer $\max\{|\alpha|\, |a_\alpha \neq 0\}$ is minimal. We denote this integer by $h(x)$. Set

$$\mathbb{F}_1 = A^D = \{u \in A | D(u) = 0\}. \tag{2.9}$$

We use this notation because it is a field extension of $\mathbb{F}$ when A is *D-simple*, i.e., A has no nontrivial D-stable ideals; see [J1].

Lemma 2.1. $A \cap A[D]D = 0$.

Proof. Suppose $x \in A \cap A[D]D$. Since $x \in A$, then the action of x on 1 is $x \cdot 1 = x$. On the other hand, since $x \in A[D]D$ and $D(1) = 0$, then the action of x on 1 is 0. So $x = 0$. Thus $A \cap A[D]D = 0$. □

Lemma 2.2. *The center $Z(A[D])$ of $A[D]$ is $\mathbb{F}_1$.*

Proof. It is clear that $\mathbb{F}_1 \subset Z(A[D])$.

Suppose $x \in Z(A[D])$. If $h(x) > 0$, write $x = X_0 + X_1$ where $X_0 \in A$ and $X_1 \in A[D]D$. Then $X_1 \neq 0$. Choose $a \in A$ such that $X_0(a) \neq 0$. It follows that

$$0 = [X, a] = [X_1, a] = X_1(a) + Y,$$

where $Y \in A[D]D$. From Lemma 2.1 we deduce that $X_1(a) = Y = 0$, a contradiction. So $h(x) = 0$. We have

$$x\partial = \partial x = x\partial + \partial(x),\ \forall \partial \in D,$$

to give $\partial(x) = 0$ for all $\partial \in D$, i.e., $x \in \mathbb{F}_1$. Therefore Lemma 2.2 follows. □

Theorem 2.3. *The associative algebra $A[D]$ is simple if and only if A is D-simple.*

Proof. "$\Rightarrow$": Suppose A is not D-simple. Choose a nonzero proper D-ideal $\mathcal{I}$. Then clearly $\mathcal{I}[D]$ is a nonzero ideal of $A[D]$. Since $A[D]$ is simple, then $\mathcal{I}[D] = A[D]$, in particular, $A \subset \mathcal{I}[D]$. From Lemma 2.1, we know that $A \subset \mathcal{I}$, a contradiction. Thus A is D-simple.

"$\Leftarrow$": Suppose L is a nonzero ideal of $A[D]$. It suffices to show that $L = A[D]$.

Claim 1. $L \cap A \neq 0$.

Suppose that $L \cap A = 0$. Choose $X \in L \setminus \{0\}$ such that $h(X)$ is minimal. So $h(X) > 0$. Write $X = X_0 + X_1$ where $X_0 \in A$ and $X_1 \in A[D]D$. Then $X_1 \neq 0$. Choose $a \in A$ such that $X_0(a) \neq 0$. It follows that

$$X' = [X, a] = [X_1, a] = X_1(a) + Y \in L,$$

where $Y \in A[D]D$. From the computation (2.6) we deduce that $h(X') = h(Y) < h(X)$. By the minimality of $h(X)$ we deduce that $X' = 0$. Applying Lemma 2.1 gives $X_1(a) = Y = 0$, a contradiction. Thus Claim 1 follows.

It is clear that $L \cap A$ is a D-ideal of A. Since A is D-simple, then $L \cap A = A$. In particular $1 \in L$. Therefore $L = A[D]$. □

3. Simple Lie algebras of Weyl type

Let A be a commutative associative algebra with an identity element 1 over $\mathbb{F}$, and let D be a nonzero $\mathbb{F}$-vector space spanned by some commuting $\mathbb{F}$-derivations of A such that A is D-simple. It is well known that $\mathbb{F}_1$ is a field extension of $\mathbb{F}$. From Theorem 2.3, we know that the associative algebra $A[D]$ is simple.

We denote the induced Lie algebra from the associative algebra $A[D]$ by $A[D]^L$.

In this section we shall first study the structure of $A[D]$ as a left A-module, then investigate the Lie structure of $A[D]$. We still use the notations in (2.1) and (2.2).

The following lemma will be often used.

