CRITICAL POINTS OF THE AREA FUNCTIONAL
OF A COMPLEX CLOSED CURVE
ON THE MANIFOLD OF KÄHLER METRICS

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ABSTRACT. We consider a compact complex manifold $M$ of dimension $n$ that admits Kähler metrics and we assume that $C \hookrightarrow M$ is a closed complex curve. We denote by $KC(1)$ the space of classes of Kähler forms $[\omega] \in H^{1,1}(M, \mathbb{R})$ that define Kähler metrics of volume 1 on $M$ and define $A_C : KC(1) \rightarrow \mathbb{R}$ by $A_C([\omega]) = \int_C \omega = \text{area of } C$ in the induced metric by $\omega$. We show how the Riemann-Hodge bilinear relations imply that any critical point of $A_C$ is the strict global minimum and we give conditions under which there is such a critical point $[\omega]$: A positive multiple of $[\omega]^{n-1} \in H^{2n-2}(M, \mathbb{R})$ is the Poincaré dual of the homology class of $C$. Applying this to the Abel-Jacobi map of a curve into its Jacobian, $C \hookrightarrow J(C)$, we obtain that the Theta metric minimizes the area of $C$ within all Kähler metrics of volume 1 on $J(C)$.

1. Introduction

Let $M$ be a compact complex manifold of dimension $n$ that admit Kähler metrics. The Kähler cone

$$ KC := \{ [\omega] \in H^{1,1}(M, \mathbb{R}) | \omega \text{ is a Kähler form} \} $$

is an open convex cone in the vector space $H^{1,1}(M, \mathbb{R})$. On $KC$ we define the function

$$ V : KC \rightarrow \mathbb{R}^+, \quad V([\omega]) = \frac{1}{n!} \int_M \omega^n = \text{Vol}(M, \omega) $$

where $\omega$ is a Kähler form representing the class $[\omega]$ and $\text{Vol}(M, \omega)$ denotes the volume of $M$ in the metric induced by $\omega$. Suppose that $C \hookrightarrow M$ is a closed complex curve in $M$. For $[\omega] \in KC$, the area of $C$ in the induced metric of $\omega$ is given by

$$ \tilde{A}_C(\omega) := \int_C \omega = \text{area of } C. $$

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Lemma 2.1. Let all Kahler metrics of volume

Corollary. The Theta metric minimizes the area of the Abel-Jacobi curve defined by the Riemann Theta divisor of canonical \( \Theta \) bundle defined by the Riemann Theta divisor of \( \Theta \) on the Jacobian \( J \) of \( M \), called the Theta metric.

Proof. This is a simple computation:

\[
\text{PD} \{ [\omega] \} = \alpha [\omega]^{n-1} \text{ for some } \alpha > 0.
\]

Let \( C \) be a compact Riemann surface of genus \( g \geq 2 \), \( J(C) \) the Jacobian of \( C \) and \( C \hookrightarrow J(C) \) the Abel-Jacobi map (\( \mathbb{P} \), p. 235). In this case we have from Poincaré's formula (\( \mathbb{P} \), p. 350) that \( \text{PD} \{ [\omega] \} = \frac{[\omega]_{\text{Hodge}}}{(g-1)!} \), where \( \omega_0 \) is the 2-form on \( J(C) \) invariant under translations representing the first Chern class of the line bundle defined by the Riemann Theta divisor of \( J(C) \). In this case \( \omega_0 \) defines a canonical flat Kahler metric on \( J(C) \), called the Theta metric.

Corollary. The Theta metric minimizes the area of the Abel-Jacobi curve \( C \) within all Kahler metrics of volume 1 on the Jacobian \( J(C) \).

2. The Hessian of \( V \) and the Riemann-Hodge Bilinear Relations

Lemma 2.1. Let \( [\omega] \in KC \). Then for \( [\eta] \in T_{[\omega]}KC = H^{1,1}(M, \mathbb{R}) \) a tangent vector, we have

\[
dV_{[\omega]}([\eta]) = \int_M *_{\omega} \omega \wedge \eta =: \langle \omega, \eta \rangle_{\omega}
\]

where \( \langle , \rangle_{\omega} \) denotes the inner product on \( H^{1,1}(M, \mathbb{R}) \) and \( *_{\omega} \) is the star operator induced by \( \omega \).

