

ULTRASTABILITY OF IDEALS OF HOMOGENEOUS POLYNOMIALS AND MULTILINEAR MAPPINGS ON BANACH SPACES

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ABSTRACT. Using the theory of full and symmetric tensor norms on normed spaces, a theorem of Kürsten and Heinrich on ultrastability and maximality of normed operator ideals is extended to ideals of n -homogeneous polynomials and n -linear mappings—scalar-valued and vector-valued. The motivation for these results is the following important special case: the “uniterated” Aron-Berner extension $\tilde{q}^L: E'' \rightarrow F''$ of an n -homogeneous polynomial $q: E \rightarrow F$ to the bidual remains in certain ideals under preservation of the norm. Moreover, Lotz’s characterization of maximal normed ideals of linear mappings through appropriate tensor norms is proved for ideals of n -homogeneous scalar-valued polynomials and ideals of n -linear mappings.

1. PRELIMINARIES

1.1. For vector spaces E_1, \dots, E_n, E, F over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} the “full” n -fold *tensor product* of (E_1, \dots, E_n) will be denoted by $E_1 \otimes \cdots \otimes E_n = \otimes(E_1, \dots, E_n) = \otimes_{j=1}^n E_j$. The universal definition of the n -fold tensor product identifies the n -linear mappings $E_1 \times \cdots \times E_n \rightarrow F$ with the linear mappings $\otimes_{j=1}^n E_j \rightarrow F$:

$$L(E_1, \dots, E_n; F) = L(\otimes_{j=1}^n E_j; F), \quad \varphi \rightsquigarrow \varphi^L.$$

Symmetric linear mappings $E^n \rightarrow F$ are linearized by the n -th *symmetric tensor product* $\otimes^{n,s} E$:

$$L_s({}^n E; F) = L(\otimes^{n,s} E; F), \quad \varphi \rightsquigarrow \varphi^{L,s}$$

(see e.g. [F1] for more details). A mapping $q: E \rightarrow F$ is, by definition, an n -homogeneous polynomial (notation: $q \in P^n(E; F)$) if there is a $\varphi \in L({}^n E; F)$ with $q(x) = \varphi(x, \dots, x)$ for all $x \in E$. Actually there is a unique $\check{q} \in L_s({}^n E; F)$ with this property:

$$P^n(E; F) = L_s({}^n E; F) = L(\otimes^{n,s} E; F), \quad q \rightsquigarrow \check{q} \rightsquigarrow q^L.$$

If the spaces are normed, then the continuous n -linear mappings and n -homogeneous polynomials are denoted by $\mathcal{L}(E_1, \dots, E_n; F)$, $\mathcal{L}_s({}^n E; F)$ or $\mathcal{P}^n(E; F)$, respectively.

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1.2. Again for fixed $n \in \mathbb{N}$ the *projective norm* π is uniquely defined by the property

$$[\otimes_{\pi}(E_1, \dots, E_n)]' \stackrel{1}{=} \mathcal{L}(E_1, \dots, E_n) \stackrel{1}{=} \mathcal{L}(E_1, \dots, E_n; \mathbb{K})$$

if E_1, \dots, E_n are normed spaces ($\stackrel{1}{=}$ means isometrically equal).

The *injective norm* ε satisfies, by definition ($\stackrel{1}{\hookrightarrow}$ stands for a metric injection),

$$\otimes_{\varepsilon}(E_1, \dots, E_n) \stackrel{1}{\hookrightarrow} (\otimes_{\pi}(E'_1, \dots, E'_n))'$$

this shows that ε is somehow dual to π . A *tensor norm* β of order n assigns to each n -tuple (E_1, \dots, E_n) a norm $\beta(\cdot; E_1, \dots, E_n)$ on $\otimes(E_1, \dots, E_n)$ (notation: $\otimes_{\beta}(E_1, \dots, E_n)$ or $\otimes_{\beta, j=1}^n E_j$) such that

- (a) $\varepsilon \leq \beta \leq \pi$,
- (b) $\|\otimes_{j=1}^n T_j : \otimes_{\beta, j=1}^n E_j \longrightarrow \otimes_{\beta, j=1}^n F_j\| \leq \prod_{j=1}^n \|T_j : E_j \longrightarrow F_j\|$ for all operators $T_j \in \mathcal{L}(E_j; F_j)$ (the metric mapping property).

Note that (a) and (b) imply that in (b) there is actually equality. β is called *finitely generated* if for all E_j and $z \in \otimes_{j=1}^n E_j$

$$\beta(z; E_1, \dots, E_n) = \inf\{\beta(z; M_1, \dots, M_n) \mid M_j \in \text{FIN}(E_j), z \in \otimes_{j=1}^n M_j\}$$

(where $\text{FIN}(E)$ denotes the set of finite-dimensional subspaces of E). ε and π are finitely generated tensor norms of order n . There is no general reference for the theory of tensor norms of order $n > 2$; many results, however, are straightforward generalizations of the case $n = 2$ which, e.g., is treated in [DF].

1.3. In the same spirit the natural *projective* and *injective* norms π_s and ε_s on the n -th symmetric tensor product satisfy

$$(\otimes_{\pi_s}^{n,s} E)' \stackrel{1}{=} \mathcal{P}^n(E; \mathbb{K}) =: \mathcal{P}^n(E),$$

$$\otimes_{\varepsilon_s}^{n,s} E \stackrel{1}{\hookrightarrow} \mathcal{P}^n(E').$$

An *s-tensor norm* α of order n (or shortly *s-tensor norm*, if $n \in \mathbb{N}$ is clear) assigns to each normed space E a norm $\alpha(\cdot; \otimes^{n,s} E)$ on $\otimes^{n,s} E$ (notation: $\otimes_{\alpha}^{n,s} E$) such that

- (a) $\varepsilon_s \leq \alpha \leq \pi_s$,
- (b) the metric mapping property $\|\otimes^{n,s} T : \otimes_{\alpha}^{n,s} E \longrightarrow \otimes_{\alpha}^{n,s} F\| \leq \|T : E \longrightarrow F\|^n$ for all $T \in \mathcal{L}(E; F)$.

α is called *finitely generated* if for all E and $z \in \otimes^{n,s} E$,

$$\alpha(z; \otimes^{n,s} E) = \inf\{\alpha(z; \otimes^{n,s} M) \mid M \in \text{FIN}(E), z \in \otimes^{n,s} M\}.$$

A detailed study of ε_s and π_s can be found in [F1]; the theory of *s-tensor norms* (in the spirit of Grothendieck’s theory of tensor norms of order 2, see [DF]) will be developed in a forthcoming paper [F2]. We do not need anything from the general theory in this paper. Note that for convenience the definitions allow n to be 1: in this case $\otimes_{\alpha}^1 E \stackrel{1}{=} E \stackrel{1}{=} \otimes_{\alpha}^{1,s} E$.

