RELATIVE BRAUER GROUPS AND \( m \)-TORSION

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Abstract. Let \( K \) be a field and \( Br(K) \) its Brauer group. If \( L=K \) is a field extension, then the relative Brauer group \( Br(L/K) \) is the kernel of the restriction map \( res_{L/K} : Br(K) \to Br(L) \). A subgroup of \( Br(K) \) is called an algebraic relative Brauer group if it is of the form \( Br(L/K) \) for some algebraic extension \( L/K \). In this paper, we consider the \( m \)-torsion subgroup \( Br_m(K) \) consisting of the elements of \( Br(K) \) killed by \( m \), where \( m \) is a positive integer, and ask whether it is an algebraic relative Brauer group. The case \( K=\mathbb{Q} \) is already interesting: the answer is yes for \( m \) squarefree, and we do not know the answer for \( m \) arbitrary. A counterexample is given with a two-dimensional local field \( K=k((t)) \) and \( m=2 \).

1. Introduction

Let \( K \) be a field and \( Br(K) \) its Brauer group. If \( L/K \) is a field extension, then the relative Brauer group \( Br(L/K) \) is the kernel of the restriction map \( res_{L/K} : Br(K) \to Br(L) \). Relative Brauer groups have been studied by Fein and Schacher (see e.g. [1, 2, 3]). Every subgroup of \( Br(K) \) is a relative Brauer group \( Br(L/K) \) for some extension \( L/K \), and the question arises as to which subgroups of \( Br(K) \) are algebraic relative Brauer groups, i.e., of the form \( Br(L/K) \) with \( L/K \) an algebraic extension. For example, if \( L/K \) is a finite extension of number fields, then \( Br(L/K) \) is infinite [1], so no finite subgroup of \( Br(K) \) is an algebraic relative Brauer group. In this paper, we consider the \( m \)-torsion subgroup \( Br_m(K) \) consisting of the elements of \( Br(K) \) killed by \( m \), where \( m \) is a positive integer, and ask when is it an algebraic relative Brauer group. For example, if \( K \) is a \((p\text{-adic})\) local field, then \( Br(K) \cong \mathbb{Q}/\mathbb{Z} \), so \( Br_m(K) \) is an algebraic relative Brauer group for all \( m \). This is not surprising, since this Brauer group is “small.” The next natural field to look at is a number field, e.g., the rational field \( \mathbb{Q} \). Here the situation is somewhat surprising: \( Br_m(\mathbb{Q}) \) is an algebraic relative Brauer group for all squarefree \( m \), and the question for arbitrary \( m \) remains open. In order to construct a counterexample, we take \( K \) to be a “two-dimensional local field” \( k((t)) \) and prove that \( Br_2(K) \) is not an algebraic relative Brauer group. We believe that the situation where the \( m \)-torsion subgroup of the Brauer group is an algebraic relative Brauer group should be exceptional for general fields.
2. REDUCTION

Lemma 2.1. Let $K$ be a field and $Br(K)$ its Brauer group. Let $m_1, m_2$ be relatively prime positive integers. Let $L_1, L_2$ be algebraic extensions of $K$ such that every prime dividing $[L_i : K]$ divides $m_i$, $i = 1, 2$. ($p$ divides $[L_i : K]$ iff $p$ divides $[F : K]$ for some finite subextension $F/K$ of $L_i/K$.) Assume that the relative Brauer group $Br(L_i/K)$ equals the $m_i$-torsion subgroup $Br_{m_i}(K)$, $i = 1, 2$. Then $Br(L_1L_2/K) = Br_{m_1m_2}(K)$.

Proof. It is clear that $Br(L_1L_2/K) \supseteq Br_{m_1m_2}(K)$. For the opposite inclusion, let $[A] \in Br(L_1L_2/K)$. Then $[A] \in Br(F/K)$ for some finite extension $F/K$, $F \subseteq L_1L_2$. Let $F = K(\alpha_1, \beta_1, \gamma_1, \ldots, \alpha_d, \beta_d, \gamma_d)$. Then $F \subseteq E_1E_2$, where $E_j = K(\{\alpha_j^{(1)}, \beta_j^{(1)}, \ldots, \gamma_j^{(1)}\})$, $E_j \subseteq L_j$, so $[A] \in Br(E_1E_2/K), [E_1 : K] = n_i$, where $p|n_i \Rightarrow p|m_i$. In particular, $(n_1, n_2) = 1$.