Lemma 3.1. (a) *If $x\partial^\alpha = 0$ for some $x \in A \setminus \{0\}, \alpha \in J$, then $\partial^\alpha = 0$.*

(b) $A \subset [A[D]^L, A[D]^L]$.

Proof. (a) Let $C = \{x \in A | x\partial^\alpha = 0\}$. It is easy to verify that C is a nonzero D-ideal of A. Since A is D-simple, then $C = A$. It yields $\partial^\alpha = 0$.

(b) From $[x\partial, y] = x\partial(y) \in [A[D]^L, A[D]^L]$ for all $x, y \in A$ and $\partial \in D$, we know that $AD(A) \subset [A[D]^L, A[D]^L]$. Since $AD(A)$ is a nonzero D-ideal of A and A is D-simple, then $A = AD(A)$. Thus $A \subset [A[D]^L, A[D]^L]$. □

If $\operatorname{char} \mathbb{F} = p > 0$, observe that for any $\partial \in \operatorname{Der}_{\mathbb{F}}(A)$, one has $\partial^p \in \operatorname{Der}_{\mathbb{F}}(A)$. Let

$$\mathcal{D} = \operatorname{Der}_{\mathbb{F}}(A) \cap \mathbb{F}_1[D], \tag{3.1}$$

where $\mathbb{F}_1[D]$ is the polynomial algebra of elements in D, and let

$$\{d_i | i \in I\} \tag{3.2}$$

be an $\mathbb{F}_1$ basis for $\mathcal{D}$. It is clear that $D \subset \mathcal{D}$. $J = \{\alpha = (\alpha_i | i \in I) | \alpha_i \in \mathbb{Z}_+, \forall i \in I$ and $\alpha_i = 0$ for all but a finite number of $i \in I\}$, where $\alpha = (\alpha_i | i \in I)$, and let

$$\begin{aligned} J_0 = \{\alpha = (\alpha_i | i \in I) | 0 \leq \alpha_i \in \leq p-1,\ \forall i \in I \\ \text{and } \alpha_i = 0 \text{ for all but a finite number of } i \in I\}, \end{aligned} \tag{3.3}$$

where $\alpha = (\alpha_i | i \in I)$ will be simply denoted by $\alpha = (\alpha_i)$. We also use the notation $d^\alpha = \prod_{i \in I} d^{\alpha_i}$, $\forall \alpha \in J_0$. If $\operatorname{char} \mathbb{F} = p$, then

$$A[D] = \sum_{\alpha \in J_0} A d^\alpha. \tag{3.4}$$

If $\operatorname{char} \mathbb{F} = 0$, we still have (3.4) by replacing J_0 with J.

Lemma 3.2. (a) *Suppose* $\operatorname{char}\mathbb{F} = p > 0$ *and* $J_1 \subset J_0$. *Then the sum* $\sum_{\alpha\in J_1} Ad^\alpha$ *is direct if and only if the sum* $\sum_{\alpha\in J_1} \mathbb{F}_1 d^\alpha$ *is direct.*

(b) *Suppose* $\operatorname{char}\mathbb{F} = 0$ *and* $J_1 \subset J$. *Then the sum* $\sum_{\alpha\in J_1} Ad^\alpha$ *is direct if and only if the sum* $\sum_{\alpha\in J_1} \mathbb{F}_1 d^\alpha$ *is direct.*

Proof. (a) "$\Rightarrow$": This direction is clear.

"$\Leftarrow$": Suppose $\sum_{\alpha\in J_1} Ad^\alpha$ is not direct. There exists $J_2 = \{\alpha^{(0)}, \alpha^{(2)}, \dots, \alpha^{(r)}\} \subset J_1$ such that $|J_2| > 1$ and $Ad^{\alpha^{(0)}} \cap \sum_{i=1}^r Ad^{\alpha^{(i)}} \neq 0$. Choose such a minimal r. Let

$$ad^{\alpha^{(0)}} = \sum_{i=1}^r a_i d^{\alpha^{(i)}} \in \left(Ad^{\alpha^{(0)}} \cap \sum_{i=1}^r Ad^{\alpha^{(i)}} \right) \setminus \{0\}$$

for some $a \in A \setminus \{0\}$, $a_i \in A$, and let $I_1 = \{0, 1, \dots, r\}$. It follows that $d^{\alpha^{(i)}} \neq 0$ for any $i \in I_1$. For any $0 \le j \le r$, let

$$B_j = \left\{ x \in A | x d^{\alpha^{(j)}} \in \sum_{i\in I_1\setminus\{j\}} Ad^{\alpha^{(i)}} \right\}.$$