1.4. Let \mathfrak{U} be an ultrafilter on a set I ; the ultraproduct (along \mathfrak{U}) of a family $(E_{\iota})_{\iota \in I}$ of Banach spaces E_{ι} will be denoted by $(E_{\iota})_{\mathfrak{U}}$ (see e.g. [DF, 18.4]). Now take $\varphi_{\iota} \in \mathcal{L}(E_{1,\iota}, \dots, E_{n,\iota}; F_{\iota})$ for all $\iota \in I$ such that $\sup_{\iota \in I} \|\varphi_{\iota}\| < \infty$. Then

$$\overline{(\varphi_{\iota})_{\mathfrak{U}}}((x_{\iota}^1)_{\mathfrak{U}}, \dots, (x_{\iota}^n)_{\mathfrak{U}}) := (\varphi_{\iota}(x_{\iota}^1, \dots, x_{\iota}^n))_{\mathfrak{U}}$$

defines an n -linear map $(E_{1,\iota})_{\mathfrak{U}} \times \cdots \times (E_{n,\iota})_{\mathfrak{U}} \longrightarrow (F_\iota)_{\mathfrak{U}}$ between the ultraproducts; it easily follows that $\|(\overline{\varphi_\iota})_{\mathfrak{U}}\| = \lim_{\mathfrak{U}} \|\varphi_\iota\|$. If $F_\iota = \mathbb{K}^m$, then

$$\overline{(\varphi_\iota)_{\mathfrak{U}}}((x_\iota^n)_{\mathfrak{U}}, \dots, (x_\iota^n)_{\mathfrak{U}}) = \lim_{\mathfrak{U}} \varphi_\iota(x_\iota^1, \dots, x_\iota^n)$$

since $(\mathbb{K}^m)_{\mathfrak{U}} = \mathbb{K}^m$; in this case we write $\lim_{\mathfrak{U}} \varphi_\iota$ for $\overline{(\varphi_\iota)_{\mathfrak{U}}}$. If $E_{1,\iota} = \cdots = E_{n,\iota} = E_\iota$, it is clear that $(\varphi_\iota)_{\mathfrak{U}}$ is symmetric if all φ_ι are; in particular, for polynomials $q_\iota \in \mathcal{P}^n(E_\iota; F_\iota)$ with $\sup_\iota \|q_\iota\| < \infty$,

$$\overline{(q_\iota)_{\mathfrak{U}}}((x_\iota)_{\mathfrak{U}}) := (q_\iota(x_\iota))_{\mathfrak{U}} = \overline{(q_\iota)_{\mathfrak{U}}}((x_\iota)_{\mathfrak{U}}, \dots, (x_\iota)_{\mathfrak{U}})$$

is an n -homogeneous polynomial $(E_\iota)_{\mathfrak{U}} \longrightarrow (F_\iota)_{\mathfrak{U}}$. Again it is rather immediate to see that

$$(\mathcal{P}^n(E_\iota; F_\iota))_{\mathfrak{U}} \longrightarrow \mathcal{P}^n((E_\iota)_{\mathfrak{U}}; (F_\iota)_{\mathfrak{U}}), \quad (q_\iota)_{\mathfrak{U}} \longrightarrow \overline{(q_\iota)_{\mathfrak{U}}}$$

is an isometry; for $F_\iota = \mathbb{K}^m$ we write $\lim_{\mathfrak{U}} q_\iota := \overline{(q_\iota)_{\mathfrak{U}}}$. The purpose of this paper is to study under which circumstances $\overline{(q_\iota)_{\mathfrak{U}}}$ (resp. $(\varphi_\iota)_{\mathfrak{U}}$) is in a certain class of polynomials (resp. n -linear mappings) if all q_ι (resp. φ_ι) are.

1.5. For this, two special properties of ultraproducts will be needed: local determination and local duality.

Proposition (local determination of ultraproducts). *Let E_ι be normed spaces for $\iota \in I$ and $\{0\} \neq M \in \text{FIN}((E_\iota)_{\mathfrak{U}})$. Then there are, for all $\iota \in I$, operators $R_\iota \in \mathcal{L}(M; E_\iota)$ such that*

- (a) $x = (R_\iota x)_{\mathfrak{U}}$ for all $x \in M$;
- (b) $\|R_\iota\| \leq 1$ for all $\iota \in I$ and there is an $\mathcal{U} \in \mathfrak{U}$ with $\|R_\iota\| = 1$ for all $\iota \in \mathcal{U}$;
- (c) for all $\varepsilon > 0$ there is an $\mathcal{U}_\varepsilon \in \mathfrak{U}$ such that the inverse $R_\iota^{-1} : R_\iota(M) \longrightarrow M$ exists and $\|R_\iota^{-1}\| \leq 1 + \varepsilon$ for all $\iota \in \mathcal{U}_\varepsilon$.

This is due to Kürsten [K, Satz 4.1] and Heinrich [H, Prop. 6.1]. Here we shall only need (a) and the first part of (b).

1.6. For the local duality, take normed spaces E_ι and denote by

$$J : (E'_\iota)_{\mathfrak{U}} \longrightarrow (E_\iota)'_{\mathfrak{U}}, \quad J(x'_\iota)_{\mathfrak{U}} = \lim_{\mathfrak{U}} x'_\iota$$

the natural map from 1.4., i.e., $\langle J(x'_\iota)_{\mathfrak{U}}, (x_\iota)_{\mathfrak{U}} \rangle = \lim_{\mathfrak{U}} \langle x'_\iota, x_\iota \rangle$, and by

$$K : (E_\iota)_{\mathfrak{U}} \hookrightarrow (E_\iota)''_{\mathfrak{U}} \xrightarrow{J'} (E'_\iota)'_{\mathfrak{U}}.$$

Proposition (local duality of ultraproducts). *Let E_ι be normed spaces for all $\iota \in I$, $N \in \text{FIN}((E_\iota)'_{\mathfrak{U}})$ and $L \in \text{FIN}((E_\iota)_{\mathfrak{U}})$. Then for every $\varepsilon > 0$ there is an operator $T \in \mathcal{L}(N; (E'_\iota)_{\mathfrak{U}})$ such that*

- (a) $\|T\| = 1$ and $\|T^{-1} : TN \longrightarrow N\| \leq 1 + \varepsilon$,
- (b) $JTx' = x'$ for all $x' \in N \cap \text{im } J$,
- (c) $\langle JTx', x \rangle = \langle Kx, Tx' \rangle = \langle x', x \rangle$ for all $x' \in N$ and $x \in L$.