Writing $E = E_1E_2$, we have, noting that $[E : E_1] = n_2$,

$$0 = core_{E_1} res_{E/K}[A] = core_{E_1} res_{E/E_1} res_{E_1/K} [A] = n_2 res_{E_1/K} [A]$$

$$= res_{E_1/K} (n_2[A]) \implies n_2[A] \in Br_{m_1}(K).$$

Hence $m_1n_2[A] = 0$. Similarly, $m_2n_1[A] = 0$. Hence $(m_1n_2, m_2n_1)[A] = 0$, and $(m_1n_2, m_2n_1) = d_1d_2$, where $d_i = (m_i, n_i)$, $i = 1, 2$, so $d_1d_2|m_1m_2$, whence $[A] \in Br_{m_1m_2}(K)$. \hfill $\Box$

Corollary 2.2. Suppose for each prime $p$ dividing $m$, $p^r$ is the exact power of $p$ dividing $m$ and there exists an algebraic extension $L^{(p)}/K$ of $p$-power degree (possibly $p^\infty$) such that $Br_{L^{(p)}}(K) = Br(L^{(p)})$. Then $Br_{m}(K) = Br(L/K)$ with $L$ equal to the composite of the $L^{(p)}$, $p|m$.

3. m-TORSION OVER $\mathbb{Q}$

Theorem 3.1. Let $l$ be an odd prime. Let $S_0$ denote the set of primes $p$ satisfying $p \not\equiv 1 \pmod{l}$, and set $S := S_0 \cup \{l\}$. Define $L$ to be the extension of $\mathbb{Q}$ generated by the $l$th roots of the elements of $S$. Then $Br(L/\mathbb{Q}) = Br_1(\mathbb{Q})$.

Proof. Note that the set $S$ is infinite by Dirichlet’s density theorem. Let $\alpha = [A] \in Br_1(\mathbb{Q}), E \subseteq L$, $E/\mathbb{Q}$ finite. We have a commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & Br(E) & \longrightarrow & \bigoplus_p \bigoplus_{p|p} Br(E_p) & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
\uparrow res & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & Br(\mathbb{Q}) & \longrightarrow & \bigoplus_p Br(\mathbb{Q}_p) & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0
\end{array}
$$

where the horizontal sequences are the fundamental exact sequences of $Br(\mathbb{Q})$, $Br(E)$, and the middle vertical arrow is for each $p$, the direct sum of the restriction maps $res_{E_p/\mathbb{Q}_p}$ for $p|p$.

We want to prove that $L$ splits $\alpha$, so we will show that some finite subextension $E$ of $L$ splits $\alpha$. If $(\alpha_p)_p$ is the image of $\alpha$ in $\bigoplus_p Br(\mathbb{Q}_p)$, we seek an $E$ such that $E_p$ splits $\alpha_p$ for all $p$ and all $p|p$. Of course we need only consider the finitely many $p$ for which $\alpha_p \neq 0$, hence if we can find, for each such $p$, a finite extension $E^{(p)} \subset L$ such that $E^{(p)}_p$ splits $\alpha_p$ for all $p|p$, then the composite $E$ of the $E^{(p)}$ will split $\alpha$. 

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There are two cases:

Case 1. \( p \in S \).

In this case, take \( E^{(p)} = \mathbb{Q}(p^{1/l}) \) which is contained in \( L \) by definition. \( p \) is totally ramified of degree \( l \) at \( p \), \( [E^{(p)}_p : \mathbb{Q}_p] = l \), hence \( E^{(p)}_p \) splits \( \alpha_p \) for every \( p | p \) (there is only one \( p | p \) in \( E^{(p)} \)).

Case 2. \( p \not\in S \).

It suffices to find a prime \( q \in S \) such that \( \mathbb{Q}(q^{1/l}) \) has local degree \( l \) at \( p \). Choose \( q \in S \) such that \( q \) is a primitive root mod \( p \), by the Chinese remainder theorem and Dirichlet’s density theorem. Since \( p \not\in S \), \( p \equiv 1 \) (mod \( l \)), so adjoining an \( l \)-th root of \( q \) to \( \mathbb{F}_p \) gives an extension of degree \( p \). This insures that \( p \) remains prime in \( \mathbb{Q}(q^{1/l}) \), so taking \( E^{(p)} = \mathbb{Q}(q^{1/l}) \), we are done in this case, similar to Case 1, since again there is only one prime of \( E^{(p)} \) above \( p \). This proves \( Br(L/Q) \rightarrow Br(Q) \).