Proof. This is a simple computation:

\[
dV_{[\omega]}(\eta) = \lim_{t \to 0} \frac{1}{t} (V(\omega + t\eta) - V(\omega)) = \lim_{t \to 0} \frac{1}{t} \left( \int_M (\omega + t\eta)^n - \int_M \omega^n \right)
\]

\[
= \frac{1}{n!} \lim_{t \to 0} \frac{1}{t} (nt \int_M (\omega^{n-1} \wedge \eta) + \binom{n}{2} \cdot t^2 \int_M (\eta^2 \wedge \omega^{n-2}) + \cdots)
\]

\[
= \frac{1}{n!} \int_M \omega^{n-1} \wedge \eta = \frac{1}{(n-1)!} \int_M \omega^{n-1} \wedge \eta.
\]

Since \( *_{\omega} \omega = \frac{\omega^{n-1}}{(n-1)!} \), we obtain that \( dV_{[\omega]}(\eta) = \langle \omega, \eta \rangle_{\omega} \).

Remark. Lemma 2.1 implies that \( \text{Kernel} dV_{\omega} = H^{1,1}_0([\omega]) \), where \( H^{1,1}_0([\omega]) \) is the space of primitive cohomology classes of \( \omega \) (\( \mathbb{P} \), p. 122). This lemma also implies
that $\mathcal{K} C(1) := \mathbf{V}^{-1}(1)$ is a smooth hypersurface and for $[\omega] \in \mathcal{K} C(1)$, $T_{[\omega]} \mathcal{K} C(1) = H_{n}^{1,1}([\omega])$.

For $[\omega] \in \mathcal{K} C$, we have the bilinear form $Q_{\omega} : H^{1,1}(M, \mathbb{R}) \times H^{1,1}(M, \mathbb{R}) \to \mathbb{R}$ defined by

$$Q_{\omega}(\eta_1, \eta_2) = \int_{M} \eta_1 \wedge \eta_2 \wedge \omega^{n-2}.$$  

**Riemann-Hodge bilinear relations** for $H^{1,1}(M, \mathbb{R})$. The form $Q_{\omega}$ is negative definite on primitive cohomology $H_{n}^{1,1}([\omega])$ ([11], p. 123).

**Lemma 2.2.** Let $[\omega] \in \mathcal{K} C$ and $[\eta], [\zeta] \in T_{[\omega]} \mathcal{K} C = H^{1,1}(M, \mathbb{R})$. Then we have

$$\text{Hess } V_{[\omega]}([\eta], [\zeta]) = \frac{1}{(n-2)!} \int_{M} \omega^{n-2} \wedge \eta \wedge \zeta$$

and $\text{Hess } V_{[\omega]}|_{H_{n}^{1,1}([\omega])}$ is negative definite.

**Proof.** By definition, $\text{Hess } V_{[\omega]} : H^{1,1}(M, \mathbb{R}) \times H^{1,1}(M, \mathbb{R}) \to \mathbb{R}$ is

$$\text{Hess } V_{[\omega]}([\eta], [\zeta]) = \lim_{t \to 0} \frac{1}{t} (dV_{\omega + t[\eta]}(\zeta) - dV_{\omega}(\zeta))$$

$$= \frac{1}{(n-1)!} \int_{M} \left( \int M (\omega + t[\eta])^{n-1} \wedge \zeta \right) - \left( \int M \omega^{n-1} \wedge \zeta \right)$$

$$= \frac{(n-1)}{(n-1)!} \int_{M} \omega^{n-2} \wedge \eta \wedge \zeta = \frac{1}{(n-2)!} \int_{M} \omega^{n-2} \wedge \eta \wedge \zeta.$$

So the lemma follows from the Riemann-Hodge bilinear relations.

3. The set $\mathcal{D}_1$ is strictly convex

Let $\mathcal{B} \subset \mathbb{R}^n$ be a set with nonempty interior: int $\mathcal{B} \neq \emptyset$. $\mathcal{B}$ is strictly convex if for all $x, y \in \mathcal{B}$, $x \neq y$, we have that $tx + (1 - t)y \in \text{int } \mathcal{B}$ for all $t \in (0, 1)$.

**Lemma 3.1.** Let $\mathcal{B} \subset \mathbb{R}^n$ be a strictly convex set with smooth boundary $\partial \mathcal{B}$. Let $T : \mathbb{R}^n \to \mathbb{R}$ be a linear functional. Consider the restriction $T := T|_{\mathcal{B}} : \mathcal{B} \to \mathbb{R}$. If $p \in \mathcal{B}$ is a strict local minimum of $T$, then $p$ is the strict global minimum of $T$.

**Proof.** This follows since $\mathcal{B} - \{p\}$ is contained in one of the half spaces obtained as the complement of the tangent hyperplane to $p \in \partial \mathcal{B}$.

**Proposition 3.2.** Let $\mathcal{B} \subset \mathbb{R}^n$ be open and convex. Let $f : \mathcal{B} \to \mathbb{R}$ be a function of class $C^2$ such that it satisfies the following conditions:

(a) for each $x \in \mathcal{B}$, $d f(x) \neq 0$,

(b) $(\text{Hess } f)(x)|_{F \ker d f(x)}$ is negative definite for all $x \in \mathcal{B}$.