This result is due to Kürsten [K], Stern (see [H] for references) and Heinrich [H]; the present formulation is taken from the proof of [H, Theorem 7.3]. We shall only need $\|T\| = 1$ and (c).

2. THE MAIN THEOREM ON ULTRASTABILITY

2.1. Every $z' \in [\otimes_{k=1}^m (\otimes^{n_k, s} E_k)]^*$ defines a $(\sum_{k=1}^m n_k)$ -linear functional φ on $\prod_{k=1}^m (E_k)^{n_k}$ which is symmetric in the n_k variables in E_k for each $k = 1, \dots, m$, and vice-versa. In the spirit of the notations of 1.1 we may define $\varphi^L := z'$. For ultraproducts and $z'_\iota = \varphi_\iota^L$ the notation

$$\lim_{\mathfrak{U}} z'_\iota := \left[\lim_{\mathfrak{U}} \varphi_\iota \right]^L$$

will be used in the following:

Theorem. *Let $m, n_1, \dots, n_m \in \mathbb{N}$, \mathfrak{U} an ultrafilter on I , normed spaces $E_{k,\iota}$, a finitely generated tensor norm β of order m and finitely generated s -tensor norms α_k of order n_k be given. If $z'_\iota \in [\otimes_{\beta, k=1}^m (\otimes_{\alpha_k}^{n_k, s} E_{k,\iota})]^\prime =: H_\iota$ for all $\iota \in I$ such that $\sup_{\iota \in I} \|z'_\iota\|_{H_\iota} < \infty$, then*

$$\lim_{\mathfrak{U}} z'_\iota \in \left[\otimes_{\beta, k=1}^m \left(\otimes_{\alpha_k}^{n_k, s} (E_{k,\iota})_{\mathfrak{U}} \right) \right]^\prime =: H$$

and $\|\lim_{\mathfrak{U}} z'_\iota\|_H \leq \lim_{\mathfrak{U}} \|z'_\iota\|_{H_\iota}$.

Proof. Since β and all α_k are finitely generated, it is enough to show that for all $M_k \in \text{FIN}((E_{k,\iota})_{\mathfrak{U}})$ and all $z \in \otimes_{k=1}^m [\otimes^{n_k, s} M_k] =: M$,

$$|\langle \lim_{\mathfrak{U}} z'_\iota, z \rangle| \leq \lim_{\mathfrak{U}} \|z'_\iota\|_{H_\iota} \cdot \beta(z; \otimes_{\alpha_1}^{n_1, s} M_1, \dots, \otimes_{\alpha_m}^{n_m, s} M_m)$$

holds. Given these M_k the local determination of ultraproducts (see 1.5) gives operators $R_{k,\iota} : M_k \rightarrow E_{k,\iota}$ with $\|R_{k,\iota}\| \leq 1$ and $(R_{k,\iota} x_k)_{\mathfrak{U}} = x_k$ for all $x_k \in M_k$; it follows that

$$\begin{aligned} & \left\langle \lim_{\mathfrak{U}} z'_\iota, \underbrace{[\otimes^{n_1} x_1] \otimes \dots \otimes [\otimes^{n_m} x_m]}_{=:x} \right\rangle \\ &= \left\langle \lim_{\mathfrak{U}} z'_\iota, [\otimes^{n_1} (R_{1,\iota} x_1)_{\mathfrak{U}}] \otimes \dots \otimes [\otimes^{n_m} (R_{m,\iota} x_m)_{\mathfrak{U}}] \right\rangle \\ &= \lim_{\mathfrak{U}} \left\langle z'_\iota, \underbrace{[\otimes^{n_1, s} R_{1,\iota}] \otimes \dots \otimes [\otimes^{n_m, s} R_{m,\iota}]}_{=:R_\iota} x \right\rangle \end{aligned}$$

and hence for all $z \in M$,

$$\begin{aligned} |\langle \lim_{\mathfrak{U}} z'_\iota, z \rangle| &= \lim_{\mathfrak{U}} |\langle z'_\iota, R_\iota z \rangle| \\ &\leq \lim_{\mathfrak{U}} \|z'_\iota\|_{H_\iota} \beta(R_\iota z; \otimes_{\alpha_1}^{n_1, s} E_{1,\iota}, \dots, \otimes_{\alpha_m}^{n_m, s} E_{m,\iota}) \\ &\leq \lim_{\mathfrak{U}} \|z'_\iota\|_{H_\iota} \|R_{1,\iota}\|^{n_1} \dots \|R_{m,\iota}\|^{n_m} \beta(z; \otimes_{\alpha_1}^{n_1} M_1, \dots, \otimes_{\alpha_m}^{n_m} M_m) \end{aligned}$$

by the metric mapping properties of β and $\alpha_1, \dots, \alpha_m$. This is the desired inequality. □

In other words, the natural map

$$\left(\left[\otimes_{\beta, k=1}^m (\otimes_{\alpha_k}^{n_k, s} E_{k,\iota}) \right]^\prime \right)_{\mathfrak{U}} \longrightarrow \left[\otimes_{\beta, k=1}^m \left(\otimes_{\alpha_k}^{n_k, s} (E_{k,\iota})_{\mathfrak{U}} \right) \right]^\prime$$

has norm ≤ 1 . It is rather immediate that the norm is 1 if all $E_\iota \neq \{0\}$ and that for $\beta = \pi$ and $\alpha_k = \pi_s$ this is even an isometry (as in the special cases of 1.4, one has to take $x_{k,\iota} \in B_{E_{k,\iota}}$ with $|\langle x'_\iota, \dots \rangle| \geq \|x'_\iota\|(1 - \varepsilon)$).

2.2. Note the special cases of $m = 1$ (polynomials) and all $n_k = 1$ (no symmetry):

- (a) If α is a finitely generated s -tensor norm of order n , then for all normed spaces E_ι the natural map

$$\left((\otimes_\alpha^{n,s} E_\iota)' \right)_\mathfrak{U} \longrightarrow \left(\otimes_\alpha^{n,s} (E_\iota)_\mathfrak{U} \right)'$$

has norm ≤ 1 .

- (b) If β is a finitely generated tensor norm of order m , then for all normed spaces $E_{k,\iota}$ the natural map

$$\left((\otimes_\beta (E_{1,\iota}, \dots, E_{m,\iota}))' \right)_\mathfrak{U} \longrightarrow \left(\otimes_\beta ((E_{1,\iota})_\mathfrak{U}, \dots, (E_{m,\iota})_\mathfrak{U}) \right)'$$

has norm ≤ 1 .