In the opposite direction, let \( \alpha \in Br(L/Q) \). Then \( \alpha \in Br(L'/Q) \) for some finite subextension \( L'/Q \) of \( L/Q \). Since every finite subextension of \( L/Q \) is contained in a finite composite of extensions \( \mathbb{Q}(q^{1/l}) \), we may assume that \( L' \) is such a composite. Observe that \( [L' : Q] = l \) is a power of \( l \); in fact, it is \( l^n \), where \( L' \) is the composite of \( n \) of the fields \( \mathbb{Q}(q^{1/l}) \). (Indeed, if we write \( L' = L''(q^{1/l}) \) with \( L'' \) a smaller composite, then \( q \) is totally ramified in \( \mathbb{Q}(q^{1/l}) \) and unramified in \( L'' \), so \( L''(q^{1/l})/L'' \) is totally ramified at \( q \).) Hence \( \alpha \) has order a power of \( l \), by a restriction-corestriction argument. To show \( \alpha \in Br_I(Q) \), it suffices to show that \( \alpha \) does not have order larger than \( l \), i.e., at most one of the local invariants has order larger than \( l \), for which it suffices to show that for all primes \( p \), with one possible exception \( p = l \), \( [L'_p : \mathbb{Q}_p] = p \) is not divisible by \( l^2 \) for at least one \( p | p \) in \( L' \). In fact, we will show this for all \( p | p \) in \( L' \).

For \( p = \infty \) this is trivial since \( l \) is odd.

Case 1. \( p \not\in S \). \( p \) is a composite of cyclic unramified extensions of degree \( l \), hence of degree dividing the least common multiple of integers \( \leq l \), hence not divisible by \( l^2 \).

Case 2. \( p \in S, p \neq l \). Without loss of generality, \( L' \) contains \( \mathbb{Q}(p^{1/l}) \), which is totally ramified of degree \( l \) at \( p \). For \( q \in S, q \neq p, q \) is an \( m \)-th power mod \( p \) since \( (m, p - 1) = 1 \) \((q \not\equiv 1 \) mod \( l \)). Hence the polynomial \( x^m - q \) has a root in \( \mathbb{Q}_p \). It follows that for every \( p | p \) in \( L' \), \( L'_p \) is a composite of \( \mathbb{Q}_p(p^{1/l}) \) with \( \mathbb{Q}_p(\zeta) \), where \( \zeta \) is some \( \ell \)-th root of unity. Hence \( [\mathbb{Q}_p(\zeta) : \mathbb{Q}_p] \) divides \( l - 1 \). Therefore, \( [L'_p : \mathbb{Q}_p] \) is not divisible by \( l^2 \) for every \( p | p \) in \( L' \).

By Theorem 3.1 and Corollary 2.2, we have

**Corollary 3.2.** If \( m \) is an odd squarefree integer, then there exists an algebraic extension \( L \) of \( \mathbb{Q} \) such that \( Br_m(Q) = Br(L/Q) \).

We now turn to the case \( m = 2 \).

**Theorem 3.3.** There is a composite \( L \) of quadratic extensions of \( \mathbb{Q} \) such that \( Br_2(Q) = Br(L/Q) \).

**Proof.** Let us call a set \( S \) of odd primes **perfect** iff:

- \( p \equiv 1 \) (mod \( 4 \)) for every \( p \in S \), and
- for any two distinct primes \( p, q \in S \), \( p \) is a quadratic residue modulo \( q \).

There exists a (nonunique) maximal perfect set \( M \) (by recursive construction or by Zorn’s Lemma). Set \( L := \mathbb{Q}((\sqrt{-1}, \{ \sqrt{p} | p \in M \})) \). We show \( Br_2(Q) = Br(L/Q) \).

**Claim.** For every prime \( p \) (including \( \infty \)), \([L_p : \mathbb{Q}_p]\) is even, and is equal to \( 2 \) if \( p \neq 2 \).
Let us first show that the claim implies the result. Consider an element in $Br(L/\mathbb{Q})$. As before, restriction-corestriction implies that the element has 2-power order. It cannot have order bigger than two since $[L_p : \mathbb{Q}_p]$ is bigger than two at only one prime. Conversely, any element of $Br_2(\mathbb{Q})$ is split by $L$, since it is split by $L$ locally at every prime.

Proof of the Claim. For $p = \infty$ it is clear since $\sqrt{-1} \in L$. For $p \in M$, $L_p = \mathbb{Q}_p(\sqrt{p})$ since $M$ is perfect. Finally, let $p \notin M$. Then $L_p/\mathbb{Q}_p$ is unramified, hence of degree 1 or 2. If $p \equiv 3 \pmod{4}$, then the degree is 2 since $\sqrt{-1} \notin \mathbb{Q}_p$, so assume that $p \equiv 1 \pmod{4}$, and contrarily that the degree is 1. Then for every $q \in M$, $q$ is a quadratic residue mod $p$, which implies, by quadratic reciprocity, that $M \cup \{p\}$ is perfect, contradicting the maximality of $M$.