Then it is easy to see that B_j is a nonzero D-ideal of A. Since A is D-simple, it follows that $B_j = A$ for any $j \in I_1$. Then

$$d^{\alpha^{(0)}} = \sum_{i\in I_1\setminus\{0\}} a_i d^{\alpha^{(i)}}, \qquad a_i \in A. \tag{3.5}$$

By taking brackets with $\partial \in \mathcal{D}$, we deduce that

$$0 = [\partial, d^{\alpha^{(0)}}] = \left[\partial, \sum_{i\in I_1\setminus\{0\}} a_i d^{\alpha^{(i)}} \right] = \sum_{i\in I_1\setminus\{0\}} \partial(a_i) d^{\alpha^{(i)}}.$$

From the minimality of r, we must have

$$\partial(a_i) d^{\alpha^{(i)}} = 0, \quad \forall \partial \in \mathcal{D}, 1 \le i \le r. \tag{3.6}$$

Suppose $D(a_i) \neq 0$ for some $i \in I_1$, say $D(a_1) \neq 0$. Lemma 3.1 yields $d^{\alpha^{(1)}} = 0$, a contradiction. Thus $D(a_i) = 0$ for all $i \in I_1$, i.e., $a_i \in \mathbb{F}_1$ in (3.5). Therefore the sum $\sum_{\alpha\in J_1} \mathbb{F}_1 d^\alpha$ is not direct. This direction of (a) follows.

The proof for part (b) is similar to that of (a). □

Lemma 3.3. (a) *If* $\operatorname{char}\mathbb{F} = p > 0$, *then* $A[D] = \bigoplus_{\alpha\in J_0} Ad^\alpha$.

(b) *If* $\operatorname{char}\mathbb{F} = 0$, *then* $A[D] = \bigoplus_{\alpha\in J} Ad^\alpha$.

Proof. (a) From (3.4), we know that it suffices to show the sum in (3.4) is direct.

Suppose the sum in (3.4) is not direct. From Lemmas 2.1, 3.1, there exists $J_2 = \{\alpha^{(0)}, \alpha^{(2)}, \dots, \alpha^{(r)}\} \subset J_0 \setminus \{0\}$ such that $\mathbb{F}_1 d^{\alpha^{(0)}} \cap \sum_{i=1}^r \mathbb{F}_1 d^{\alpha^{(i)}} \neq 0$. Let $I_1 = \{0, 1, \dots, r\}$ and $h(J_2) = \max\{|\alpha^{(i)}| \,|\, i \in I_1\}$. Then $h(J_2) > 1$. Choose such a J_2 such that $h(J_2)$ is minimal, and then r is minimal. Let

$$d^{\alpha^{(0)}} = \sum_{i=1}^r a_i d^{\alpha^{(i)}} \in \left(\mathbb{F}_1 d^{\alpha^{(0)}} \cap \sum_{i=1}^r \mathbb{F}_1 d^{\alpha^{(i)}} \right) \setminus \{0\} \tag{3.7}$$

for some $a_i \in \mathbb{F}_1$, $1 \le i \le r$. Denote $a_0 = 1$. It follows that $a_i d^{\alpha^{(i)}} \neq 0$ for any $i \in I_1$. Let

$$J_0(\alpha) = \{\beta \in J_0 \setminus \{0, \alpha\} | \beta_i \le \alpha_i,\ \forall i \in I\}. \tag{3.8}$$