2.3. It is clear that the same reasoning gives that the natural map

$$\left(\left[\otimes_\alpha^{m,s} \left(\otimes_{\beta,k=1}^n E_{k,\iota} \right) \right]' \right)_\mathfrak{U} \longrightarrow \left[\otimes_\alpha^{m,s} \left(\otimes_{\beta,k=1}^n (E_{k,\iota})_\mathfrak{U} \right) \right]'$$

has norm ≤ 1 .

3. SCALAR-VALUED IDEALS OF POLYNOMIALS AND MULTILINEAR MAPPINGS

3.1. A subclass $\mathcal{Q} \subset \mathcal{P}^n$ of n -homogeneous continuous scalar-valued polynomials on Banach spaces is called an *ideal*, if

- (a) $\mathcal{Q}(E) := \mathcal{P}^n(E) \cap \mathcal{Q}$ is a linear subspace of $\mathcal{P}^n(E)$ for all Banach spaces E ,
- (b) if $T \in \mathcal{L}(E; F)$ and $q \in \mathcal{Q}(F)$, then $q \circ T \in \mathcal{Q}$,
- (c) $[\mathbb{K} \ni z \rightsquigarrow z^n \in \mathbb{K}] \in \mathcal{Q}$.

If $\|\cdot\|_{\mathcal{Q}} : \mathcal{Q} \rightarrow [0, \infty]$ satisfies

- (a') $\|\cdot\|_{\mathcal{Q}}|_{\mathcal{Q}(E)}$ is a norm for all Banach spaces E ,
- (b') $\|q \circ T\|_{\mathcal{Q}} \leq \|T\|^n \|q\|_{\mathcal{Q}}$ in the situation of (b),
- (c') $\|\mathbb{K} \ni z \rightsquigarrow z^n \in \mathbb{K}\|_{\mathcal{Q}} = 1$,

then $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is called a *normed ideal of n -homogeneous polynomials*. It can easily be seen that always $\|q\| \leq \|q\|_{\mathcal{Q}}$ and that $\mathcal{Q}(M) = \mathcal{P}^n(M) = \otimes^{n,s} M'$ for all finite-dimensional M . It would also be possible to define ideals of polynomials on normed spaces (not only Banach spaces) – but this would not make much of a difference.

It is rather immediate to see that for each s -tensor norm α of order n

$$\mathcal{Q}(E) := (\otimes_\alpha^{n,s} E)'$$

defines a normed ideal of n -homogeneous polynomials. For $\alpha = \varepsilon_s$ one obtains the integral polynomials (see e.g. [F1, chap. 3] for their properties). It is not difficult to see that all *extendible* n -homogeneous polynomials (i.e., those $q \in \mathcal{P}^n(E)$ such that for all super spaces $G \supset E$ there is an extension $\tilde{q} \in \mathcal{P}^n(G)$ of G ; see Kirwan and Ryan [KR]) are also of this form; in this case the s -tensor norm α of order n is the “injective associate” of the projective s -norm π_s , i.e., satisfies

$$\otimes_\alpha^{n,s} E \xrightarrow{1} \otimes_{\pi_s}^{n,s} \ell_\infty(B_{E'}) ;$$

we omit the details.

3.2. For $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ and $q \in \mathcal{P}^n(E)$ define

$$\|q\|_{\mathcal{Q}_{\max}} := \sup\{\|q|_M\|_{\mathcal{Q}} \mid M \in \text{FIN}(E)\} \in [0, \infty].$$

$(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is called *maximal* if every $q \in \mathcal{P}^n(E)$ with $\|q\|_{\mathcal{Q}_{\max}} < \infty$ is in \mathcal{Q} and $\|q\|_{\mathcal{Q}} = \|q\|_{\mathcal{Q}_{\max}}$. $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is called *ultrastable* if for $q_i \in \mathcal{Q}(E_i)$ with $\sup_{i \in I} \|q_i\|_{\mathcal{Q}} < \infty$ one has $\lim_{\mathfrak{U}} q_i \in \mathcal{Q}((E_i)_{\mathfrak{U}})$ and $\|\lim_{\mathfrak{U}} q_i\|_{\mathcal{Q}} \leq \sup \|q_i\|_{\mathcal{Q}}$ (and hence $\leq \lim_{\mathfrak{U}} \|q_i\|_{\mathcal{Q}}$).

Theorem. For each normed ideal $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ of n -homogeneous (scalar-valued) polynomials the following statements are equivalent:

- (1) $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is maximal.
- (2) $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is ultrastable.
- (3) There is a finitely generated s -tensor norm α of order n such that

$$\mathcal{Q}(E) \stackrel{1}{=} (\otimes_{\alpha}^{n,s} E)'$$

for all Banach spaces E .

It is clear that this result can be conjectured when knowing the K\"ursten-Heinrich characterization of maximal Banach operator ideals [H, Theorem 8.1] and Lotz' representation with tensor norms of order 2 (see [DF, 17.5]). It follows from (3) that $\mathcal{Q}(E)$ is a Banach space if \mathcal{Q} is maximal.

Proof. (1) \curvearrowright (3). For finite-dimensional Banach spaces M define α by

$$\otimes_{\alpha}^{n,s} M := \mathcal{Q}(M)'$$

and for arbitrary normed spaces E by

$$\alpha(z; \otimes^{n,s} E) := \inf\{\alpha(z; \otimes^{n,s} M) \mid M \in \text{FIN}(E), z \in \otimes^{n,s} M\}.$$

It is straightforward to see that α is a finitely generated s -tensor norm of order n with $(*) \mathcal{Q}(M) \stackrel{1}{=} (\otimes_{\alpha}^{n,s} M)'$ if M is finite-dimensional. Now

$$\begin{aligned} \|q^L\|_{(\otimes_{\alpha}^{n,s} E)'} &= \sup\{|\langle q^L, z \rangle| \mid \alpha(z; \otimes^{n,s} E) < 1\} \\ &= \sup\{|\langle (q|_M)^L, z \rangle| \mid M \in \text{FIN}(E), z \in \otimes^{n,s} M, \alpha(z; \otimes^{n,s} M) < 1\} \\ &\stackrel{(*)}{=} \sup\{\|q|_M\|_{\mathcal{Q}} \mid M \in \text{FIN}(E)\} = \|q\|_{\mathcal{Q}} \end{aligned}$$

by the maximality of \mathcal{Q} .

(3) \curvearrowright (2). This is the special case 2.2(a) of the main theorem.