Corollary 3.4. If $m$ is a positive squarefree integer, then there exists an algebraic extension $L$ of $\mathbb{Q}$ such that $Br_m(\mathbb{Q}) = Br(L/\mathbb{Q})$.

4. A COUNTEREXAMPLE

It is conceivable that for any number field $K$ and any $m$, there exists an algebraic extension $L/K$ such that $Br_m(K) = Br(L/K)$; in any event, we have no counterexample to this for $K$ a number field. We therefore give a counterexample with $K$ a “two-dimensional local field”.

Let $K$ be a Laurent series field $k((t))$, where $k$ is any nonarchimedean local field containing $\sqrt{-1}$. We show that there is no algebraic extension $L$ of $K$ such that $Br_2(K) = Br(L/K)$. Suppose $L$ were such an extension. By a theorem of Witt [16 p. 186],

$$Br(K) \cong Br(k) \oplus Hom(G_k, \mathbb{Q}/\mathbb{Z})$$

where $G_k$ denotes the absolute Galois group of $k$. Extracting 2-torsion,

$$Br_2(K) \cong Br_2(k) \oplus Hom(G_k^{(2)}, \mathbb{Z}/2)$$

where $G_k^{(2)}$ denotes the maximal elementary abelian 2-quotient of $G_k$. These are finite groups by local class field theory, hence, without loss of generality, $L/K$ is a finite extension. Let $L_1/K$ denote the maximal subextension of $L/K$ which is unramified (constant) with respect to $t$. Then $L_1 = \ell_1((t))$, $\ell_1/k$ finite. We claim $[\ell_1 : k] = 2$. If $[\ell_1 : k] > 2$, $L_1$ would split a constant algebra (coming from $Br(k)$) of order $> 2$, hence so would $L$, contrary to hypothesis. If $[\ell_1 : k] = 1$, then $L/K$ would be totally ramified, $L = k((u))$ ($u$ a local uniformizer for $L$), and $L$ would not split a constant algebra of order 2.

$L/L_1$ is totally (and tamely) ramified, so $L = L_1(\sqrt{\pi})$, where $\pi$ is a local uniformizer of $L_1 = \ell_1((t))$ as above, so $\pi = ct$, $c \in \ell_1^\times$. Now $e = [L : L_1]$ is even, for otherwise, $Br_2(K)$ would equal $Br(L_1/K)$. This is impossible as follows: write $\ell_1 = k(\sqrt{a})$ and choose $b \in k^*$ so that $a,b$ are multiplicatively independent in $k^*/k^{*2}$ (such a $b$ exists!). Then $L_1$ does not split the quaternion algebra $(b, t)$.

Since $e = [L : L_1]$ is even, $L$ contains $L_1(\sqrt{d}) =: L_2$. Consider the fourth power symbol algebra $(a, t)_4$ over $K = k((t))$. If $L$ splits this algebra which has exponent four, we have a contradiction. Suppose not. By [14 p. 261], $(a, t)_4 \otimes_K L_1$ is Brauer equivalent to the quaternion algebra $(\sqrt{a}, t)$ over $L_1$. Tensoring this up to $L_2$ gives

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1. We thank the referee for observing that the above proof holds for $k$ any nonarchimedean local field containing $\sqrt{-1}$; in the original version $k$ was $\mathbb{Q}_p$ with $p \equiv 1 \pmod{4}$. 

\((\sqrt{a}, t) = (\sqrt{a}, c^{-1} t) \sim (\sqrt{a}, c^{-1})(\sqrt{a}, c t) \sim (\sqrt{a}, c^{-1})\). By assumption this is not split. \((\sqrt{a}, c^{-1})\) is a constant algebra, defined over \(\ell_1\). But \(Br_2(\ell_1) \cong \mathbb{Z}/2\mathbb{Z}\) (\(\ell_1\) is a local field). Take an algebra class \([A]\) of exponent four in \(Br(k)\). Its restriction to \(L_1\) has exponent two, hence is equivalent to \((\sqrt{a}, c^{-1})\). Let \([A]\) also denote the corresponding (constant) algebra class in \(Br(K)\), and set \([B] := [A]^{-1}[(a, t)_4] \in Br(K)\). Then \(L_2\) splits \(B\), whereas \([B]^2 = [A]^{-2}[(a, t)_4]^2 = [A]^{-2}[(a, t)]\) which is not split because the first factor is a constant algebra class of order two and the second is a nonconstant algebra class of order two. Thus \([B]\) is an algebra class of order four in \(Br(K)\) which is split by \(L\), contradiction. We conclude \(Br_2(K) \neq Br(L/K)\) for all algebraic extensions \(L/K\).

**References**


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