Rewrite (3.7) to give

$$\sum_{i=0}^{r} a_i d^{\alpha^{(i)}} = 0.$$

For any $x \in A$, it is clear that $\sum_{i=0}^{r} a_i d^{\alpha^{(i)}}(x) = 0$; then

$$\begin{aligned} 0 &= \left[\sum_{i=0}^{r} a_i d^{\alpha^{(i)}}, x\right] = \sum_{i=0}^{r} \sum_{\gamma \in J_0(\alpha^{(i)})} a_i \binom{\alpha^{(i)}}{\gamma} d^{\alpha^{(i)}-\gamma}(x) d^{\gamma} \\ &= \sum_{\gamma \in J_0, 0<|\gamma|<h(J_2)} \left(\sum_{i=0}^{r} a_i \binom{\alpha^{(i)}}{\gamma} d^{\alpha^{(i)}-\gamma}(x)\right) d^{\gamma}. \end{aligned}$$

From the minimality of $h(J_2)$, it follows that

$$\sum_{i=0}^{r} a_i \binom{\alpha^{(i)}}{\gamma} d^{\alpha^{(i)}-\gamma}(x) d^{\gamma} = 0$$

for all $x \in A$, and all $\gamma \in J$ with $0 < h(\gamma) < h(J_2)$. Since $d^{\gamma} \neq 0$, from Lemma 3.1 we see that

$$\sum_{i=0}^{r} a_i \binom{\alpha^{(i)}}{\gamma} d^{\alpha^{(i)}-\gamma}(x) = 0$$

for all $x \in A$, and all $\gamma \in J$ with $0 < h(\gamma) < h(J_2)$. So

$$\sum_{i=0}^{r} a_i \binom{\alpha^{(i)}}{\gamma} d^{\alpha^{(i)}-\gamma} = 0$$

for all $\gamma \in J$ with $0 < h(\gamma) < h(J_2)$. Since $\alpha_j^{(i)} \le p-1$, then $a_i \binom{\alpha^{(i)}}{\gamma} \neq 0$ if $\gamma \in J_0(\alpha^{(i)})$. This contradicts the minimality of $h(J_2)$. Therefore $A[D] = \bigoplus_{\alpha \in J_0} A d^{\alpha}$. Part (a) is true.

The proof of part (b) is similar to that of part (a). □

Lemma 3.4. (a) *If* $\operatorname{char} \mathbb{F} = p > 0$, *then* $d^{\alpha} \neq 0$, *for any* $\alpha \in J_0$.

(b) *If* $\operatorname{char} \mathbb{F} = 0$, *then* $d^{\alpha} \neq 0$, *for any* $\alpha \in J$.

Proof. (a) It is obvious that $d^{\alpha} \neq 0$ for any $\alpha \in J_0$ with $|\alpha| < 2$.

Suppose $d^{\alpha} = 0$ for some $\alpha \in J_0$ with $|\alpha| \ge 2$. Choose an α such that $|\alpha|$ is minimal. For any $x \in A$, we see $d^{\alpha}(x) = 0$. Then

$$0 = [d^{\alpha}, x] = \sum_{\beta \in J_0(\alpha)} \binom{\alpha}{\beta} d^{\alpha-\beta}(x) d^{\beta} = 0, \quad \forall x \in A,$$

where $J_0(\alpha)$ is as in (3.8). Applying Lemma 3.3 and noting that $\binom{\alpha}{\beta} \neq 0$ for any $\beta \in J_0(\alpha)$, since $|\alpha|$ is minimal we get $d^{\alpha-\beta}(x) d^{\beta} = 0$, $\forall x \in A, \beta \in J_0(\alpha)$. Applying Lemma 3.1 and noting that $d^{\beta} \neq 0$ for any $\beta \in J_0(\alpha)$, we obtain $d^{\alpha-\beta}(x) = 0$, $\forall x \in A$, i.e., $d^{\alpha-\beta} \neq 0$ for any $\beta \in J_0(\alpha)$. This contradicts the minimality of $|\alpha|$. Therefore (a) follows.