(2) \curvearrowright (1). Let $q \in \mathcal{P}^n(E)$ with $\|q\|_{\mathcal{Q}_{\max}} < \infty$ and let \mathfrak{U} be an ultrafilter finer than the order filter in $\text{FIN}(E)$. For $q|_M$ on $M \in \text{FIN}(E)$ one obtains $\|q|_M\|_{\mathcal{Q}} \leq \|q\|_{\mathcal{Q}_{\max}}$ and $\lim_{\mathfrak{U}} q|_M : (M)_{\mathfrak{U}} \rightarrow \mathbb{K}$ extends q (via the natural isometric embedding $E \hookrightarrow (M)_{\mathfrak{U}}$). It follows from (2) that $q \in \mathcal{Q}$ and

$$\|q\|_{\mathcal{Q}} \leq \|\lim_{\mathfrak{U}} q|_M\|_{\mathcal{Q}} \leq \sup \|q|_M\|_{\mathcal{Q}} = \|q\|_{\mathcal{Q}_{\max}}$$

and hence $\|q\|_{\mathcal{Q}} = \sup \|q|_M\|_{\mathcal{Q}}$ since always $\|\cdot\|_{\mathcal{Q}_{\max}} \leq \|\cdot\|_{\mathcal{Q}}$. □

3.3. Every $q \in \mathcal{P}^n(E)$ has an extension to $\mathcal{P}^n(E'')$: the Aron-Berner extension (see [AB]) can easily be seen as an iterated limit along local ultrafilters of E (see [LR] or [F1, 6.9]). This motivated Dineen and Timoney [DT] and Lindström and Ryan [LR] independently to define an “uniterated” Aron-Berner extension (as it is called in [F1]) as follows: for $\iota := (M, N, \varepsilon) \in I := \text{FIN}(E'') \times \text{FIN}(E') \times]0, 1]$ choose with the strong principle of local reflexivity an operator $T_\iota \in \mathcal{L}(M; E)$ with $T_\iota x = x$ for all $x \in M \cap E$ such that $\|T_\iota\| \leq 1 + \varepsilon$ and $\langle T_\iota x'', x' \rangle$ for all $(x'', x') \in M \times N$; for $x'' \in E''$ define $f_\iota(x'') := T_\iota x''$ if $x'' \in M$ and $f_\iota(x'') := 0$ otherwise. Take a local ultrafilter \mathfrak{U} on I , i.e., an ultrafilter which is finer than the order filter on I , then the natural mappings

$$\begin{aligned} J_E : E'' &\longrightarrow (E)_{\mathfrak{U}}; & x'' &\rightsquigarrow (f_\iota(x''))_{\mathfrak{U}}, \\ Q_E : (E)_{\mathfrak{U}} &\longrightarrow E''; & (x_\iota)_{\mathfrak{U}} &\rightsquigarrow \lim_{\iota, \mathfrak{U}} x_\iota \end{aligned}$$

($\sigma(E'', E')$ -limit) have the following properties: J_E is an isometry which extends the natural embedding $E \hookrightarrow (E)_{\mathfrak{U}}$, the mapping Q_E has norm 1 (if $E \neq \{0\}$) and $J_E Q_E : (E)_{\mathfrak{U}} \longrightarrow \text{im } J_E \stackrel{1}{=} E''$ is a norm-1-projection. For $q \in \mathcal{P}^n(E)$ define

$$\bar{q}^{\mathfrak{U}} := \left[\lim_{\iota, \mathfrak{U}} q \right] \circ J_E \in \mathcal{P}^n(E'')$$

hence $\bar{q}^{\mathfrak{U}}(x'') = \lim_{\iota, \mathfrak{U}} q(f_\iota(x''))$; it is clear that $\bar{q}^{\mathfrak{U}}$ extends q . The foregoing theorem implies the following:

Theorem. *Let \mathcal{Q} be a maximal normed ideal of n -homogeneous polynomials, $q \in \mathcal{P}^n(E)$ and \mathfrak{U} a local ultrafilter of E . Then $q \in \mathcal{Q}(E)$ if and only if $\bar{q}^{\mathfrak{U}} \in \mathcal{Q}(E'')$; in this case $\|\bar{q}^{\mathfrak{U}}\|_{\mathcal{Q}} = \|q\|_{\mathcal{Q}}$.*

In particular, this applies to the class of integral polynomials. Note that it took considerable effort to prove this result for the iterated Aron-Berner extension in the cases $\mathcal{Q} := \mathcal{P}^n$ (Davie-Gamelin [DG]) and $\mathcal{Q} := \{\text{integral polynomials}\}$ (due to [CZ]; in [F1, 6.8] there is an alternative proof).

3.4. It is clear that a theorem like 3.1 holds also for normed ideals of n -linear functionals originally defined by Pietsch [P] in 1983: a subclass \mathcal{A} of all n -linear continuous functionals on Banach spaces is an *ideal* if (for all Banach spaces E_j and F_j)

- (a) $\mathcal{A}(E_1, \dots, E_n) := \mathcal{A} \cap \mathcal{L}(E_1, \dots, E_n)$ is a linear subspace of $\mathcal{L}(E_1, \dots, E_n)$,
- (b) if $T_j \in \mathcal{L}(E_j; F_j)$ and $\varphi \in \mathcal{A}(E_1, \dots, E_n)$, then $\varphi \circ (T_1, \dots, T_n) \in \mathcal{A}$,
- (c) $[\mathbb{K}^n \ni (x_1, \dots, x_n) \rightsquigarrow x_1 \cdots x_n \in \mathbb{K}] \in \mathcal{A}$.

If $\|\cdot\|_{\mathcal{A}} : \mathcal{A} \longrightarrow [0, \infty]$ satisfies

- (a') $\|\cdot\|_{\mathcal{A}}|_{\mathcal{A}(E_1, \dots, E_n)}$ is a norm on $\mathcal{A}(E_1, \dots, E_n)$,
- (b') $\|\varphi \circ (T_1, \dots, T_n)\|_{\mathcal{A}} \leq \|T\| \cdots \|T_n\| \|\varphi\|_{\mathcal{A}}$ in the situation of (b),
- (c') $\|\mathbb{K}^n \ni (x_1, \dots, x_n) \rightsquigarrow x_1 \cdots x_n \in \mathbb{K}\|_{\mathcal{A}} = 1$,

then $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called a *normed ideal of n -linear functionals*. It is obvious from 3.2 how to define that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is maximal or ultrastable. The same kind of ideas as in the proof of Theorem 3.2 gives the following:

Theorem. *For every normed ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ of n -linear functionals the following statements are equivalent:*

- (1) $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is maximal.
- (2) $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is ultrastable.