Then the set $\mathcal{B}_1 := \{ x \in \mathcal{B} \mid f(x) \geq 1 \}$ is strictly convex.

**Proof.** Let $x, y \in \mathcal{B}_1$ be $x \neq y$, and consider the line $r(t) = tx + (1 - t)y$, $t \in [0, 1]$. Let $F : [0, 1] \to \mathbb{R}$ be defined by $F(t) = f(r(t))$. We have the following cases:

(i) If $F'(t) > 0$ in $(0, 1)$, then for all $t \in (0, 1)$, $F(t) > F(0) \geq 1$, hence $r((0, 1)) \in \text{int } \mathcal{B}_1$.

(ii) If $F'(t) < 0$ in $(0, 1)$, then for all $t \in (0, 1)$, $F(t) > F(1) \geq 1$, hence $r((0, 1)) \in \text{int } \mathcal{B}_1$. 

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By hypothesis we have
\[ F'(t_0) = D^2 f(r(t_0))(x - y, x - y) < 0, \]
that is, \( t_0 \) is a local maximum. Hence, at any critical point \( F \) has a local maximum, which clearly implies that \( t_0 \) is the unique global maximum of \( F \). This implies that for all \( t \in [0, t_0) \), \( F'(t) > 0 \), that is, \( F \) is increasing in \( [0, t_0) \); and for all \( t \in (t_0, 1] \), \( F'(t) < 0 \), then \( F \) is decreasing in \( (t_0, 1] \). This implies that \( F(t) > 1 \) for all \( t \in (0, 1) \), in other words, \( r((0, 1)) \subsetint \mathcal{B}_1 \).

Then (i), (ii), (iii) prove that \( \mathcal{B}_1 \) is strictly convex. \( \square \)

**Corollary 3.3.** The set \( \mathcal{D}_1 := \{ \omega \in \mathcal{K}\mathcal{C}|V(\omega) \geq 1 \} \) is strictly convex.

**Proof.** By Lemmas 2.1 and 2.2 we have that the function \( V \) satisfies the hypothesis of Proposition 3.2. \( \square \)

4. **Minimizing \( A_C \) on \( \mathcal{D}_1 \)**

**Lemma 4.1.** If \( [\omega] \in \mathcal{K}\mathcal{C}(1) \) is a critical point of \( A_C \), then \( [\omega] \) is the global minimum of \( A_C \) on \( \mathcal{D}_1 \). In particular \( [\omega] \) is the global minimum of \( A_C \) on \( \mathcal{K}\mathcal{C}(1) \).

**Proof.** Suppose that \( [\omega] \in \mathcal{K}\mathcal{C}(1) \) is a critical point of \( A_C \). This implies that there exists a \( \lambda_0 \in \mathbb{R} \) such that \( (\lambda_0, [\omega]) \) is a critical point of the Lagrange function

\[ L : \mathbb{R} \times \mathcal{K}\mathcal{C} \to \mathbb{R}, \quad L(\lambda, \omega) = \hat{A}_C(\omega) - \lambda V(\omega). \]

It is easy to see that \( \lambda_0 > 0. \) We have that the Hessian of \( \hat{A}_C \) is zero since it is linear. From Lemma 2.2, \( \text{Hess} \ V_{\omega} |_{H^{1,1}_0([\omega])} \) is negative definite; then

\[ \text{Hess} \ L(\lambda_0, [\omega]) |_{H^{1,1}_0([\omega])} = -\lambda_0 \text{Hess} \ V_{\omega} |_{H^{1,1}_0([\omega])} \]

is positive definite. Using the second derivative test criteria we have that \( [\omega] \) is a local strict minimum of \( A_C \) on \( \mathcal{K}\mathcal{C}(1) \). This implies that there exists an open neighbourhood \( U \) of \( [\omega] \) in \( \mathcal{K}\mathcal{C}(1) \) such that \( A_C([\omega]) < A_C([\bar{\omega}]) \) for all \( \bar{\omega} \in U - \{[\omega]\} \). Note that \( W = \{t[\omega]| t \geq 1, [\bar{\omega}] \in U \} \) is an open set in \( \mathcal{D}_1 \). By linearity we have that \( A_C([\omega]) < t A_C([\bar{\omega}]) \) for all \( t \bar{\omega} \in W \). Then \( [\omega] \) is a strict local minimum of \( A_C \) on \( \mathcal{D}_1 \). Applying Lemma 3.1 we have that \( [\omega] \) is the global minimum of \( A_C \) on \( \mathcal{D}_1 \). In particular \( [\omega] \) is the global minimum of \( A_C \) on \( \mathcal{K}\mathcal{C}(1) \). \( \square \)

**Proof of the Theorem.** Lemma 4.1 proves part (i).

To prove part (ii) suppose that \( \text{PD} \ [C] = \alpha [\omega]^{n-1}, [\omega] \in \mathcal{K}\mathcal{C}(1), \alpha > 0. \) Then it is easy to see that \( d(\hat{A}_C)[\omega] - \lambda_0 dV[\omega] = 0, \lambda_0 = \alpha(n-1)! \). Then by Lagrange multipliers we have that \( [\omega] \) is a critical point of \( A_C \).