The proof of part (b) is similar to that of part (a). □

Theorem 3.5. *Suppose A is D-simple. Then $A[D]$ is a free A-module. Furthermore,*

(a) *if* $\operatorname{char}\mathbb{F} = p > 0$, *then $A[D]$ is a free A-module with a basis $\{d^\alpha | \alpha \in J_0\}$;*

(b) *if* $\operatorname{char}\mathbb{F} = 0$, *then $A[D]$ is a free A-module with a basis $\{d^\alpha | \alpha \in J\}$.*

Proof. This theorem directly follows from Lemmas 3.2, 3.3 and 3.4. □

Lemma 3.6. (a) *Suppose that* $\operatorname{char}\mathbb{F} = p > 0$, *and that $\{d_i | 1 \le i \le n\} \subset \mathcal{D}$ are $\mathbb{F}_1$ linearly independent. Then, for any integers $0 \le r_1, r_2, \dots, r_n \le p-1$ with $r_1 + r_2 + \cdots + r_n \le n(p-1) - 1$, we have*

$$Ad_1^{r_1} d_2^{r_2} \cdots d_n^{r_n} \subset [A[D]^L, A[D]^L].$$

(b) *Suppose that* $\operatorname{char}\mathbb{F} = 0$, *and that $\{d_i | 1 \le i \le n\} \subset \mathbb{F}_1 D$ are $\mathbb{F}_1$ linearly independent. Then, for any integers $r_1, r_2, \dots, r_n \ge 0$, we have*

$$Ad_1^{r_1} d_2^{r_2} \cdots d_n^{r_n} \subset [A[D]^L, A[D]^L].$$

Proof. (a) In this proof, we define $J_0 = \{(r_1, r_2, \dots, r_n) | 0 \le r_1, r_2, \dots, r_n \le p-1\}$. Let $\beta = (p-1, p-1, \dots, p-1) \in J_0, J_1 = J_0 \setminus \{0\}$ and $J_2 = J_1 \setminus \{\beta\}$. Denote $W = [A[D]^L, A[D]^L]$. For any $x, y \in A$, we deduce

$$[xd_1^{p-1} d_2^{p-1} \cdots d_n^{p-1}, y] = \sum_{\alpha \in J_1} \binom{\beta}{\alpha} xd^\alpha(y) d^{\beta-\alpha} \in W.$$

By using Lemma 3.1 we obtain that

$$\sum_{\alpha \in J_2} \binom{\beta}{\alpha} xd^\alpha(y) d^{\beta-\alpha} \in W, \quad \forall x, y \in A. \tag{3.9}$$

Note that the coefficients $\binom{\beta}{\alpha}$ are not zero. Since $\{d^\alpha | \alpha \in J_2\}$ is A linearly independent (Theorem 3.5), there exist $a_\alpha \in A$ for all $\alpha \in J_2$ such that the $(p^n-2) \times (p^n-2)$ determinant

$$w = \det(d^\alpha(a_\gamma))_{\alpha, \gamma \in J_2} \in A \setminus \{0\}. \tag{3.10}$$

In (3.9), if we take y to be a_γ respectively, we obtain $p^n - 2$ equations

$$x \sum_{\alpha \in J_2} \binom{\beta}{\alpha} d^\alpha(a_\gamma) d^{\beta-\alpha} \in W, \quad \forall x \in A, \gamma \in J_2. \tag{3.11}$$

Applying linear algebra knowledge to (3.11), we deduce that $Awd^\alpha \in L$ for all $\alpha \in J_2$. Let $X_\alpha = \{x \in A | xd^\alpha \in L\}$. Then $Aw \subset X_\alpha$, and $D(X_\alpha) \subset X_\alpha$. By induction on $|\alpha|$ we can easily show that $Ad^\alpha(w) \subset X_\alpha$. Thus $A[D](w) \subset X_\alpha$. Note that $A[D](w)$ is a nonzero D-ideal of A. Since A is D-simple, we must have $A[D](w) = A$, so $X_\alpha = A$ for all $\alpha \in J_2$. Therefore $Ad^\alpha \subset L$ for all $\alpha \in J_2$. Part (a) follows.

The proof of part (b) is similar to that of part (a). □

Theorem 3.7. (a) *Suppose that* $\operatorname{char}\mathbb{F} = p > 0$, *and that $\{d_i | 1 \le i \le n\}$ is an $\mathbb{F}_1$ basis of $\mathcal{D}$ with $n < \infty$. Let $\beta = (p-1, p-1, \dots, p-1) \in J_0$. Then*

$$W = [A[D]^L, A[D]^L] = \bigoplus_{\alpha \in J_0 \setminus \{\beta\}} Ad^\alpha \oplus D(A)d^\beta. \tag{3.12}$$

(b) *Suppose that* $\operatorname{char}\mathbb{F} = 0$, *or that* $\dim \mathcal{D} = \infty$ *if* $\operatorname{char}\mathbb{F} = p > 0$. *Then* $A[D]^L = [A[D]^L, A[D]^L]$.