(3) *There exists a finitely generated tensor norm β of order n with*

$$(\otimes_{\beta}(E_1, \dots, E_n))' \stackrel{1}{=} \mathcal{A}(E_1, \dots, E_n)$$

for all Banach spaces E_1, \dots, E_n .

4. VECTOR-VALUED IDEALS OF POLYNOMIALS AND MULTILINEAR MAPPINGS

4.1. *A normed ideal of n -homogeneous continuous vector-valued polynomials on Banach spaces* is, by definition, a pair $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ such that:

- (a) $\mathcal{Q}(E; F) := \mathcal{Q} \cap \mathcal{P}^n(E; F)$ is a linear subspace of $\mathcal{P}^n(E; F)$ and $\|\cdot\|_{\mathcal{Q}}|_{\mathcal{Q}(E; F)}$ is a norm on it.
- (b) If $T \in \mathcal{L}(E_1; E_2)$, $q \in \mathcal{Q}(E_2; E_3)$ and $S \in \mathcal{L}(E_3; E_4)$, then $S \circ q \circ T \in \mathcal{Q}$ and $\|S \circ q \circ T\|_{\mathcal{Q}} \leq \|T\|^n \|q\|_{\mathcal{Q}} \|S\|$.
- (c) $[\mathbb{K} \ni z \rightsquigarrow z^n \in \mathbb{K}]$ is in \mathcal{Q} and has norm 1.

4.2. We are only interested in ideals coming from tensor norms. For this the following notation will be used:

$$\mathcal{L}(E; G') = (E \otimes_{\pi} G) ', \quad T \rightsquigarrow \beta_T,$$

$$\mathcal{P}^n(E; G') = \mathcal{L}(\otimes_{\pi_s}^{n,s} E; G') = ((\otimes_{\pi_s}^{n,s} E) \otimes_{\pi} G) ', \quad q \rightsquigarrow \beta_q := \beta_{qL}.$$

Let α be a finitely generated s -tensor norm of order n and β a finitely generated tensor norm of order 2. We define $q \in \mathcal{P}_{(\alpha, \beta)}^n(E; F)$ if $\beta_{(\varkappa_F \circ q)} \in ((\otimes_{\alpha}^{n,s} E) \otimes_{\beta} F) '$ and in this case $\|q\|_{(\alpha, \beta)} := \|\beta_{\varkappa_F \circ q}\|(\dots)$ (here $\varkappa_F : F \hookrightarrow F''$ is the natural injection). It is easy to see that $(\mathcal{P}_{(\alpha, \beta)}^n, \|\cdot\|_{(\alpha, \beta)})$ is a normed ideal in the sense of 4.1. If \mathcal{C} is the maximal Banach operator ideal associated with the dual tensor norm β' of β , then from the representation theorem [DF, 17.5] we get

$$\mathcal{P}_{(\alpha, \beta)}^n(E; F) \stackrel{1}{=} \mathcal{C}(\tilde{\otimes}_{\alpha}^{n,s} E; F).$$

A consequence of this is that $\mathcal{P}_{(\alpha, \beta)}^n$ is *regular*, i.e., $q \in \mathcal{P}_{(\alpha, \beta)}^n(E; F)$ if (and only if) $\varkappa_F \circ q \in \mathcal{P}_{(\alpha, \beta)}^n(E; F'')$ and $\|q\|_{(\alpha, \beta)} = \|\varkappa_F \circ q\|_{(\alpha, \beta)}$. It is also rather routine to show that $\mathcal{P}_{(\alpha, \beta)}^n$ is *maximal* in the following sense: $q \in \mathcal{P}^n(E; F)$ is in $\mathcal{P}_{(\alpha, \beta)}^n$ if

$$\sup\{\|Q_L^F \circ q\|_{M(\alpha, \beta)} \mid L \in \text{COFIN}(E), M \in \text{FIN}(E)\} < \infty$$

and then this number is $\|q\|_{(\alpha, \beta)}$ (where $\text{COFIN}(E) := \{L \subset E/L \text{ finite-codimensional, closed subspace of } E\} = \{N^0 \mid N \in \text{FIN}(E')\}$ and $Q_L^F : F \rightarrow F/L$ the canonical quotient map). However, it seems to be unlikely that every maximal and regular $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is of the form $\mathcal{P}_{(\alpha, \beta)}^n$, contrary to the scalar case.

Just one example: Since $[\otimes_{\varepsilon_s}^{n,s} E] \otimes_{\varepsilon} F \xrightarrow{1} \mathcal{P}^n(E'; F)$, one can deduce from [JM, Lemma 2.1.] (see also [A2] for the reflexive case) that $\mathcal{P}_{\varepsilon_s, \varepsilon}^n(E; F')$ is the space of integral n -homogeneous polynomials $E \rightarrow F'$ in the sense of Alencar [A1].

Theorem. *The ideal $\mathcal{P}_{(\alpha, \beta)}^n$ is ultrastable, i.e., if \mathfrak{U} is an ultrafilter on I and $q_i \in \mathcal{P}_{(\alpha, \beta)}^n(E_i; F_i)$ such that $\sup \|q_i\|_{(\alpha, \beta)} < \infty$, then $(\overline{q_i})_{\mathfrak{U}} \in \mathcal{P}_{(\alpha, \beta)}^n((E_i)_{\mathfrak{U}}; (F_i)_{\mathfrak{U}})$ and $\|(q_i)_{\mathfrak{U}}\|_{(\alpha, \beta)} \leq \lim_{\mathfrak{U}} \|q_i\|_{(\alpha, \beta)}$.*

See 1.4 for the notation.