Now suppose that \( [\omega] \in \mathcal{K}\mathcal{C}(1) \) is a critical point of \( A_C \), that is, \( dV[\omega] \wedge (d\hat{A}_C)[\omega] = 0. \) Then for \( [\eta], [\zeta] \in T_{[\omega]} \mathcal{K}\mathcal{C} \) we have

\[ dV_{[\omega]}([\eta])d(\hat{A}_C)[\omega]([\zeta]) = dV_{[\omega]}([\zeta])d(\hat{A}_C)[\omega]([\eta]). \tag{4.1} \]

In particular we can take \( [\zeta] = [\omega] \) and \( [\eta] \in H^{1,1}_0([\omega]) \), that is, \( \langle \omega, \eta \rangle [\omega] = 0. \) By Lemma 2.1 and (4.1) we have that \( d(\hat{A}_C)[\omega]([\eta]) = 0. \) By linearity of \( \hat{A}_C \),

\[ d(\hat{A}_C)[\omega]([\eta]) = \hat{A}_C(\hat{\eta}), \]

then

\[ 0 = \int_C \eta = \int_M \eta \wedge = \omega (\star_\omega (\text{PD} \ [C])) = \langle \eta, \star_\omega (\text{PD} \ [C]) \rangle \]
for all \([\eta] \in H^{1,1}_0([\omega])\). Hence there exists \(r_1 \in \mathbb{R}\) such that \(*_\omega \text{PD} ([C]) = r_1 [\omega]\). From the fact that \(C\) is a complex curve there exists a positive real number \(r_2\) such that \(\text{PD} [C] = r_2 [\omega]^{n-1}/(n-1)!\), we take \(\alpha = r_2/(n-1)!\).

**Proof of the Corollary.** By Poincaré’s formula we have that
\[
\text{PD}[C] = [\omega_0]^{n-1}/(n-1)!
\]
Applying the Theorem we obtain the Corollary.

**Example 1.** We consider the projective line \(\mathbb{CP}^1\) and \(p \in \mathbb{CP}^1\). Let \(M = \mathbb{CP}^1 \times \mathbb{CP}^1\) and define
\[
j: \mathbb{CP}^1 \hookrightarrow M, \quad j(x) = (x, p).
\]
\(C := j(\mathbb{CP}^1)\). We have projections \(\pi_1, \pi_2 : M \to \mathbb{CP}^1\), \(\pi_1(x, y) = x, \pi_2(x, y) = y\), and let \(\omega\) be the Fubini-Study form of \(\mathbb{CP}^1\). \(\omega_t := t\pi_1^* (\omega) + \frac{1}{t}\pi_2^* (\omega), t > 0\), are Kähler forms on \(M\) of volume 1. Clearly \(\int_C \omega_t = t\). In this case we have that the infimum \(\{A_C(\omega_t)\}\) is \(0\). Hence \(A_C\) has no minimum on \(\mathcal{K}C(1)\).

However when we consider the diagonal map \(\Delta : \mathbb{CP}^1 \hookrightarrow M, \quad \Delta(x) = (x, x)\), and
\(C := \Delta(\mathbb{CP}^1)\), we have \(\text{PD} [C] = \pi_1^* (\omega) + \frac{1}{2}\pi_2^*(\omega)\), and by the Theorem the minimum of \(A_C\) is \(1\) and is obtained in \(\frac{1}{2}[\pi_1^*(\omega) + \pi_2^*(\omega)]\). This example can be generalized to \(\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1\) \(k\)-times.

**Example 2.** Let \(L \to M\) be an ample line bundle on a complex \(n\)-dimensional manifold \(M\) and \(c_1(L) = [\omega]\) be the first Chern class of \(L\). Then there exist complex closed curves \(C \hookrightarrow M\) such that \(\text{PD} [C] = \alpha [\omega]^{n-1}, \alpha > 0\): Embed \(M \hookrightarrow \mathbb{CP}^N\) with a multiple of \(L\), then intersect \(M\) with generic hyperplanes of \(\mathbb{CP}^N\) until one obtains a curve \(C\) as desired.

**References**


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