Proof. (a) Let $\delta_i = (\delta_{1,i}, \delta_{2,i}, \ldots, \delta_{n,i}) \in J_0$. From Lemma 3.6 we know that $\bigoplus_{\alpha \in J_0 \setminus \{\beta\}} Ad^\alpha \subset W$. The computation $[Ad^\beta, D] = D(A)d^\beta$ yields $D(A)d^\beta \subset W$. Note that $D(A) = \mathcal{D}(A)$. For any $\alpha \in J_0 \setminus \{\beta\}$, and any $x, y \in A$, we obtain that

$$\begin{aligned}[xd^\alpha, yd^{\beta-\alpha+\delta_i}] &\equiv (x\alpha_i d_i(y) - y(p-\alpha_i)d_i(x))d^\beta \pmod L \\ &\equiv \alpha_i d_i(xy)d^\beta \pmod L,\end{aligned}$$

where $L = [A[D]^L, A[D]^L]$. Thus (3.12) holds.

Part (b) directly follows from Lemma 3.6. □

Theorem 3.8. *Suppose A is D-simple. Then the Lie algebra*

$$\overline{A[D]} = [A[D]^L, A[D]^L]/(\mathbb{F}_1 \cap [A[D]^L, A[D]^L]) \tag{3.13}$$

is simple except when $\operatorname{char}\mathbb{F} = 2, A = \mathbb{F}_1[t]/(t^2 - a)$ *and* $A[D] = A \oplus \mathbb{F}_1(d/dt) \oplus \mathbb{F}_1(td/dt)$ *for some* $a \in \mathbb{F}_1^*$ *where d/dt is the ordinary differential.*

Proof. From Lemma 3.1 we see that

$$A \subset [A[D]^L, A[D]^L], \qquad \overline{A[D]} \neq 0.$$

From [H], we know that the Lie algebra

$$\overline{A[D]} = [A[D]^L, A[D]^L]/(\mathbb{F}_1 \cap [A[D]^L, A[D]^L])$$

is simple except when $\operatorname{char}\mathbb{F} = 2, \dim_{\mathbb{F}_1} A[D] = 4$.

Next suppose $\operatorname{char}\mathbb{F} = 2, \dim_{\mathbb{F}_1} A[D] = 4$.

From Lemma 2.1, we see that $\dim_{\mathbb{F}_1} A = 2, 3$. If $\dim_{\mathbb{F}_1} A = 3$, let $A = \mathbb{F}_1 \oplus \mathbb{F}_1 x_1 \oplus \mathbb{F}_1 x_2$. Then $\mathbb{F}_1 D = \mathbb{F}_1 \partial$. We must have $A\partial = \mathbb{F}_1\partial$. It follows that $\partial(A) = AD(A) = A$, contrary to $\partial(\mathbb{F}_1) = 0$. Thus $\dim_{\mathbb{F}_1} A = 2$.

Let $A = \mathbb{F}_1 \oplus \mathbb{F}_1 x$. Let $x^2 = ax + b$ where $a, b \in \mathbb{F}_1$. For any $\partial \in D \setminus \{0\}$, we have $0 = \partial(x^2) = a\partial(x)$, to give $a = 0$ since $\partial(x) \neq 0$. Thus $x^2 - b = 0$, i.e., $A \simeq \mathbb{F}_1(t)/(t^2 - b)$. It follows that $\operatorname{Der}_{\mathbb{F}_1} A = \mathbb{F}_1(d/dt) + \mathbb{F}_1(td/dt)$. If $b = 0$, we see that A is not $\operatorname{Der}(A)$-simple. So $b \neq 0$. Then we must have $A[D] = A \oplus \mathbb{F}_1(d/dt) \oplus \mathbb{F}_1(td/dt)$. □

We call the Lie algebra $\overline{A[D]}$ a *Lie algebra of Weyl type.*

From Theorem 3.7 and Theorem 3.8, we have

Theorem 3.9. (a) *Suppose A is D-simple. Then $A[D]^L/(\mathbb{F}_1 \cap [A[D]^L, A[D]^L])$ is a simple Lie algebra if and only if one of the following conditions holds*:

(I) $\operatorname{char}\mathbb{F} = 0$;
(II) $\operatorname{char}\mathbb{F} = p > 0$ *and* $\dim \mathcal{D} = \infty$;
(III) $\operatorname{char}\mathbb{F} = p > 0, \dim \mathcal{D} < \infty$ *and* $D(A) = A$.