Proof. Let $V \in \mathcal{L}(\otimes_{\pi_s}^{n,s}(E_l)_{\mathfrak{U}}; (F_l)_{\mathfrak{U}})$ be the operator associated with $\overline{(q_l)}_{\mathfrak{U}}$. One has to show that $\tilde{V} \in \mathcal{C}(\tilde{\otimes}_{\alpha}^{n,s}(E_l)_{\mathfrak{U}}; (F_l)_{\mathfrak{U}})$ where $(\mathcal{C}, \mathbf{C})$ is the Banach operator ideal associated with β' and \tilde{V} is the extension of V to the completion $\tilde{\otimes}_{\alpha}^{n,s}(E_l)_{\mathfrak{U}}$. From the main theorem in Section 2 we get

$$z' := \lim_{\mathfrak{U}} \beta_{\mathfrak{z}_{F_l} \circ q_l} \in \left[(\otimes_{\alpha}^{n,s}(E_l)_{\mathfrak{U}}) \otimes_{\beta} (F_l)_{\mathfrak{U}} \right]' =: H$$

and $\|z'\|_H \leq \lim_{\mathfrak{U}} \|q_l\|_{(\alpha,\beta)}$. If $z' = \beta_U$ with $U \in \mathcal{L}(\tilde{\otimes}_{\alpha}^{n,s}(E_l)_{\mathfrak{U}}; (F_l)_{\mathfrak{U}}')$ (hence $\mathbf{C}(U) = \|z'\|_H$), then $U = K \circ \tilde{V}$ where $K : (F_l)_{\mathfrak{U}} \rightarrow (F_l)_{\mathfrak{U}}' =: G'$ is the natural mapping from 1.6 (check on $[\otimes^n(x_l)_{\mathfrak{U}}] \otimes (x_l)_{\mathfrak{U}}$). Using maximality it is enough to show that

$$\|M \hookrightarrow \tilde{\otimes}_{\alpha}^{n,s}(E_l)_{\mathfrak{U}} \xrightarrow{\tilde{V}} F \xrightarrow{Q_{N^0}^F} F/N^0 = N'\| \leq \|z'\|_H$$

for all $M \in \text{FIN}(\tilde{\otimes}_{\alpha}^{n,s}(E_l)_{\mathfrak{U}})$ and $N \in \text{FIN}(F')$. With $L := \tilde{V}(M) \subset F$ the local duality 1.6 of ultraproducts gives an operator $T \in \mathcal{L}(N; G)$ with $\|T\| = 1$ and

$$\langle K\tilde{V}(u), Tx' \rangle_{G',G} = \langle x', \tilde{V}(u) \rangle_{F',F} = \langle Q_{N^0}^F \tilde{V}(u), x' \rangle_{N',N}$$

for all $u \in M$ and $x' \in N$. This means $Q_{N^0}^F \circ \tilde{V}|_M = T' \circ K \circ \tilde{V}|_M$. It follows that $\mathbf{C}(Q_{N^0}^F \circ \tilde{V}|_M) \leq \|T'\| \mathbf{C}(K \circ \tilde{V}|_M) = \mathbf{C}(U|_M) \leq \|z'\|_H$ and therefore $\|\overline{(q_l)}_{\mathfrak{U}}\|_{(\alpha,\beta)} = \mathbf{C}(\tilde{V}) \leq \lim_{\mathfrak{U}} \|q_l\|_{(\alpha,\beta)}$. \square

4.3. Note the special case $\beta = \pi$ and $q_l^0 : E_l \rightarrow \otimes_{\alpha}^{n,s} E_l$ being the ‘‘canonical’’ polynomial $q_l^0(x) := \otimes^n x$, i.e., $(q_l^0)^L = \text{id}_{\otimes_{\alpha}^{n,s} E_l}$ and $\|q_l^0\|_{(\alpha,\pi)} = 1$. Then $\overline{(q_l^0)}_{\mathfrak{U}}$ is in $\mathcal{P}_{\alpha,\pi}(\otimes_{\alpha}^{n,s}(E_l)_{\mathfrak{U}}; (\tilde{\otimes}_{\alpha}^{n,s} E_l)_{\mathfrak{U}})$ which means that the natural map

$$\otimes_{\alpha}^{n,s}(E_l)_{\mathfrak{U}} \rightarrow (\otimes_{\alpha}^{n,s} E_l)_{\mathfrak{U}}$$

defined by $\otimes^n(x_l)_{\mathfrak{U}} \rightsquigarrow (\otimes^n x_l)_{\mathfrak{U}}$ has norm ≤ 1 , if α is a finitely generated s -tensor norm.

4.4. For any ultrafilter \mathfrak{U} and normed space F the map $Q_F : (F)_{\mathfrak{U}} \rightarrow F''$ from 3.3 is well-defined, extends the natural embedding $F \hookrightarrow (F)_{\mathfrak{U}}$ and satisfies

$$\langle Q_F((y_l)_{\mathfrak{U}}), y' \rangle_{F'',F'} = \lim_{l, \mathfrak{U}} \langle y_l, y' \rangle_{F,F'}$$

If \mathfrak{U} is a local ultrafilter of another space E and $q \in \mathcal{P}^n(E; F)$, then (see 3.3 for the notation)

$$\overline{q}^{\mathfrak{U}} := Q_F \circ \overline{(q)}_{\mathfrak{U}} \circ J_E : E'' \rightarrow F''$$

is an n -homogeneous polynomial which extends q and can be calculated as follows:

$$\langle \overline{q}^{\mathfrak{U}}(x''), y' \rangle_{F'',F'} = \lim_{l, \mathfrak{U}} \langle q(f_l(x'')), y' \rangle$$

Theorem 4.2 and the regularity of $\mathcal{P}_{(\alpha,\beta)}^n$ imply the following:

Corollary. *Take $q \in \mathcal{P}^n(E; F)$ and a local ultrafilter on E . Then the extension $\overline{q}^{\mathfrak{U}}$ is in $\mathcal{P}_{(\alpha,\beta)}^n(E''; F'')$ if and only if q is; in this case $\|q\|_{(\alpha,\beta)} = \|\overline{q}^{\mathfrak{U}}\|_{(\alpha,\beta)}$.*

4.5. For n -linear operators $E_1 \times \dots \times E_n \rightarrow F$ the same ideas apply: a *normed ideal of n -linear continuous operators between Banach spaces* is a pair $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ such that

- (a) $\mathcal{A}(E_1, \dots, E_n; F) = \mathcal{A} \cap \mathcal{L}(E_1, \dots, E_n; F)$ is linear and $\|\cdot\|_{\mathcal{A}}|_{\mathcal{A}(E_1, \dots, E_n; F)}$ is a norm,
- (b) if $T_j \in \mathcal{L}(G_j; E_j)$, $\varphi \in \mathcal{A}(E_1, \dots, E_n; F)$ and $S \in \mathcal{L}(F; G)$, then the composition $S \circ \varphi \circ (T_1, \dots, T_n)$ is in \mathcal{A} and

$$\|S \circ \varphi \circ (T_1, \dots, T_n)\|_{\mathcal{A}} \leq \|S\| \|\varphi\|_{\mathcal{A}} \|T_1\| \cdots \|T_n\|,$$

- (c) $[\mathbb{K}^n \ni (x_1, \dots, x_n) \rightsquigarrow x_1 \cdots x_n \in \mathbb{K}]$ is in \mathcal{A} and $\|\cdots\|_{\mathcal{A}} = 1$.

Every tensor norm β of order $n + 1$ defines an ideal \mathcal{A}_{β} as follows: an n -linear map φ is in $\mathcal{A}_{\beta}(E_1, \dots, E_n; F)$ if and only if the $(n + 1)$ -linear form associated with $\varkappa_F \circ \varphi$ is in $[\otimes_{\beta}(E_1, \dots, E_n, F)]'$.

An ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is called *maximal* if

$$\|\varphi\|_{\mathcal{A}^{\max}} := \sup\{\|Q_L^F \circ \varphi|_{M_1 \times \dots \times M_n}\|_{\mathcal{A}} \mid M_j \in \text{FIN}(E_j), L \in \text{COFIN}(F)\} < \infty$$

implies $\varphi \in \mathcal{A}$ and $\|\varphi\|_{\mathcal{A}} = \|\varphi\|_{\mathcal{A}^{\max}}$ holds. The ideal is called *regular* if $\varkappa_F \circ \varphi \in \mathcal{A}(E_1, \dots, E_n; F'')$ implies $\varphi \in \mathcal{A}$ and $\|\varphi\|_{\mathcal{A}} = \|\varkappa_F \circ \varphi\|_{\mathcal{A}}$; it is easy to see that the ideals \mathcal{A}_{β} are regular. \mathcal{A} is *ultrastable* if for $\varphi_{\iota} \in \mathcal{A}(E_{1,\iota}, \dots, E_{n,\iota}; F_{\iota})$ with $\|\varphi_{\iota}\|_{\mathcal{A}} \leq c$ the operator $\overline{(\varphi_{\iota})}_{\mathfrak{U}}$ is in $\mathcal{A}((E_{1,\iota})_{\mathfrak{U}}, \dots, (E_{n,\iota})_{\mathfrak{U}}; (F_{\iota})_{\mathfrak{U}})$ and $\|\overline{(\varphi_{\iota})}_{\mathfrak{U}}\|_{\mathcal{A}} \leq \sup \|\varphi_{\iota}\|_{\mathcal{A}}$.

Theorem. Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a normed ideal of n -linear continuous mappings between Banach spaces. Then the following statements are equivalent:

- (1) $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is maximal.
- (2) $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is ultrastable and regular.
- (3) There is a finitely generated tensor norm β of order $n + 1$ such that

$$\begin{aligned} \mathcal{A}(E_1, \dots, E_n; F') &\stackrel{1}{=} (\otimes_{\beta}(E_1, \dots, E_n, F))', \\ \mathcal{A}(E_1, \dots, E_n; F) &\stackrel{1}{=} (\otimes_{\beta}(E_1, \dots, E_n, F'))' \cap \mathcal{L}(E_1, \dots, E_n; F). \end{aligned}$$

Proof. (1) \curvearrowright (3) runs exactly as in the case $n = 1$ (see [DF, 17.5.], the extension lemma holds also for finitely generated tensor norms of arbitrary order) with a construction of β as in the proof of Theorem 3.2.

(3) \curvearrowright (2). Ultrastability follows from the main theorem as in the proof of Theorem 4.2; the regularity follows immediately when looking at the $(n + 1)$ -linear functionals appearing in the two formulae in (3).

(2) \curvearrowright (1). Take $\varphi \in \mathcal{L}(E_1, \dots, E_n; F)$ with $\|\varphi\|_{\mathcal{A}^{\max}} < \infty$. Following Heinrich's proof for the case $n = 1$ (see [H]) consider $I := \text{FIN}(E_1) \times \dots \times \text{FIN}(E_n) \times \text{FIN}(F')$ and let \mathfrak{U} be an ultrafilter finer than the order filter. For $\iota = (M_1, \dots, M_n, N)$ define $E_{k,\iota} := M_k$ and $F_{\iota} := F/N^0$ and metric embeddings

$$J_k : E_k \rightarrow (E_{k,\iota})_{\mathfrak{U}} \quad x_k \rightsquigarrow (x_{k,\iota})_{\mathfrak{U}}$$

where $x_{k,\iota} = x_k$ if $x_k \in M_k$ and $= 0$ otherwise. Moreover, define a mapping $Q : (F_{\iota})_{\mathfrak{U}} \rightarrow F''$ by

$$\langle Q(y_{\iota})_{\mathfrak{U}}, y' \rangle := \lim_{\mathfrak{U}} \langle y_{\iota}, y'_{\iota} \rangle$$

where $y'_{\iota} = y'$ if $y' \in N$ and $= 0$ otherwise; Q has norm ≤ 1 .

For $\varphi_{\iota} := Q_{N^0}^F \circ \varphi|_{M_1 \times \dots \times M_n}$ one obtains

$$\varkappa_F \circ \varphi = Q \circ \overline{(\varphi_{\iota})}_{\mathfrak{U}} \circ (J_1, \dots, J_n) \in \mathcal{A}$$

with $\|\varkappa_F \circ \varphi\|_{\mathcal{A}} \leq \lim_t \|\varphi\|_{\mathcal{A}} \leq \|\varphi\|_{\mathcal{A}^{\max}}$; the regularity gives $\varphi \in \mathcal{A}$, and $\|\varphi\|_{\mathcal{A}} = \|\varkappa_F \circ \varphi\|_{\mathcal{A}} \leq \|\varphi\|_{\mathcal{A}^{\max}}$. \square

For $n = 1$ the equivalence (1) \curvearrowright (2) is the Heinrich-Kürsten result for Banach operator ideals and (2) \curvearrowright (3) Lotz' representation theorem. Note that the result implies that all $\mathcal{A}(E_1, \dots, E_n; F)$ are Banach spaces if $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is maximal.

4.6. As in 4.3 one obtains that the natural map

$$\otimes_{\gamma, k=1}^n (E_{k,t})_{\mathfrak{U}} \longrightarrow \left(\otimes_{\gamma, k=1}^n E_{k,t} \right)_{\mathfrak{U}}$$

defined by $(x'_t)_{\mathfrak{U}} \otimes \cdots \otimes (x''_t)_{\mathfrak{U}} \rightsquigarrow (x'_t \otimes \cdots \otimes x''_t)_{\mathfrak{U}}$ has norm ≤ 1 if γ is a finitely generated tensor norm or order n . For a proof define the tensor norm β of order $n + 1$ by

$$\otimes_{\beta}(E_1, \dots, E_n, E_{n+1}) := [\otimes_{\gamma}(E_1, \dots, E_n)] \otimes_{\pi} E_{n+1},$$

and apply the theorem to $\otimes : E_{1,t} \times \cdots \times E_{n,t} \longrightarrow \widetilde{\otimes}_{\gamma}(E_{1,t}, \dots, E_{n,t}) =: F_t$.

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