(b) *Suppose* $\operatorname{char}\mathbb{F} = p > 0$ *and* $\dim \mathcal{D} < \infty$. *Let* $\{d_i | 1 \leq i \leq n\}$ *be an* $\mathbb{F}_1$ *basis of* $\mathcal{D}$ *with* $n < \infty$. *Let* $\beta = (p-1, p-1, \ldots, p-1) \in J_0$. *Then the subquotient* $(\bigoplus_{\alpha \in J_0 \setminus \{\beta\}} Ad^\alpha \oplus D(A)d^\beta)/\mathbb{F}_1$ *is a simple Lie algebra except when* $\operatorname{char}\mathbb{F} = 2, A = \mathbb{F}_1[t]/(t^2 - a)$ *and* $A[D] = A \oplus \mathbb{F}_1(d/dt) \oplus \mathbb{F}_1(td/dt)$ *for some* $a \in \mathbb{F}_1^*$.

Remark. If $\operatorname{char}\mathbb{F} = 0$, then in Theorem 3.9 we have $A[D] = A \otimes \mathbb{F}[D]$, and $\dim A[D] = \infty$.

The converse of Theorem 3.9(a) is generally not true. See the following example.

Example 3.10. Suppose $\operatorname{char}\mathbb{F} = 0$. Let $B = \mathbb{F}[t]$ be the polynomial algebra in the invariable t, $D' = \mathbb{F}(d/dt)$. We know that $B[D']/\mathbb{F}$ is a simple Lie algebra. Let $A = B \times \mathbb{F} = \{(x,a)|x \in A, a \in \mathbb{F}\}$ be the associative algebra with the product $(x,a)(y,b) = (xy + bx + ay, ab)$. Then $(0,1)$ is the identity element in A. Let $D = \mathbb{F}\partial$, where $\partial((x,a)) = (\frac{dx}{dt}, 0)$. It is not difficult to verify that ∂ is a derivation of A, $(B,0)$ is a D-ideal, and $A^D = (\mathbb{F}, \mathbb{F})$. So A is not D-simple. But we can prove that $A[D]/A^D \simeq B[D']/\mathbb{F}$ is a simple Lie algebra. □

Using Theorem 3.9, one can obtain some interesting examples of simple (associative or Lie) algebras of Weyl type which cannot be obtained by the method in [SZ1].

Example 3.11. Let $\operatorname{char}\mathbb{F} = p > 0$, let $A = \mathbb{F}[t_i|0 \le i \le n]/(t_i^p - b_i|0 \le i \le n)$ where $0 < n < \infty$ and $b_i \in \mathbb{F}$, and let $D = \operatorname{span}\{\frac{\partial}{\partial t_i}|0 \le i \le n\}$. Then

$$D(A) = \operatorname{span}\{t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n} | 0 \le i_1, i_2, \ldots, i_n \le p-1, \\ (i_1, i_2, \ldots, i_n) \ne (p-1, p-1, \ldots, p-1)\}.$$

Then $\overline{A[D]} = [A[D]^L, A[D]^L]/\mathbb{F}$ is a simple Lie algebra of dimension $p^{2n} - 2$. □

Example 3.12. In Example 3.11, if $n = \infty$, then $A[D]^L/\mathbb{F}$ is a simple Lie algebra of infinite dimension. □

We would like to conclude this paper by several questions.

Questions. 1. Are there any commutative associative algebras A over a field F without identity element such that A is D-simple for some commutative subalgebra $D \subset \operatorname{Der}_{\mathbb{F}}(A)$?

2. If $A[D]/A^D$ is a simple Lie algebra, what conditions do A and D satisfy?

3. If the commutative associative algebra A over a field F does not have an identity element, and A is D simple, is $A[D]$ still simple?

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